Measuring the noncommutativity of DG-algebras

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Many constructions, which successfully work for commutative DG-algebras, fail in the noncommutative case. There exists the classical tool which measures the noncommutativity of a DG-algebra $(A, d, \cdot)$, namely the Steenrod's $\cup_1$product, satisfying the condition

$$d(a \cup_1 b) = da \cup_1 b + a \cup_1 db + a \cdot b - b \cdot a; \tag{1}$$

(the signs are ignored in the whole text). The existence of $\cup_1$ guarantees the commutativity of $H(A)$. But this structure is too pure for most applications. A $\cup_1$ product satisfying just the condition (1) can not compensate the commutativity. In many constructions some more deep properties of $\cup_1$, for example the compatibility with the product of A (the Hirsch formula)

$$a \cup_1 (b \cdot c) = b \cdot (a \cup_1 c) + (a \cup_1 b) \cdot c. \tag{2}$$

are needed.

In this article, as a tool which compensates the commutativity of A, we use a multiplication in the bar construction

$$\mu : BA \otimes BA \to BA$$

which turns DG-coalgebra BA into a DG-Hopf algebra. In fact each such multiplication is uniquely determined by a collection of operations

$$\{E_{pq} : (\otimes^p A) \otimes (\otimes^q A) \to A,\ p, q = 0, 1, 2, 3, ...\ \}$$

subject of certain compatibility conditions. Particularly the binary component $E_{11} : A \otimes A \to A$ satisfies the condition (1), so it can be regarded as a sort of $\cup_1$product, measuring the noncommutativity of A. For convenience we call such an object $(A, \cdot, d, \{E_{pq}\})$ *Hirsch algebra* since the defining properties of operations E_{pq} in fact generalize the classical Hirsch formula (2). Actually this structure is the particular case of the notion of B_∞-algebra ([2], [7]) which is defined as a structure on A, granting that BA becomes a DG-Hopf algebra. In fact this structure consists of new differential $\widetilde{d} : BA \to BA$ and new multiplication $\widetilde{\mu} : BA \otimes BA \to BA$. A Hirsch algebra is the case when the standard differential of bar construction remains unchanged.

The extremely important particular case of Hirsch algebra is the structure, which is known as *Homotopy G-algebra* [6], [25]. This is the case when all E_{pq}-s except E_{01} and $E_{1k}, k = 0, 1, 2, 3, ...$ are zero. Thus it is a DG-algebra with $\cup_1$product and certain tail which consists of a sequence of cochain operations $\{E_{1,k} : A \otimes (\otimes^k A) \to A, \quad k = 1, 2, 3, ..., \quad E_{1,1} = \cup_1\}$, satisfying certain compatibility conditions. Some constructions and results, valid for commutative DG-algebras, are valid for homotopy G-algebras too. This structure arises in some important cases, namely there exist *explicit* formulae for operations $E_{1,k}$

(i) in the cochain complex of a topological space $C^*(X)$;

(ii) in the Hochschild cochain complex $C^*(A, A)$ of an algebra A;

(iii) in the cobar construction $\Omega\mathcal{H}$ of a DG-Hopf algebra $\mathcal{H}$, particularly in the cobar construction of the bar construction ΩBA of an algebra A.

We remark here that in all this three cases the starting operation $E_{11} = \cup_1$ is classical: the Seenrods $\cup_1$ product in $C^*(X)$, the Gerstenhabers circle product in $C^*(A, A)$ [5] and the Adams's $\cup_1$ product in $\Omega\mathcal{H}$ [1]. The suitable tails, i.e. the higher operations E_{1k} in $C^*(X)$ actually where constructed in [2], in $C^*(A, A)$ in [13], [7], [6], [25].

Bellow we shall give some applications of these structures.

The first section starts with the study of the structure of a product in the bar construction, which motivates notion of Hirsch algebra. Then the comparisons of this structure with $B(\infty)$-algebra structure, DG-Lie algebra structure, homotopy G-algebra structure and *shc* (strong homotopy commutative) algebra structure are given. The section ends with two versions of the notion of twisting element (the first controls deformations of algebras and the second the degeneracy of A_∞-algebra structures) and the suitable notion of their (gauge) transformation in a homotopy G-algebra.

