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RAPIDLY GROWING AND BLOW-UP SOLUTIONS TO HIGHER ORDER NONLINEAR DELAY ORDINARY DIFFERENTIAL EQUATIONS

Dedicated to Professor T. Kusano

Abstract. For higher order nonlinear delay ordinary differential equations, sufficient conditions for the existence of multi-parameter sets of rapidly growing and blow-up solutions are established and the asymptotic estimates of such solutions are obtained.

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1 Statement of the problem and formulation of the main results

Problems on the existence and asymptotic estimates of rapidly growing and blow-up solutions occupy an important place in the qualitative theory of ordinary differential equations and they have been investigated in sufficient detail for a wide class of nonlinear non-autonomous differential equations (see, [1–12] and the references therein). However, for delay ordinary differential equations these problems still remain unstudied. The present paper is devoted to filling this existing gap.

We consider the differential equation

$$u^{(n)}(t) = f(t, u(\tau(t))), \tag{1.1}$$

where n is an arbitrary natural number, $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function, $\mathbb{R}_+ = [0, +\infty[$, and $\tau : \mathbb{R}_+ \to \mathbb{R}$ is a continuous function such that

$$\tau(t) \le t \ \text{ for } \ t \in \mathbb{R}_+, \quad \lim_{t\to+\infty} \tau(t) = +\infty.$$

Definition 1.1. Let $a \in \mathbb{R}_+$, $b \in]a, +\infty[$, and

$$a_0 = \min\{\tau(t) : \ t \ge a\} < a \quad \big(a_0 = \min\{\tau(t) : \ a \le t \le b\} < a\big).$$

An n-times continuously differentiable function $u : [a, +\infty[\to \mathbb{R}_+$ ($u : [a, b[\to \mathbb{R}_+$) is said to be a solution to equation (1.1) in the interval $[a, +\infty[$ (in the interval $[a, b[$) if there exists a continuous function $u_0 : [a_0, a] \to \mathbb{R}_+$ such that equality (1.1) is satisfied in that interval, where

$$u(t) = u_0(t) \ \text{ for } \ a_0 \le t \le a.$$

Definition 1.2. A solution u to equation (1.1), defined in some infinite interval $[a, +\infty[\subset \mathbb{R}_+$, is said to be **rapidly growing** (**slowly growing**) if

$$\lim_{t\to+\infty} u^{(n-1)}(t) = +\infty \quad \big(\lim_{t\to+\infty} u^{(n-1)}(t) < +\infty\big).$$

Definition 1.3. A solution u to equation (1.1), defined in some finite interval $[a, b[\subset \mathbb{R}_+$, is said to be **blow-up** (**bounded**) if

$$\lim_{t\to b} u(t) = +\infty \quad \big(\lim_{t\to b} u(t) < +\infty\big).$$

For equation (1.1), in the nonnegative semi-axis $\mathbb{R}_+$ and in some finite interval $[0, b[\subset \mathbb{R}_+$, the Cauchy problem with the initial data

$$u(t) = u_0(t) \ \text{ for } \ t_* \le t < 0, \ \ u^{(i-1)}(0) = c_i \ \ (i = 1, \dots, n) \tag{1.2}$$

is investigated in the cases where the function τ satisfies the conditions

$$\tau(t) < t \ \text{ for } \ t \in \mathbb{R}_+, \quad \lim_{t\to+\infty} \tau(t) = +\infty \tag{1.3}$$

and

$$\tau(t) < t \ \text{ for } \ t \in [0, b[, \quad \tau(b) = b, \tag{1.4}$$

respectively.

Here

$$t_* = \min\{\tau(t) : \ t \in \mathbb{R}_+\} \quad \big(t_* = \min\{\tau(t) : \ 0 \le t < b\}\big),$$

$(c_1, \dots, c_n) \in \mathbb{R}^n_+$, while $u_0 : [t_*, 0] \to \mathbb{R}_+$ is a continuous function such that $u_0(0) = c_1$.

A solution u to equation (1.1), defined in the interval $\mathbb{R}_+$ (in the interval $[0, b[$) and satisfying the initial conditions (1.2), is said to be a **solution to problem** (1.1)**,** (1.2) in that interval.

Condition (1.3) (condition (1.4)) guarantees the existence of a unique solution to problem (1.1), (1.2) in the interval $\mathbb{R}_+$ (in the interval $[0, b[$). Our aim is to find conditions under which the above mentioned solution is, respectively, rapidly growing or slowly growing (blow-up or bounded).

Put

$$f^*(t, x) = \max\{f(t, y) : \ 0 \le y \le x\} \ \text{ for } \ (t, x) \in \mathbb{R}^2_+, \quad \|u_0\| = \max\{u_0(t) : \ t_* \le t \le 0\}.$$

Theorem 1.1. *Let along with* (1.3) *the conditions*

$$f(t,x) \geq f_0(t,x) \ \ for \ \ t \geq t_0, \ \ x \geq 0, \tag{1.5}$$

$$(c_1,\dots,c_n) \in \mathbb{R}_+^n, \ \ c_1 = u_0(0), \ \ c_n > 0 \tag{1.6}$$

hold, where $t_0 \geq 0$ *and* $f_0 : [t_0,+\infty[\,\times \mathbb{R}_+ \to \mathbb{R}_+$ *is a nondecreasing in the second argument continuous function such that*

$$\int_{t_0}^{+\infty} f_0\big(t, |\tau(t)|^{n-1}x\big)\,dt = +\infty \ \ for \ \ x > 0. \tag{1.7}$$

Then the solution to problem (1.1), (1.2) *is rapidly growing. If along with* (1.3) *the condition*

$$\lim_{x\to 0} \int_0^{+\infty} \frac{f^*\big(t,(1+|\tau(t)|)^{n-1}x\big)}{x}\,dt = 0 \tag{1.8}$$

is satisfied, then there exists $\varepsilon > 0$ *such that in the case, where*

$$\|u_0\| \leq \varepsilon, \ \ c_1 = u_0(0), \ \ 0 \leq c_i < \varepsilon \ \ (i = 1,\dots,n), \tag{1.9}$$

the solution to problem (1.1), (1.2) *is slowly growing.*

Corollary 1.1. *Let along with* (1.3) *the condition*

$$p(t)x^\lambda \leq f(t,x) \leq \ell p(t)x^\lambda \ \ for \ \ (t,x) \in \mathbb{R}_+^2 \tag{1.10}$$

hold, where $\lambda > 1$, $\ell > 1$, *and* $p : \mathbb{R}_+ \to \mathbb{R}_+$ *is a continuous function. Then the condition*

$$\int_0^{+\infty} |\tau(t)|^{(n-1)\lambda} p(t)\,dt = +\infty \tag{1.11}$$

is necessary and sufficient for the solution to problem (1.1), (1.2) *to be rapidly growing for any continuous function* $u_0 : [t_*, 0] \to \mathbb{R}_+$ *and the initial data, satisfying condition* (1.6).