In the second section the above mentioned three examples of homotopy G-algebra are given.

The third section is dedicated to some applications: multiplicative twisted tensor products, deformation of algebras and degeneracy of $A(\infty)-$algebras.

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1 Hirsch algebras

1.1 Products in bar construction

Let $(A, d, \cdot)$ be a DG-algebra with differential $d : A^* \to A^{*+1}$(cochain algebra) and let

$$BA = T^c(s^{-1}A) = \Lambda + s^{-1}A + s^{-1}A \otimes s^{-1}A + s^{-1}A \otimes s^{-1}A \otimes s^{-1}A + ...$$

be its bar construction (here $s^{-1}A$ is the desuspension of A, i.e. $(s^{-1}A)^n = A^{n+1}$ and T^c is the tensor coalgebra functor). By definition BA is a *DG*-coalgebra with the differential

$$d(a_1 \otimes ... \otimes a_n) = \sum_k \pm a_1 \otimes ... \otimes da_k \otimes ... \otimes a_n + \sum_k \pm a_1 \otimes ... \otimes a_k \cdot a_{k+1} \otimes ... \otimes a_n,$$

the coproduct $\nabla : BA \to BA \otimes BA$ given by

$$\nabla(a_1 \otimes ... \otimes a_n) = \sum_{k=0}^{n} (a_1 \otimes ... \otimes a_k) \otimes (a_{k+1} \otimes ... \otimes a_n)$$

and with the counit $1_\Lambda \in \Lambda \subset BA$.

We are interested in the structure of multiplications

$$\mu : BA \otimes BA \to BA,$$

turning BA into a DG-Hopf algebra, i.e. we require that μ must be

- a DG-coalgebra map;
- which has the unit element $1_\Lambda \in \Lambda \subset BA$;
- is associative.

Because of the cofreeness of the tensor coalgebra $BA = T^c(s^{-1}A)$, each map of graded coalgebras

$$\mu : BA \otimes BA \to BA,$$

is uniquely determined by the projection

$$E = pr \cdot \mu : BA \otimes BA \to BA \to A.$$

Moreover, each homomorphism $E : BA \otimes BA \to A$ of degree +1 determines a graded coalgebra map $\mu_E : BA \otimes BA \to BA$ given by

$$\mu_E = \sum_{k=0}^{\infty} (E \otimes ... \otimes E) \nabla^k_{BA \otimes BA},$$

where $\nabla^k_{BA\otimes BA} : BA \otimes BA \to \otimes^k (BA \otimes BA)$ is the k-fold iteration of the standard coproduct

$$\nabla_{BA\otimes BA} = (id \otimes T \otimes id)(\nabla \otimes \nabla) : BA \otimes BA \to \otimes^2 (BA \otimes BA),$$

here $T : BA \otimes BA \to BA \otimes BA$ is interchange map and ∇^k is the k-fold iteration of a coproduct ∇:

$$\nabla^0 = \epsilon,\ \nabla^1 = id,\ \nabla^2 = \nabla,\ \nabla^k = (\nabla^{k-1} \otimes id)\nabla.$$

The map μ_E is a *chain map* (i.e. it is a map of DG-coalgebras) if and only if E is a twisting cochain in the sense of E. Brown, i.e. satisfies the condition $dE + Ed_{BA\otimes BA} = E \cup E$ (here the $\cup$-product in $Hom(BA \otimes BA, A)$ is given by $f \cup g = \mu(f \otimes g)\nabla)$: again because of the cofreeness of the tensor coalgebra $BA = T^c(s^{-1}A)$ the condition $d_{BA}\mu_E = \mu_E d_{BA\otimes BA}$ is satisfied if and only if it is satisfied after the projection on A, i.e. if $pr \cdot d_{BA}\mu_E = pr \cdot \mu_E d_{BA\otimes BA}$ but this condition is nothing else than the Brown's condition.

The same argument shows that the product μ_E is associative if and only if the following condition is satisfied:

$$pr \cdot \mu_E(\mu_E \otimes id) = pr \cdot \mu_E(id \otimes \mu_E),$$

or, having in mind $E = pr \cdot \mu_E$

$$E(\mu_E \otimes id) = E(id \otimes \mu_E).$$

Thus we can summarize that any multiplication $\mu : BA \otimes BA \to BA$ which specifies on BA a structure of DG-Hopf algebra is induced by a homomorphism of degree +1

$$E : BA \otimes BA \to BA$$

which satisfies the following conditions:

$$d_A E + E(d_{BA} \otimes id + id \otimes d_{BA}) = E \cup E, \tag{3}$$

i.e. E is a twisting cochain, and

$$E(\mu_E \otimes id) = E(id \otimes \mu_E), \tag{4}$$

this implies the associativity of μ.

Each twisting cochain $E : BA \otimes BA \to BA$ has *components*

$$\begin{array}{ccccccc} & & E_{01} & & E_{10} & & \\ & E_{02} & & E_{11} & & E_{20} & \\ E_{03} & & E_{12} & & E_{21} & & E_{30} \\ .. & ... & ... & ... & ... & ... & ... \end{array}$$

where E_{pq} is the restriction of E on $(\otimes^p s^{-1}A) \otimes (\otimes^q s^{-1}A)$. Thus a twisting cochain can be regarded as a collection of *multioperations*

$$E_{pq} : (\otimes^p s^{-1}A) \otimes (\otimes^q s^{-1}A) \to A.$$

The value of E_{pq} on the element $(s^{-1}a_1 \otimes ... \otimes s^{-1}a_p) \otimes (s^{-1}b_1 \otimes ... \otimes s^{-1}b_q) \in (\otimes^p s^{-1}A) \otimes (\otimes^q s^{-1}A)$ we denote by $E_{pq}(a_1 \otimes ... \otimes a_p | b_1 \otimes ... \otimes b_q)$.

The above requirements on μ_E imply some restrictions on the collection $\{E_{pq}\}$.

First of all it is not hard to check that the element $1_\Lambda \in \Lambda \subset BA$ is the unit for a multiplication μ_E if and only if

$$E_{01} = E_{10} = id; \quad E_{0k} = E_{k0} = 0 \;\; k > 1. \tag{5}$$

Thus each multiplication on BA with unit 1_Λ is induced by a collection of operations

$$\begin{array}{ccccccc} & & E_{01} = id & & E_{10} = id & & \\ & & & E_{11} & & & \\ & & E_{12} & & E_{21} & & \\ & E_{13} & & E_{22} & & E_{31} & \\ .. & ... & ... & ... & ... & ... & ..., \end{array} \tag{6}$$

Let us look, what the condition (3) gives for low dimensional E_{pq}-s.

The restriction on $(\otimes^1 A)\otimes(\otimes^1 A)$ gives:

$$d_A E_{11}(a|b) + E_{11}(d_A a|b + a|d_A b) = a\cdot b - b\cdot a,$$

i.e. the operation E_{11} is sort of $\cup_1$ product, which measures the noncommutativity of A.

The restriction on $(\otimes^1 A)\otimes(\otimes^2 A)$ gives:

$$\begin{gathered} d_A E_{12}(a|b\otimes c) + E_{12}(d_A a|b\otimes c + a|d_A b\otimes c + a|b\otimes d_A c) = \\ E_{11}(a|bc) + bE_{11}(a|c) + E_{11}(a|b)c, \end{gathered} \qquad (7)$$

or, using the notation $E_{11}(a|b) = a\cup_1 b$:

$$\begin{gathered} d_A E_{12}(a|b\otimes c) + E_{12}(d_A a|b\otimes c + a|d_A b\otimes c + a|b\otimes d_A c) = \\ a\cup_1 (bc) + b(a\cup_1 c) + (a\cup_1 b)c, \end{gathered}$$

this means, that this $\cup_1$ satisfies what is called the *right Hirsch formula* up to homotopy and the appropriate homotopy is the operation E_{12}.