The question arises: may equation (1.1) have a rapidly growing solution if condition (1.8) is satisfied? Theorems 1.2 and 1.3 below and their corollaries give a positive answer to this question. According to these statements, equation (1.1) may have an n-parametric set of rapidly growing solutions even in the case where condition (1.10) is satisfied but condition (1.11) is violated.

Theorem 1.2. *Let conditions* (1.3) *and* (1.5) *hold, where* t_0 *is a positive number, while* $f_0 : [t_0,+\infty[\,\times \mathbb{R}_+ \to \mathbb{R}_+$ *is a continuous and nondecreasing in the second argument function such that the differential equation*

$$v^{(n)}(t) = f_0(t, v(\tau(t))) \tag{1.12}$$

in the interval $[t_0,+\infty[$ *has a rapidly growing solution* v. *Then there exist numbers* $r > 0$ *and* $a \geq t_0$ *such that if*

$$(c_1,\dots,c_n) \in \mathbb{R}_+^n, \ \ c_1 = u_0(0), \ \ c_n \geq r, \tag{1.13}$$

then the solution to problem (1.1), (1.2) *is rapidly growing and admits the estimates*

$$u^{(i-1)}(t) \geq v^{(i-1)}(t) \ \ for \ \ t \geq a \ \ (i = 1,\dots,n). \tag{1.14}$$

Corollary 1.2. *Let the functions* f *and* τ *satisfy the inequalities*

$$f(t,x) \geq t^\mu x^\lambda \ \ for \ \ t \geq t_0, \ \ x \geq 0, \tag{1.15}$$

$$t^\alpha - \delta \leq \tau(t) < t \ \ for \ \ t \geq 0, \tag{1.16}$$

where

$$\lambda > 1, \ \ \mu \in \mathbb{R}, \ \ \alpha \in \,]\lambda^{-1}, 1[\,, \ \ \delta \geq 1, \ \ t_0 \geq 1.$$

Then for any $\gamma > n-1$ *there is a positive number* $r = r(\gamma)$ *such that if condition* (1.13) *is fulfilled, then the solution to problem* (1.1), (1.2) *is rapidly growing and admits the estimate*

$$\inf\big\{t^{-\gamma}u(t) : \ t \geq t_0\big\} > 0. \tag{1.17}$$

Corollary 1.3. *Let the functions f and τ satisfy the inequalities*

$$f(t,x) \geq \exp(\mu t)x^{\lambda} \ \text{ for } t \geq t_0, \ \ x \geq 0, \tag{1.18}$$
$$\alpha t - \delta \leq \tau(t) < t \ \text{ for } t \geq 0, \tag{1.19}$$

where

$$\lambda > 1, \ \ \mu \in \mathbb{R}, \ \ \alpha \in]\lambda^{-1}, 1], \ \ \delta > 0, \ \ t_0 > 0.$$

Then for any $\gamma > 0$ there is a positive number $r = r(\gamma)$ such that if condition (1.13) is fulfilled, then the solution to problem (1.1), (1.2) is rapidly growing and admits the estimate

$$\inf\Big\{ \exp(-\gamma t)u(t) : \ t \geq t_0 \Big\} > 0. \tag{1.20}$$

Corollary 1.4. *Let the function f satisfy the inequality*

$$\exp(\mu t)x^{\lambda} \leq f(t,x) \leq t^{\nu}x^{\lambda} \ \text{ for } t \geq t_0, \ \ x \geq 0, \tag{1.21}$$

and let the function τ satisfy inequality (1.19), where

$$t_0 > 0, \ \ \lambda > 1, \ \ \mu < 0, \ \ \nu < -(n-1)\lambda - 1, \ \ \alpha \in]\lambda^{-1}, 1], \ \ \delta > 0.$$

Then there exists $\varepsilon > 0$ such that if condition (1.9) is fulfilled, then the solution to problem (1.1), (1.2) is slowly growing. On the other hand, for any $\gamma > 0$ there is a number $r = r(\gamma) > \varepsilon$ such that if condition (1.13) is satisfied, then the solution to problem (1.1), (1.2) is rapidly growing and admits estimate (1.20).

Theorem 1.2 is not applicable in the case where the function f does not have a nondecreasing in the second argument nontrivial nonnegative minor. Theorem 1.3 below deals with this case.

Let $n \geq 2$, $t_0 > 0$, and the function f admit estimate (1.5), where $f_0 : [t_0, +\infty[\times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function. Put

$$F_n(t,x) = \left(\frac{n}{(n-2)!} \int\limits_0^x (x-y)^{n-2} f_0(t,y)\, dy \right)^{\frac{1}{n}} \ \text{ for } t \geq t_0, \ \ x \geq 0, \tag{1.22}$$

and consider the differential equation

$$v'(t) = F_n(t, v(\tau(t))). \tag{1.23}$$

Theorem 1.3. *Let $n \geq 2$ and there exist a positive number t_0 and a nonincreasing in the first argument continuous function $f_0 : [t_0, +\infty[\times \mathbb{R}_+ \to \mathbb{R}_+$ such that the function τ is continuously differentiable in the interval $[t_0, +\infty[$,*

$$\tau(t_0) = 0, \ \ \tau(t) > 0, \ \ 0 \leq \tau'(t) \leq 1 \ \text{ for } t \geq t_0, \tag{1.24}$$

and conditions (1.3), (1.5) hold. Let, moreover, the differential equation (1.23) in the interval $[t_0, +\infty[$ has a solution v, satisfying the equality

$$\lim_{t \to +\infty} \frac{v(t)}{t^{n-1}} = +\infty. \tag{1.25}$$

Then for any continuous function $u_0 : [t_*, 0] \to \mathbb{R}_+$ there are numbers $r = r(u_0) > 0$, $a = a(u_0) > t_0$ such that if condition (1.13) holds, then the solution to problem (1.1), (1.2) is rapidly growing and admits the estimate

$$u(t) \geq v(t) \ \text{ for } t \geq a. \tag{1.26}$$

Corollary 1.5. *Let $n \geq 2$ and there exist numbers*

$$t_0 \geq 1, \ \ \lambda > 1, \ \ \alpha \in \Big] \frac{n}{n-1+\lambda}, 1 \Big[, \ \ \mu \in \mathbb{R}_+, \tag{1.27}$$

and a continuous function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\tau(t) = t^\alpha - t_0^\alpha \ \ \textit{for} \ \ t \in \mathbb{R}_+ \tag{1.28}$$

$$f(t,x) \geq t^\mu \omega(x) \ \ \textit{for} \ \ t \geq t_0, \ \ x \in \mathbb{R}_+, \tag{1.29}$$

$$\int\limits_0^x \omega(s)\,ds \geq x^{\lambda+1} - 1 \ \ \textit{for} \ \ x \in \mathbb{R}_+. \tag{1.30}$$