The restriction of (3) on $(\otimes^2 A)\otimes(\otimes^1 A)$ gives:

$$\begin{gathered} d_A E_{21}(a\otimes b|c) + E_{21}(d_A a\otimes b|c + a\otimes d_A bc + a\otimes b|d_A c) = \\ E_{11}(ab|c) + aE_{11}(b|c) + E_{11}(a|c)b, \end{gathered}$$

or, using the notation $E_{11}(a|b) = a\cup_1 b$:

$$\begin{gathered} d_A E_{21}(a\otimes b|c) + E_{21}(d_A a\otimes b|c + a\otimes d_A b|c + a\otimes b|d_A c) = \\ (ab)\cup_1 c + a(b\cup_1 c) + (a\cup_1 c)b, \end{gathered} \qquad (8)$$

this means, that this $\cup_1$ satisfies what is called the *left Hirsch formula* up to homotopy and the appropriate homotopy is the operation E_{21}.

Generally the restriction of (3) on $(\otimes^m A)\otimes(\otimes^n A)$ gives:

$$\begin{gathered} d_A E_{m,n}(a_1\otimes ...\otimes a_m|b_1\otimes ...\otimes b_n) + \\ \sum_i E_{m,n}(a_1\otimes ...\otimes d_A a_i\otimes ...\otimes a_m|b_1\otimes ..\otimes b_n) \\ +\sum_i E_{m,n}(a_1\otimes ...\otimes a_m|b_1\otimes ...\otimes d_A b_i\otimes ..\otimes b_n) = \\ a_1 E_{m-1,n}(a_2\otimes ...\otimes a_m|b_1\otimes ...\otimes b_n) + \\ E_{m-1,n}(a_1\otimes ...\otimes a_{m-1}|b_1\otimes ...\otimes b_n)a_m \\ +b_1 E_{m,n-1}(a_1\otimes ...\otimes a_m|b_2\otimes ...\otimes b_n) + \\ E_{m,n-1}(a_1\otimes ...\otimes a_m|b_1\otimes ...\otimes b_{n-1})b_m + \\ \sum_i E_{m-1,n}(a_1\otimes ...\otimes a_i\cdot a_{i+1}\otimes ...\otimes a_m|b_1\otimes ...\otimes b_n) + \\ \sum_i E_{m,n-1}(a_1\otimes ...\otimes a_m|b_1\otimes ...\otimes b_i\cdot b_{i+1}\otimes ...\otimes b_n) + \\ \sum_{p=1,...,m-1;\ q=1,...,n-1} E_{p,q}(a_1\otimes ...\otimes a_p|b_1\otimes ..\otimes b_q)\cdot \\ E_{m-p,n-q}(a_{p+1}\otimes ...\otimes a_m|b_{q+1}\otimes ...\otimes b_n). \end{gathered} \qquad (9)$$

Now let us look, what the associativity condition (4) gives for the components E_{pq}.

The restriction on $(\otimes^1 A)\otimes(\otimes^1 A)\otimes(\otimes^1 A)$ gives:

$$E_{11}(E_{11}(a_1|a_2)|a_3) - E_{11}(a_1|E_{11}(a_2|a_3)) = E_{12}(a_1|a_2\otimes a_3) + E_{12}(a_1|a_3\otimes a_2) - E_{21}(a_1\otimes a_2|a_3) + E_{21}(a_2\otimes a_1|a_3),$$

or

$$(a_1\cup_1 a_2)\cup_1 a_3 - a_1\cup_1(a_2\cup_1 a_3) = E_{12}(a_1|a_2\otimes a_3) + E_{12}(a_1|a_3\otimes a_2) - E_{21}(a_1\otimes a_2|a_3) + E_{21}(a_2\otimes a_1|a_3). \tag{10}$$

Note that this condition will play important role in the definition on the desuspension of a Hirsch algebra of DG-Lia algebra structure, see below.

Remark. Thus the operations E_{12} and E_{21}, which initially are tools to measure the deviations from Hirsch formulae, see (7) and (8), simultaneously measure the deviation from associativity of the $\cup_1$ product.

Generally the restriction of (3) on $(\otimes^k A)\otimes(\otimes^l A)\otimes(\otimes^m A)$ gives:

$$\begin{array}{c}\sum_{r=1}^{l+m}\sum_{l_1+...+l_r=l,m_1+...+m_r=m}\\ E_{kr}(a_1\otimes...\otimes a_k|E_{l_1m_1}(b_1\otimes...\otimes b_{l_1}|c_1\otimes...\otimes c_{m_1})\otimes...\otimes\\ E_{l_rm_r}(b_{l_1+...+l_{r-1}+1}\otimes...\otimes b_l|c_{m_1+...+m_{r-1}+1}\otimes...\otimes c_m) =\\ \sum_{s+1}^{k+l}\sum_{k_1+...+k_s=k,l_1+...+l_s=l}\\ E_{sm}(E_{k_1l_1}(a_1\otimes...\otimes a_{k_1}|b_1\otimes...\otimes b_{l_1})\otimes...\otimes\\ E_{k_sl_s}(a_{k_1+...+k_{s-1}+1}\otimes...\otimes a_k|b_{l_1+...+l_{s-1}+1}\otimes...\otimes b_l)|c_1\otimes...\otimes c_m)\end{array} \tag{11}$$

All above can be summarized as the

Theorem 1 *A multiplication $\mu : BA\otimes BA\to BA$, which turns the bar construction BA into a DG-Hopf algebra, specifies on A the set of multioperations (6) $E_{m,n} : (\otimes^m A)\otimes(\otimes^n A)\to A$ which satisfy the conditions (5), (9) and (11).*

In particular, the operation $E_{1,1}$ is a sort of $\cup_1$ product, which measures the noncommutativity of A and satisfies both (left and right) Hirsch formulae up to homotopy.

A DG-algebra endowed with such structure we call *Hirsch algebra*. This name is inspired by the fact that the defining conditions (9) and (9) and (11) can be regarded as generalizations of classical Hirsch formula

$$(a\cdot b)\cup_1 c = a\cdot(b\cup_1 c) + (a\cup_1 c)\cdot b.$$

This structure is the particular case of B_∞-algebra, see bellow.

1.2 Levels of noncommutativity

We distinguish various levels of "noncommutativity" of A according to the form of the appropriate twisting cochain E.

Level 1. If the twisting cochain E looks as

$$\begin{array}{ccccccccccc} & & & E_{0,1}=id & & & & E_{1,0}=id & & & \\ & & & & & 0 & & & & & \\ & & & 0 & & & & 0 & & & \\ & 0 & & & & 0 & & & & 0 & \\ .. & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & \cdots\ , \end{array}$$

i.e. E has just two nonzero components $E_{0,1} = id$ and $E_{1,0} = id$, then, it follows from (1), A is *strictly* commutative DG-algebra.

Level 2. Suppose E looks as

$$\begin{array}{ccccccccccc} & & & E_{0,1}=id & & & & E_{1,0}=id & & & \\ & & & & & E_{11} & & & & & \\ & & & 0 & & & & 0 & & & \\ & 0 & & & & 0 & & & & 0 & \\ .. & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & \cdots, \end{array}$$

i.e. E has just three nonzero components $E_{0,1} = id$, $E_{1,0} = id$ and $E_{1,1}$. In this case A is endowed by a "strict" $\cup_1$ product $a \cup_1 b = E_{1,1}(a \otimes b)$, the condition (9) here gives

$$d_A(a \cup_1 b) = d_A a \cup_1 b + a \cup_1 d_A b + ab - ba,$$

$$a \cup_1 (bc) = b(a \cup_1 c) + (a \cup_1 b)c,$$
$$(ab) \cup_1 c = a(b \cup_1 c) + (a \cup_1 c)b,$$
$$(a \cup_1 c) \cdot (b \cup_1 d) = 0,$$

and the condition (11) degenerates to the associativity $\cup_1$:

$$a \cup_1 (b \cup_1 c) = (a \cup_1 b) \cup_1 c.$$

As we see here we have very strong restrictions on $\cup_1$-product. The trivial example of DG-algebra with such strict $\cup_1$ product is $(H^*(SX, Z_2), d = 0)$

with $a\cup_1 b=0$ if $a\neq b$ and $a\cup_1 a=Sq^{|a|-1}a$. Another example (see [19]) is $C^*(SX,CX)$, where SX is the suspension and CX is the cone of a space X.