Then for any continuous function $u_0 : [t_*, 0] \to \mathbb{R}_+$ *and any number* $\gamma > n-1$ *there is a positive number* $r = r(u_0, \gamma)$ *such that if condition* (1.13) *holds, then the solution to problem* (1.1), (1.2) *is rapidly growing and admits estimate* (1.17).

Example 1.1. Let t_0, λ, α, and μ be numbers satisfying condition (1.27), $n \geq 2$,

$$\delta_k = 2^{-1-k}(1+\lambda)^{-1}k^{-\lambda} \ \ (k = 1, 2, \dots),$$

and let $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function whose restriction to an arbitrary interval $[k-1, k]$ has the form

$$\omega(x) = \begin{cases} (\lambda+1)x^\lambda & \text{for} \ \ k-1 \leq x \leq k - 2\delta_k, \\[2mm] \dfrac{|x-k+\delta_k|}{\delta_k}(\lambda+1)x^\lambda & \text{for} \ \ k - 2\delta_k < x \leq k. \end{cases} \tag{1.31}$$

Consider the differential equation

$$u^{(n)}(t) = t^\mu \omega(u(t^\alpha - t_0^\alpha)). \tag{1.32}$$

The function ω in the interval $\mathbb{R}_+$ does not have a positive nondecreasing minor since

$$\omega(k - \delta_k) = 0 \ \ (k = 1, 2, \dots). \tag{1.33}$$

Thus Theorem 1.2 leaves open the question on the existence of rapidly growing solutions to equation (1.32).

On the other hand, in view of (1.31), for an arbitrarily fixed natural number m we have

$$\int\limits_0^x \omega(y)\,dy > (\lambda+1)\int\limits_0^x y^\lambda dy - (\lambda+1)\sum_{k=1}^m \int\limits_{k-2\delta_k}^k \Big(1 - \frac{|y-k+\delta_k|}{\delta_k}\Big) y^\lambda dy$$

$$> x^{\lambda+1} - 2(\lambda+1)\sum_{k=1}^m k^\lambda \delta_k = x^{\lambda+1} - \sum_{k=1}^m 2^{-k} > x^{\lambda+1} - 1 \ \ \text{for} \ \ m-1 \leq x < m.$$

Consequently, inequality (1.30) holds. However, by virtue of Corollary 1.5 this inequality guarantees the existence of an n-parametric set of rapidly growing solutions to equation (1.32).

Note that if

$$\mu < -(n-1)\alpha\lambda,$$

then by Theorem 1.1 equation (1.32) along with rapidly growing solutions has an n-parametric set of slowly growing solutions as well.

Corollary 1.6. *Let* $n \geq 2$ *and there exist numbers*

$$t_0 > 0, \ \ \lambda > 1, \ \ \alpha \in \Big]\frac{n}{n-1+\lambda}, 1\Big], \ \ \mu \in \mathbb{R}, \tag{1.34}$$

and a continuous function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ *such that along with inequality* (1.30) *the conditions*

$$\tau(t) = \alpha(t - t_0) \ \ \textit{for} \ \ t \in \mathbb{R}_+,$$

$$f(t,x) \geq \exp(\mu t)\omega(x) \ \ \textit{for} \ \ t \geq t_0, \ \ x \in \mathbb{R}_+$$

are satisfied. Then for any continuous function $u_0 : [t_*, 0] \to \mathbb{R}_+$ *and any positive number* γ *there is a positive number* $r = r(u_0, \gamma)$ *such that if condition* (1.13) *holds, then the solution to problem* (1.1), (1.2) *is rapidly growing and admits estimate* (1.20).

Example 1.2. Suppose t_0, λ, α, and μ are numbers satisfying condition (1.34), $n \geq 2$, $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function whose restriction to an arbitrary interval $[k-1,k]$ has form (1.31).

Consider the differential equation

$$u^{(n)}(t) = \exp(\mu t)\omega(u(\alpha(t-t_0))). \tag{1.35}$$

In view of equalities (1.33), Theorem 1.2 is not applicable to this equation. On the other hand, as shown above, the function ω satisfies inequality (1.30). Hence by Corollary 1.6 it follows the existence of an n-parametric set of rapidly growing solutions to equation (1.35). If $\mu < 0$, then by Theorem 1.1 this equation along with rapidly growing solutions has an n-parametric set of slowly growing solutions as well.

Theorems 1.4, 1.5 and their corollaries given at the end of this section contain conditions guaranteeing the existence of an n-parametric set of blow-up solutions to equation (1.1).

Theorem 1.4. *Let the function τ satisfy condition (1.4), and let the function f satisfy the inequality*

$$f(t,x) \geq f_0(t,x) \ \text{ for } \ t_0 \leq t \leq b, \ \ x \in \mathbb{R}_+, \tag{1.36}$$

where $t_0 \in]0,b[$ and $f_0 : [t_0,b] \times \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing in the second argument continuous function such that the differential equation (1.12) in the interval $[t_0,b[$ has a blow-up solution v. Then there are numbers $r > 0$ and $a \in]t_0,b[$ such that if condition (1.13) holds, then the solution to problem (1.1), (1.2) is blow-up and admits the estimates

$$u^{(i-1)}(t) \geq v^{(i-1)}(t) \ \text{ for } \ a \leq t < b \ \ (i=1,\dots,n).$$