Level 3. This is the "one line" case, when E looks as

$$\begin{array}{ccccccccc} & & & E_{0,1}=id & & E_{1,0}=id & & & \\ & & & & E_{11} & & & & \\ & & & E_{1,2} & & 0 & & & \\ & & E_{1,3} & & 0 & & 0 & & \\ .. & ... & & ... & ... & ... & ... & ..., & \end{array} \tag{12}$$

i.e. when all components of E are zero except $E_{0,1},E_{1,0}$ and $E_{1,k},\quad k=1,2,3,...$. We remark here that this case is of special interest in this article.

In this case the condition (9) degenerates into two conditions: at $(\otimes^1 A)\otimes(\otimes^k A)$

$$\begin{array}{c} d_AE_{1,k}(a|b_1\otimes...\otimes b_k)+E_{1,k}(d_Aa|b_1\otimes...\otimes b_k)+ \\ \sum_i E_{1k}(a|b_1\otimes...\otimes d_Ab_i\otimes...\otimes b_k)= \\ b_1E_{1k}(a|b_2\otimes...\otimes b_k)+\sum_i E_{1k}(a|b_1\otimes...\otimes b_ib_{i+1}\otimes...\otimes b_k)+ \\ E_{1k}(a|b_1\otimes...\otimes b_{k-1})b_k, \end{array} \tag{13}$$

and at $(\otimes^2 A)\otimes(\otimes^k A)$

$$\begin{array}{c} a_1E_{1,k}(a_2|b_1\otimes...\otimes b_k)+E_{1,k}(a_1\cdot a_2|b_1\otimes..\otimes b_k)+E_{1,k}(a_1|b_1\otimes...\otimes b_k)a_2= \\ \sum_{p=1,...,k-1}E_{1,p}(a_1|b_1\otimes...\otimes b_p)\cdot E_{1,m-p}(a_2|b_{p+1}\otimes...\otimes b_k), \end{array} \tag{14}$$

at $(\otimes^{n>2}A)\otimes(\otimes^k A)$ the condition is trivial.

The associativity condition (11) in this case looks as

$$\begin{array}{c} E_{1,n}(E_{1,m}(a;b_1,...,b_m);c_1,...,c_n)= \\ \sum_{0\leq i_1\leq...\leq i_j\leq n}\sum_{0\leq n_1+...+n_r\leq n} \\ E_{1,n-(n_1+...+n_j)+j}(a;c_1,...,c_{i_1},E_{1,n_1}(b_1;c_{i_1+1},...,c_{i_1+n_1}),c_{i_1+n_1+1},..., \\ c_{i_2},E_{1,n_2}(b_2;c_{i_2+1},...,c_{i_2+n_2}),c_{i_2+n_2+1},..., \\ c_{i_j},E_{1,n_j}(b_j;c_{i_j+1},...,c_{i_j+n_j}),c_{i_j+n_j+1},...,c_n), \end{array} \tag{15}$$

Actually the structure of this level coincides with the notion of Homotopy G-algebra, see bellow.

Note that as the **case 3'** can be considered the case

$$\begin{array}{ccccccccccc} & & & E_{0,1}=id & & & & E_{1,0}=id & & & \\ & & & & & E_{11} & & & & & \\ & & & 0 & & & & E_{21} & & & \\ & 0 & & & & 0 & & & & E_{31} & \\ .. & ... & & ... & & ... & & ... & & ... & ..., \end{array}$$

with suitable conditions.

Level 4. As our last level we consider the case of a twisting cochain $E = \{E_{pq}\}$ with no other restrictions but (5), (9) and (11).

This is nothing else than the Hirsch algebra structure, defined above.

1.3 B_∞-algebra

The notion of $B_\infty-$algebra was introduced in [2], [7] as an additional structure on a DG-algebra $(A, \cdot, d)$ which turns the tensor coalgebra $T^c(s^{-1}A) = BA$ into a DG-Hopf algebra. So it requires a new differential

$$\tilde{d} : BA \to BA$$

(which should be a coderivation with respect to standard coproduct of BA) and a new associative multiplication

$$\tilde{\mu} : (BA, \tilde{d}) \otimes (BA, \tilde{d}) \to (BA, \tilde{d})$$

which should be a map of DG-coalgebras, with $1_\Lambda \in \Lambda \subset BA$ as a unit element.

It is known (see for example [11], [20], [14]) that such $\tilde{d}$ specifies on A a structure of A_∞-algebra in the sense of Stasheff [22], namely a sequence of operations $\{m_i : \otimes^i A \to A, i = 1, 2, 3, ...\}$ subject of appropriate conditions.

As for the new multiplication $\tilde{\mu}$, it follows from the above considerations, that it is induced by a sequence of operations $\{E_{pq}\}$ satisfying (5), (11) and the modified condition (9) with involved A_∞-algebra structure $\{m_i\}$.

Thus the Hirsch algebra structure (the above mentioned level 4 and, consequently, other levels) in fact is the particular B_∞-algebra structure on A when the standard differential of the bar construction $d_B : BA \to BA$ *does not change*, i.e. $\tilde{d} = d_B$ (in this case the corresponding A_∞-algebra structure is degenerate: $\{m_1 = d_A, m_2 = \mu_A, m_3 = 0, m_4 = 0, ...\}$).

Let us mention, that a sequence of cochain operations $\{E_{pq}\}$ satisfying (5) and (9), (but not (11) i.e. the induced product in the bar construction is not strictly associative), was constructed in [16] for the singular cochain complex of a topological space $C^*(X)$ using acyclic models, the starting condition $E_{0,1} = E_{1,0} = id$, and $E_{0,k} = E_{k,0} = 0$ for $k > 1$, determines a twisting cochain E uniquely up to equivalence of twisting cochains in this case.

1.4 DG-Lie algebra structure in a Hirsch algebra

Let $(A, d, \cdot, \{E_{pq}\})$ be a Hirsch algebra, then in the desuspension $s^{-1}A$ there appears a structure of DG-Lie algebra: although the $\cup_1 = E_{11}$ is not associative, the condition (10), which is the particular case of the condition 11, implies the pre-Jacobi identity

$$a \cup_1 (b \cup_1 c) - (a \cup_1 b) \cup_1 c = a \cup_1 (c \cup_1 b) - (a \cup_1 c) \cup_1 b$$

which guarantees that the commutator

$$[a, b] = a \cup_1 b - b \cup_1 a$$

satisfies the Jacobi identity. Besides the condition (1) implies that $[\ ,\ ] : A^p \otimes A^q \to A^{p+q-1}$ is a chain map.

The structure of Hirsch algebra on A induces on homology $H(A)$ the structure of Gerstenhaber algebra (G-algebra) [6] which is defined as a commutative graded algebra $(H, \cdot)$ together with a lie bracket of degree -1

$$[\ ,\ ] : H^p \otimes H^q \to H^{p+q-1}$$

i.e. a graded Lie algebra structure on the desuspension $s^{-1}H$ which is a biderivation: $[ab, c] = a[b, c] + [a, c]b$ (this is a sort of graded version of Poisson algebra).

The existence of this structure in the homology $H(A)$ of a Hirsch algebra $(A, d, \cdot, \{E_{1k}\})$ is seen by the following argument. As it is mentioned above $s^{-1}A$ is a DG-Lie algebra when A is a Hirsch algebra. Thus on $s^{-1}H(A)$ there appears the structure of graded Lie algebra. The up to homotopy Hirsh formulae (7) and (8) imply that the induced Lie bracket is a biderivation.

1.5 Homotopy G-algebra

A Hirsch algebra of particular type of level 3 in the literature is known as a Homotopy G-algebra.

A *Homotopy G-algebra* in [6] and [25] is defined as a DG-algebra $(A, d, \cdot)$ with a given sequence of multibraces $a\{a_1, ..., a_k\}$ which, in our notation, we regard as a sequence of operations

$$E_{1,k} : A \otimes (\otimes^k A) \to A, \quad k = 1, 2, 3, ...$$

which, together with $E_{01} = id$ satisfies the conditions (5), (13), (14) and (15).