Corollary 1.7. *Let the functions τ and f satisfy the inequalities*

$$\alpha(t-b) + b \leq \tau(t) < t \ \text{ for } \ 0 \leq t < b,$$
$$f(t,x) \geq (b-t)^{\mu} x^{\lambda} \ \text{ for } \ t_0 \leq t < b, \ \ x \in \mathbb{R}_+,$$

where

$$\alpha > 1, \ \ t_0 > 0, \ \ \mu \in \mathbb{R}, \ \ \lambda > 1.$$

Then for any positive number γ there is a positive number $r = r(\gamma)$ such that if condition (1.13) holds, then the solution to problem (1.1), (1.2) is blow-up and admits the estimate

$$\inf\big\{(b-t)^{\gamma} u(t) : \ t_0 \leq t < b\big\} > 0. \tag{1.37}$$

Theorem 1.5. *Let $n \geq 2$, $t_0 \in]0,b[$, let the function τ have the form*

$$\tau(t) = \frac{b(t-t_0)}{b-t_0} \ \text{ for } \ 0 \leq t \leq b, \tag{1.38}$$

and let the function f satisfy inequality (1.36), where $f_0 : [t_0,b[\times \mathbb{R}_+ \to \mathbb{R}_+$ is a nonincreasing in the first argument continuous function. Let, moreover, the differential equation

$$v'(t) = \Big(\frac{b-t_0}{b}\Big)^{\frac{n-1}{n}} F_n(t, v(\tau(t))),$$

where F_n is a function given by equality (1.22), in the interval $[t_0,b[$ has a blow-up solution v. Then for any continuous function $u_0 : [t_*,0] \to \mathbb{R}_+$ there are numbers $r = r(u_0) > 0$, $a = a(u_0) \in]t_0,b[$ such that if condition (1.13) holds, then the solution to problem (1.1), (1.2) is blow-up and admits the estimate

$$u(t) \geq v(t) \ \text{ for } \ a \leq t < b.$$

Corollary 1.8. *Let $n \geq 2$ and there exist numbers*

$$t_0 > 0, \ \ \lambda > 1, \ \ \mu \geq 0$$

and a continuous function $\omega:\mathbb{R}_+\to\mathbb{R}_+$ such that identity (1.38) *holds and along with* (1.30) *the inequality*

$$f(t,x)\geq (b-t)^{\mu}\omega(x)\ \text{ for }\ t_0\leq t\leq b,\ \ x\in\mathbb{R}_+$$

is satisfied. Then for any continuous function $u_0:[t_*,0]\to\mathbb{R}_+$ and any positive number γ there is a positive number $r=r(u_0,\gamma)$ such that if condition (1.13) *holds, then the solution to problem* (1.1), (1.2) *is blow-up and admits estimate* (1.37).

Example 1.3. Let

$$n\geq 2,\ \ t_0>0,\ \ \lambda>1,\ \ \mu\geq 0,\ \ \delta_k=2^{-1-k}(1+\lambda)^{-1}k^{-\lambda}\ \ (k=1,2,\dots),$$

and let $\omega:\mathbb{R}_+\to\mathbb{R}_+$ be a continuous function whose restriction to an arbitrary interval $[k-1,k]$ has form (1.31). Consider the differential equation

$$u^{(n)}(t)=(b-t)^{\mu}\omega\Big(u\Big(\frac{b(t-t_0)}{b-t_0}\Big)\Big). \tag{1.39}$$

In view of condition (1.33), Theorem 1.4 leaves open the question on the existence of blow-up solutions of that equation. On the other hand, by virtue of condition (1.30) and Corollary 1.8 equation (1.39) has an n-parametric set of blow-up solutions.

2 Auxiliary propositions

Along with problem (1.1), (1.2) we consider the problem

$$v^{(n)}(t)=f_0(t,v(\tau(t))), \tag{2.1}$$

$$v(t)=v(t_0)\ \text{ for }\ t_*\leq t<0,\ \ v^{(i-1)}(0)=c_{0i}\ \ (i=1,\dots,n). \tag{2.2}$$

Moreover, we assume that $f:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$, $f_0:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$, $\tau:\mathbb{R}_+\to\mathbb{R}$, $u_0:[t_*,0]\to\mathbb{R}_+$, $v_0:[t_*,0]\to\mathbb{R}_+$ are continuous functions,

$$c_1=u_0(0),\ \ (c_1,\dots,c_n)\in\mathbb{R}_+^n,\ \ c_{01}=v_0(0),\ \ (c_{01},\dots,c_{0n})\in\mathbb{R}_+^n.$$

If condition (1.3) (condition (1.4)) holds, then there exists an increasing sequence of positive numbers $(t_i)_{i=1}^{+\infty}$ such that

$$\tau(t)<0\ \text{ for }\ 0\leq t<t_1,\ \ \tau(t_1)=0,$$
$$\tau(t)<t_i\ \text{ for }\ t_i\leq t<t_{i+1},\ \ \tau(t_{i+1})=t_i\ \ (i=1,2,\dots),$$
$$\lim_{t\to+\infty}t_i=+\infty\ \ \big(\lim_{t\to+\infty}t_i=b\big).$$

From this fact it immediately follows the validity of the following lemma.

Lemma 2.1. *If condition* (1.3) (*condition* (1.4)) *holds, then problem* (1.1), (1.2) *in the interval $\mathbb{R}_+$* (*in the interval $[0,b[$*) *has a unique solution u and for any natural k the equality*

$$u(t)=u_k(t)\ \text{ for }\ 0\leq t\leq t_k \tag{2.3}$$

is satisfied, where

$$u_k(t)=u_0(t)\ \text{ for }\ t_*\leq t<0,\quad u_k(t)=\sum_{i=1}^{n}\frac{c_i}{(i-1)!}t^{i-1}$$
$$+\frac{1}{(n-1)!}\int_0^t(t-s)^{n-1}f(s,u_{k-1}(\tau(s)))ds\ \text{ for }\ 0\leq t\leq t_k\ \ (k=1,2\dots). \tag{2.4}$$

Now we consider the case where one of the following two inequalities is satisfied:

$$f(t,x) \geq f_0(t,x) \text{ for } t \in \mathbb{R}_+, \ \ x \in \mathbb{R}_+, \tag{2.5}$$

$$f(t,x) \geq f_0(t,x) \text{ for } 0 \leq t < b, \ \ x \in \mathbb{R}_+. \tag{2.6}$$

Lemma 2.2. *Let along with conditions* (1.3) *and* (2.5) (*along with conditions* (1.4) *and* (2.6)) *the inequalities*

$$u_0(t) \geq v_0(t) \text{ for } t_* \leq t < 0, \ \ c_i \geq c_{0i} \ \ (i = 1, \dots, n) \tag{2.7}$$

be satisfied. If, moreover, one of the functions f and f_0 is nondecreasing in the second argument, then in the interval $\mathbb{R}_+$ (in the interval $[0, b[$) the inequalities

$$u^{(i-1)}(t) \geq v^{(i-1)}(t) \ \ (i = 1, \dots, n) \tag{2.8}$$

hold, where u and v are solutions to problems (1.1), (1.2) *and* (2.1), (2.2), *respectively.*

Proof. According to Lemma 2.1, problem (2.1), (2.2) is uniquely solvable and its solution for any natural k admits the representation

$$v(t) = v_k(t) \text{ for } 0 \leq t \leq t_k, \tag{2.9}$$

where

$$v_k(t) = v_0(t) \text{ for } t_* \leq t < 0, \quad v_k(t) = \sum_{i=1}^{n} \frac{c_{0i}}{(i-1)!} t^{i-1}$$

$$+ \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} f_0(s, v_{k-1}(\tau(s)))ds \text{ for } 0 \leq t \leq t_k \ \ (k = 1, 2 \dots). \tag{2.10}$$

If along with conditions (2.5) and (2.7) (along with conditions (2.6) and (2.7)) we take into account the fact that one of the functions f and f_0 is nondecreasing in the second argument, then the validity of the inequality

$$u_1(t) = \sum_{i=1}^{n} \frac{c_i}{(i-1)!} t^{i-1} + \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} f(s, u_0(\tau(s)))ds$$

$$\geq \sum_{i=1}^{n} \frac{c_{0i}}{(i-1)!} t^{i-1} + \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} f_0(s, v_0(\tau(s)))ds = v_1(t) \text{ for } 0 \leq t \leq t_1$$

becomes evident. By virtue of this inequality, representations (2.3), (2.4) and (2.9), (2.10) yield estimate (2.8). □

Definition 2.1. An n-times continuously differentiable function $u : \mathbb{R}_+ \to \mathbb{R}_+$ ($u : [0, b[\to \mathbb{R}_+$) is said to be **a solution of the differential inequality**

$$u^{(n)}(t) \geq f_0(t, u(\tau(t))) \tag{2.11}$$

in the interval $\mathbb{R}_+$ (in the interval $[0, b[$ **)** if there exists a continuous function $u_0 : [t_*, 0] \to \mathbb{R}_+$ such that in this interval inequality (2.11) holds, where

$$u(t) = u_0(t) \text{ for } t_* \leq t \leq 0.$$

Definition 2.2. A function $u : \mathbb{R}_+ \to \mathbb{R}_+$ ($u : [0, b[\to \mathbb{R}_+$) is said to be **a solution of problem** (2.11), (1.2) **in the interval $\mathbb{R}_+$ (in the interval** $[0, b[$ **)** if it is a solution of the differential equation (2.11) in that interval, satisfying the initial conditions (1.2).

Lemma 2.3. *Let conditions* (1.3), (2.7) (*conditions* (1.4), (2.7)) *hold, the function* f_0 *do not decrease in the second argument, and let* u *and* v *be solutions of problems* (2.11), (1.2) *and* (2.1), (2.2) *in the interval* $\mathbb{R}_+$ (*in the interval* $[0,b[\,$), *respectively. Then inequalities* (2.8) *holds in that interval.*

Proof. Assume

$$\delta(t)=u^{(n)}(t)-f_0(t,u(\tau(t))).$$

In view of (2.11), δ is a continuous and nonnegative function defined in the interval $\mathbb{R}_+$ (in the interval $[0,b[\,$). On the other hand, the function u is a solution of the differential equation

$$u^{(n)}(t)=f_0(t,u(\tau(t)))+\delta(t)$$

under the initial conditions (1.2). If now we apply Lemma 2.2, then the validity of Lemma 2.3 becomes evident. ☐

Lemma 2.4. *Let* $n\geq 2$ *and conditions* (1.3), (1.5) (*conditions* (1.4), (1.36)) *hold, where* $t_0>0$ ($t_0\in\,]0,b[\,$), *and let* $f_0:[t_0,+\infty[\,\times\mathbb{R}_+\to\mathbb{R}_+$ ($f_0:[t_0,b[\,\times\mathbb{R}_+\to\mathbb{R}_+$) *be nonincreasing in the first argument continuous function. Let, moreover, the function* τ *be continuously differentiable in the interval* $[t_0,+\infty[$ (*in the interval* $[t_0,b[\,$) *and satisfy the condition*

$$\tau(t_0)=0,\ \ \tau(t)>0\ \text{ for }\ t>t_0,\ \ 0\leq\tau'(t)\leq\alpha \tag{2.12}$$

in that interval, where α *is a positive constant. Then for any continuous function* $u_0:[t_*,0]\to\mathbb{R}_+$ *there is a positive number* $r_0=r_0(u_0(0))$ *such that if*

$$(c_1,\dots,c_n)\in\mathbb{R}^n_+,\ \ c_1=u_0(0),\ \ c_n>r_0, \tag{2.13}$$

then the solution to problem (1.1), (1.2) *in the interval* $[t_0,+\infty[$ (*in the interval* $[t_0,b[\,$) *satisfies the differential inequality*

$$u'(t)\geq\alpha^{-\frac{n-1}{n}}F_n(t,u(\tau(t))), \tag{2.14}$$

where F_n *is a function defined by equality* (1.22).

Proof. We prove the lemma for the case where $n\geq 3$ and conditions (1.3), (1.5) are satisfied. The case, where $n=2$ or conditions (1.4), (1.36) are satisfied, can be proved analogously.

Put

$$\delta=\begin{cases}(n!)^{-n}t_0^{n-2} & \text{for }\ t_0\geq 1,\\ (n!)^{-n}t_0^{n(n-1)} & \text{for }\ t_0<1,\end{cases}$$

$$r_0=1+\delta^{-1}\Big(\sum_{k=1}^{n-1}\alpha^{-k}c_1^{k-1}\Big)\int_0^{c_1}f_0(t_0,y)\,dy,$$

$$f_k(t,x)=\frac{\alpha^{-k}}{(k-1)!}\int_0^x(x-y)^{k-1}f_0(t,y)\,dy\ \ (k=1,\dots,n-1).$$

By Lemma 2.1 problem (1.1), (1.2) has a unique solution u. In view of the nonnegativeness of the functions f, u_0, and conditions (2.12), (2.13), the function u admits the estimates

$$u^{(n-k)}(t_0)(u'(t_0))^k\geq\frac{t_0^{(n-1)k-1}}{((n-2)!)^k(k-1)!}\,c_n^{k+1}>\delta r_0\ \ (k=1,\dots,n-2), \tag{2.15}$$

$$(u'(t_0))^n\geq\frac{t_0^{(n-2)n}}{((n-2)!)^n}\,c_n^n>n\delta r_0, \tag{2.16}$$

$$u'(t)\geq\alpha^{-1}\tau'(t)u'(\tau(t))\ \text{ for }\ t\geq t_0. \tag{2.17}$$

On the other hand, according to the definitions of the number r_0 and the functions f_k $(k = 1, \dots, n-1)$, we have

$$f_k(t, c_1) \le \alpha^{-k} c_1^{k-1} \int_0^{c_1} f_0(t_0, y)\, dy < \delta r_0 \ \text{ for } \ t \ge t_0 \ \ (k = 1, \dots, n-1). \tag{2.18}$$

From estimates (1.5), (2.17) it follows that

$$u^{(n)}(t) u'(t) = f(t, u(\tau(t))) u'(t) \ge \alpha^{-1} f_0(t, u(\tau(t))) u'(\tau(t)) \tau'(t) \ \text{ for } \ t \ge t_0.$$

If we integrate this inequality from t_0 to t and take into account the fact that f_0 is nonincreasing in the first argument, we get

$$\begin{aligned} u^{(n-1)}(t) u'(t) &= \int_{t_0}^{t} u^{(n-1)}(s) u''(s)\, ds + u^{(n-1)}(t_0) u'(t_0) + \alpha^{-1} \int_{t_0}^{t} f_0(s, u(\tau(s)))\, du(\tau(s)) \\ &\ge u^{(n-1)}(t_0) u'(t_0) - f_1(t_0, c_1) + f_1(t, u(\tau(t))) \ \text{ for } \ t \ge t_0. \end{aligned}$$

Hence, due to estimates (2.15), (2.18), we get

$$u^{(n-1)}(t) u'(t) > f_1(t, u(\tau(t))) \ \text{ for } \ t \ge t_0. \tag{2.19}$$

Now we have to prove that for any $k \in \{1, \dots, n-2\}$ the inequality

$$u^{(n-k)}(t) (u'(t))^k > f_k(t, u(\tau(t))) \ \text{ for } \ t \ge t_0 \tag{2.20}$$

holds. In view of (2.19), it remains to consider the case, where $n \ge 4$.

Assume that for some $k \in \{1, \dots, n-3\}$ inequality (2.20) holds. Then by virtue of estimate (2.17) we have

$$u^{(n-k)}(t) (u'(t))^{k+1} > \alpha^{-1} f_k(t, u(\tau(t))) u'(\tau(t)) \tau'(t) \ \text{ for } \ t \ge t_0.$$

Hence, due to estimates (2.15), (2.18), it follows that

$$\begin{aligned} u^{(n-k-1)}(t) (u'(t))^{k+1} &\ge (k+1) \int_{t_0}^{t} u^{(n-k-1)}(s) (u'(s))^k u''(s)\, ds + u^{(n-k-1)}(t_0) (u'(t_0))^{k+1} \\ &- f_{k+1}(t_0, c_1) + f_{k+1}(t, u(\tau(t))) > f_{k+1}(t, u(\tau(t))) \ \text{ for } \ t \ge t_0. \end{aligned}$$

Thus it is proved that inequality (2.20) holds for any $k \in \{1, \dots, n-2\}$, and consequently,

$$u''(t) (u'(t))^{n-2} > f_{n-2}(t, u(\tau(t))) \ \text{ for } \ t \ge t_0.$$

Therefore,

$$u''(t) (u'(t))^{n-1} > \alpha^{-1} f_{n-2}(t, u(\tau(t))) u'(\tau(t)) \tau'(t) \ \text{ for } \ t \ge t_0.$$

If we integrate this inequality from t_0 to t and take into account estimates (2.16) and (2.18), we get

$$\frac{(u'(t))^n}{n} \ge \frac{(u'(t_0))^n}{n} - f_{n-1}(t_0, c_1) + f_{n-1}(t, u(\tau(t))) > f_{n-1}(t, u(\tau(t))) \ \text{ for } \ t \ge t_0,$$

i.e.,

$$u'(t) > \big(n f_{n-1}(t, u(\tau(t)))\big)^{\frac{1}{n}} \ \text{ for } \ t \ge t_0.$$

Hence in view of (1.22) it follows that the function u is a solution of the differential inequality (2.14) in the interval $[t_0, +\infty[\,$. □

3 Proof of the main results

Proof of Theorem 1.1. First we consider the case where along with (1.3) conditions (1.5)–(1.7) hold. By Lemma 2.1, problem (1.1), (1.2) has a unique solution u. On the other hand, in view of (1.3) and (1.6) we have

$$u(t) \geq xt^{n-1} \ \text{ for } \ t \geq 0, \ \ \tau(t) > 0 \ \text{ for } \ t > t_0,$$

where $x = c_n/(n-1)!$, and t_0 is a sufficiently large positive number. Hence by virtue of conditions (1.5) and (1.7) we get

$$u^{(n-1)}(t) \geq \int_{t_0}^{t} f_0\big(s, (\tau(s))^{n-1}x\big)ds \to +\infty \ \text{ as } \ t \to +\infty.$$

Consequently, u is a rapidly growing solution.

Now consider the case where conditions (1.3) and (1.8) hold.

Put

$$\tau_*(t) = \big(\tau(t) + |\tau(t)|\big)/2,$$

and choose $\varepsilon > 0$ such a small that the inequality

$$\int_0^{+\infty} f^*\big(s, (n+2)(1 + |\tau(s)|)^{n-1}\varepsilon\big)\, ds < \varepsilon \tag{3.1}$$

is satisfied.

Let condition (1.9) be fulfilled. Then the solution u of problem (1.1), (1.2) admits the estimates

$$u^{(n-1)}(0) < \varepsilon, \ \ u(\tau(t)) < (1 + |\tau(t)|)^{n-1}\big(n\varepsilon + u^{(n-1)}(\tau_*(t))\big).$$

Therefore,

$$u^{(n-1)}(t) < \varepsilon + \int_0^t f^*\big(s, (1 + |\tau(s)|)^{n-1}(n\varepsilon + u^{(n-1)}(\tau_*(s)))\big)\, ds \ \text{ for } \ t \geq 0. \tag{3.2}$$

Our aim is to prove that

$$u^{(n-1)}(t) < 2\varepsilon \ \text{ for } \ t \geq 0. \tag{3.3}$$

Assume the contrary. Then there exists $t_0 > 0$ such that

$$u^{(n-1)}(t) < 2\varepsilon \ \text{ for } \ 0 \leq t < t_0, \ \ u^{(n-1)}(t_0) = 2\varepsilon.$$

Thus inequalities (3.1) and (3.2) yield

$$2\varepsilon < \varepsilon + \int_0^{t_0} f^*\big(s, (n+2)(1 + |\tau(s)|)^{n-1}\varepsilon\big)\, ds < 2\varepsilon.$$

The contradiction obtained proves the validity of estimate (3.3). Consequently, the solution u is slowly growing. □

To be convinced of the validity of Corollary 1.1, it is enough to note that if along with (1.3) conditions (1.10), (1.11) hold, then conditions (1.5), (1.7) are satisfied as well, where

$$f_0(t, x) = p(t)x^\lambda.$$

If condition (1.10) is satisfied but condition (1.11) is violated, then condition (1.8) holds.

Proof of Theorem 1.2. By virtue of condition (1.3), there exists a number $a \in]t_0, +\infty[$ such that

$$a = \min\big\{\tau(t) : \ t \geq t_0\big\}.$$

Put

$$\begin{aligned}\widetilde{u}_0(t) = u(t) \ \text{ for } \ t_0 \leq t \leq a, \ \ \widetilde{c}_i = u^{(i-1)}(a) \ \ (i = 1, \dots, n),\\ \widetilde{v}_0(t) = v(t) \ \text{ for } \ t_0 \leq t \leq a, \ \ \widetilde{c}_{0i} = v^{(i-1)}(a) \ \ (i = 1, \dots, n),\end{aligned}$$

and choose $r > 1$ such that the inequality

$$\frac{r\, t_0^{n-i}}{(n-i)!} \geq \widetilde{c}_{0i} \ \ (i = 1, \dots, n) \tag{3.4}$$

is satisfied. Then the restrictions of the functions u and v to the interval $[a, +\infty[$ are solutions to the differential equations (1.1) and (1.12), respectively, under the initial conditions

$$\begin{aligned}u(t) = \widetilde{u}_0(t) \ \text{ for } \ t_0 \leq t < a, \ \ u^{(i-1)}(a) = \widetilde{c}_i \ \ (i = 1, \dots, n),\\ v(t) = \widetilde{v}_0(t) \ \text{ for } \ t_0 \leq t < a, \ \ v^{(i-1)}(a) = \widetilde{c}_{0i} \ \ (i = 1, \dots, n).\end{aligned}$$

On the other hand, due to (1.13) and (3.4) we have

$$\widetilde{u}_0(t) \geq \frac{c_n t_0^{n-1}}{(n-1)!} \geq \widetilde{c}_{01} \geq \widetilde{v}_0(t) \ \text{ for } \ t_0 \leq t \leq a, \tag{3.5}$$

$$\widetilde{c}_i \geq \frac{c_n a^{n-i}}{(n-i)!} > \frac{r\, t_0^{n-i}}{(n-i)!} > \widetilde{c}_{0i} \ \ (i = 1, \dots, n). \tag{3.6}$$

By Lemma 2.2, inequalities (1.5), (3.5), and (3.6) imply estimates (1.14). □

Proof of Corollary 1.2. By virtue of Theorem 1.2, to prove Corollary 1.2 it is enough to state that for any $\gamma > n - 1$ the differential equation

$$v^{(n)}(t) = t^{\mu} v^{\lambda}(\tau(t)) \tag{3.7}$$

in the interval $[t_0, +\infty[$ has a solution v, admitting the estimate

$$\inf\big\{t^{-\gamma} v(t) : \ t \geq t_0\big\} > 0. \tag{3.8}$$

Without loss of generality we can assume that

$$-(\lambda\alpha - 1)\gamma - n \leq \mu. \tag{3.9}$$

Let

$$\rho = \bigg[\Big(1 + \big(1 + \frac{\delta}{t_0}\big)^{\gamma-n}\Big) \prod_{i=0}^{n-1} (\gamma - i)\bigg]^{\frac{1}{\lambda-1}}.$$

We introduce the function

$$p(t) = \rho^{1-\lambda} \Big(\prod_{i=0}^{n-1} (\gamma - i)\Big) (t + \delta)^{\gamma-n} (\tau(t) + \delta)^{-\lambda\gamma} \ \text{ for } \ t \geq t_0. \tag{3.10}$$

It is clear that

$$(t + \delta)^{\gamma-n} < \Big(1 + \big(1 + \frac{\delta}{t_0}\big)^{\gamma-n}\Big) t^{\gamma-n} \ \text{ for } \ t \geq t_0.$$

On the other hand, in view of (1.16) we have

$$(\tau(t) + \delta)^{-\lambda\gamma} \leq t^{-\lambda\alpha\gamma} \ \text{ for } \ t \geq t_0.$$

According to condition (3.9), the last two inequalities yield the estimate

$$p(t) \leq t^{\mu} \;\text{ for }\; t \geq t_0. \tag{3.11}$$

Consider the differential equation

$$w^{(n)}(t) = p(t)w^{\lambda}(\tau(t)).$$

By identity (3.10), this equation in the interval $[t_0, +\infty[$ has a rapidly growing solution

$$w(t) = \rho(t+\delta)^{\gamma} \;\text{ for }\; t \geq t_0.$$

Let

$$t_1 = \min\big\{\tau(t) :\; t \geq t_0\big\}.$$

By virtue of Lemmas 2.1, 2.2 and inequality (3.11), the differential equation (3.7) in the interval $[t_0, +\infty[$ has a unique solution v, satisfying the initial conditions

$$v(t) = \rho(t+\delta)^{\gamma} \;\text{ for }\; t_1 \leq t < t_0, \;\; v^{(i-1)}(t_0) = w^{(i-1)}(t_0) \;\; (i = 1, \dots, n),$$

and this solution admits the estimate

$$v(t) \geq w(t) \;\text{ for }\; t \geq t_0.$$

Thus

$$t^{-\gamma}v(t) \geq \rho\Big(1 + \frac{\delta}{t}\Big)^{\gamma} \geq \rho \;\text{ for }\; t \geq t_0,$$

and, consequently, estimate (3.8) is valid. □

Corollary 1.3 can be proved analogously to Corollary 1.2.
Corollary 1.4 immediately follows from Theorem 1.1 and Corollary 1.4.

Proof of Theorem 1.3. Let $u_0 : [t_*, 0] \to \mathbb{R}_+$ be an arbitrarily fixed continuous function. Then by Lemma 2.4 there exists a positive number r_0 such that for any initial values $c_2, \dots, c_n$, satisfying the inequalities

$$c_i \geq 0 \;\; (i = 2, \dots, n), \;\; c_n \geq r_0, \tag{3.12}$$

the solution u of problem (1.1), (1.2) in the interval $[t_0, +\infty[$ satisfies the differential inequality

$$u'(t) \geq F_n(t, u(\tau(t))) \;\text{ for }\; t \geq t_0. \tag{3.13}$$

Due to condition (1.3), there exists a number $a > t_0$ such that

$$\min\big\{\tau(t) :\; t \geq a\big\} = a.$$

Put

$$r = (n-1)!\, t_0^{1-n} v(a) + r_0.$$

Then, if condition (1.13) holds, condition (3.12) holds as well, and according to the above said, the solution u of problem (1.1), (1.2) satisfies the differential inequality (3.13).

On the other hand, in view of (1.13) we have

$$u(t_0) \geq \frac{t_0^{n-1}}{(n-1)!}c_n \geq \frac{t_0^{n-1}}{(n-1)!}r > v(a).$$

The last inequality yields the inequality

$$u(t) > v(t) \;\text{ for }\; t_0 \leq t \leq a, \tag{3.14}$$

since u and v are nondecreasing in the interval $[t_0, a]$ functions.

However, by Lemma 2.3 conditions (1.3), (3.13), and (3.14) guarantee the validity of estimate (1.26) since F_n is a nondecreasing in the second argument function. From (1.25) and (1.26) it follows that the solution u is rapidly growing. □

Proof of Corollary 1.5. Let

$$\Omega_n(x)=\Big(\frac{n}{(n-2)!}\int\limits_0^x (x-y)^{n-2}\omega(y)\,dy\Big)^{\frac{1}{n}} \ \text{ for } \ x\geq 0.$$

By Theorem 1.3 it is enough to state that for any $\gamma>n-1$ the differential equation

$$v'(t)=t^{\frac{\mu}{n}}\Omega_n(v(\tau(t))) \tag{3.15}$$

in the interval $[t_0,+\infty[$ has a solution v admitting estimate (3.8).

First note that estimate (1.30) implies the estimate

$$\begin{aligned}\frac{1}{(n-2)!}\int\limits_0^x (x-y)^{n-2}\omega(y)\,dy &> (\lambda+1)\prod_{i=1}^{n-1}(\lambda+i)^{-1}x^{\lambda+n-1}-\frac{1}{(n-2)!}\,x^{n-2}\\ &> 2\ell^{-1}x^{\lambda+n-1}\Big(1-\frac{\ell}{2}\,x^{-\lambda-1}\Big)>\ell^{-1}x^{\lambda+n-1} \ \text{ for } \ x>\ell^{\frac{1}{\lambda+1}},\end{aligned}$$

where

$$\ell=\prod_{i=1}^{n-1}(\lambda+i).$$

Thus

$$\Omega_n(x)>\Big(\frac{n}{\ell}\Big)^{\frac{1}{n}}x^{\frac{\lambda+n-1}{n}} \ \text{ for } \ x>\ell^{\frac{1}{\lambda+1}}. \tag{3.16}$$

Without loss of generality, we will assume below that

$$-\big((\lambda+n-1)\alpha-n\big)\gamma-n\leq\mu. \tag{3.17}$$

Let

$$\rho=\ell^{\frac{1}{\lambda+1}}+\Big(\gamma\Big(\frac{\ell}{n}\Big)^{\frac{1}{n}}\big(1+t_0^{\alpha-1}\big)^{\gamma-1}\Big)^{\frac{n}{\lambda-1}}. \tag{3.18}$$

Introduce the function

$$p(t)=\rho\gamma(t+t_0^{\alpha})^{\gamma-1}/\Omega_n\big(\rho(\tau(t)+t_0^{\alpha})^{\gamma}\big), \tag{3.19}$$

and consider the initial value problem

$$\begin{gathered}w'(t)=p(t)\Omega_n(w(\tau(t))),\\ w(t)=\rho(t+t_0^{\alpha})^{\gamma} \ \text{ for } \ 0\leq t<t_0, \ \ w(t_0)=\rho(t_0+t_0^{\alpha})^{\gamma}.\end{gathered}$$

It is evident that this problem has a solution

$$w(t)=\rho(t+t_0^{\alpha})^{\gamma} \ \text{ for } \ t\geq t_0, \tag{3.20}$$

which by Lemma 2.1 is unique.

Due to (1.28) and (3.18), we have

$$\rho\big(\tau(t)+t_0^{\alpha}\big)^{\gamma}\geq\rho t^{\alpha\gamma}>\ell^{\frac{1}{\lambda+1}} \ \text{ for } \ t\geq t_0,$$

according to which from (3.16) and (3.19) we get

$$\Omega_n\big(\rho(\tau(t)+t_0^{\alpha})^{\gamma}\big)>\Big(\frac{n}{\ell}\Big)^{\frac{1}{n}}\big(\rho(\tau(t)+t_0^{\alpha})^{\gamma}\big)^{\frac{\lambda+n-1}{n}}\geq\Big(\frac{n}{\ell}\Big)^{\frac{1}{n}}\rho^{\frac{\lambda+n-1}{n}}t^{\frac{(\lambda+n-1)\alpha}{n}\gamma} \ \text{ for } \ t\geq t_0,$$

and

$$p(t)\leq\rho^{\frac{1-\lambda}{n}}\Big(\frac{\ell}{n}\Big)^{\frac{1}{n}}\gamma(1+t_0^{\alpha-1})^{\gamma-1}t^{-\frac{((\lambda+n-1)\alpha-n)\gamma+n}{n}} \ \text{ for } \ t\geq t_0.$$

Hence by conditions (3.17) and (3.18) we get the inequality

$$p(t) \leq t^{\frac{\mu}{n}} \text{ for } t \geq t_0. \tag{3.21}$$

By virtue of Lemmas 2.1, 2.2, and inequality (3.21), the differential equation (3.15) in the interval $[t_0, +\infty[$ has a unique solution v, satisfying the initial condition

$$v(t) = \rho(t + t_0^\alpha)^\gamma \text{ for } 0 \leq t < t_0, \;\; v(t_0) = \rho(t_0 + t_0^\alpha)^\gamma,$$

and admitting the estimate $v(t) \geq w(t)$ for $t \geq t_0$. Therefore estimate (3.8) is satisfied as well. □

Corollary 1.6 can be proved analogously to Corollary 1.5.

Theorem 1.4 and Corollary 1.7 can be proved analogously to Theorem 1.2 and Corollary 1.2, while the proofs of Theorem 1.5 and Corollary 1.8 are analogous to those of Theorem 1.3 and Corollary 1.5.

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