The name *Homotopy G-algebra* is motivated by the fact that this structure induces on homology $H(A)$ the structure of Gerstenhaber algebra (G-algebra) (as we have seen in the previous section appears even in the homology of a Hirsch algebra).

As it was mentioned above, such a sequence defines a twisting cochain

$$E : BA \otimes BA \to A,$$

the conditions (13) and (14) mean nothing other than that E satisfies the condition (3), and, consequently defines a product on the bar construction $\mu_E : BA \otimes BA \to BA$. But, we emphasize, that this twisting cochain E is of special type, it is of level 3, i.e. it is a “one line” twisting cochain, like (12): all it's components, except maybe $E_{1,k}$, are zero.

1.6 Strong homotopy commutative algebras

The notion of Strong homotopy commutative algebra (shc-algebra), as a tool for measuring of noncommutativity of DG-algebras, was used in many papers: [9], [17], [24], etc.

A shc-algebra is a DG-algebra $(A, d, \cdot)$ with a given twisting cochain

$$\Phi : B(A \otimes A) \to A$$

which satisfies certain up to homotopy conditions of associativity and commutativity (actually Φ induces a DG-algebra map $\Omega B(A \otimes A) \to A$).

We remark here that the fact that shc structure measures the noncommutativity of A is result *just of existence* of twisting cochain Φ and not of homotopy commutativity of it: in [17], Proposition 4.8 the $\cup_1$ product in A is defined in terms of Φ by the formula

$$a \cup_1 b = \Phi[(1 \otimes a) \otimes (b \otimes 1) + (a \otimes 1) \otimes (1 \otimes b)].$$

There is the shuffle map (a DG-coalgebra map)

$$Sh : BA \otimes BA \to B(A \otimes A),$$

thus each shc-algebra structure, i.e. a twisting cochain Φ induces a twisting cochain $E = \Phi \circ Sh : BA \otimes BA \to A$ of level 4 in the above description, which, in fact is an almost Hirsch algebra structure on A: we can not expect the strict associativity of the product in BA induced by this E, since Φ is associative just up to homotopy.

Conversely, the shuffle map Sh is a weak equivalence of DG-coalgebras, thus it induces a bijection between *equivalence classes* of twisting cochains $E : BA \otimes BA \to A$ and $\Phi : B(A \otimes A) \to A$. It means that to a twisting cochain E (to a Hirsch algebra structure) corresponds a class of twisting cochains Φ (class of shc-algebra structures) such that $E \sim \Phi \circ Sh$.

We remark here that as a rule a shc-algebra structure (i.e. the twisting cochain Φ) is constructed using acyclic models, so it is not uniquely determined, thus there is no guarantee, that the induced $E = \Phi \circ Sh$ will be of level 3 (i.e. of "one line" form, consisting just of components $E_{1,k}$), so the induced structure will not be generally a homotopy G-algebra. We emphasize that for the homotopy G-algebra structure (for the twisting cochain E) there are explicit formulae in the concrete cases mentioned above.

2 Twisting elements

2.1 Twisting elements in a homotopy G-algebra

There is a very useful notion of *twisting element*, introduced by E. Brown in [4]: roughly speaking for a cochain algebra C^* a twisting element is defined as a 1-dimensional element $a \in C^1$ such that $da = aa$. Later in [3] N. Berikashvili (see also [**?**], [17] in various contexts) had introduced an equivalence relation in the set of all twisting elements $Tw(C^*)$, namely he introduced the action of the group G of invertible elements from C^0 on the set

$Tw(C^*)$, given by the formula

$$g * a = g \cdot a \cdot g^{-1} + dg \cdot g^{-1}.$$

The set of orbits $D(C^*) = Tw(C^*)/G$ is a functor on C^* (with very interesting properties, for example D sends weak equivalences to isomorphisms), which has important applications in the homology theory of fibrations.

In this section we construct the analog of the notion of twisting element for Homotopy G-algebra (and the appropriate group action), replacing in the equation $da = aa$ the product by the $\cup_1$ product.

Let $(C^{*,*}, d, \cdot, \{E_{1,k}\})$ be a *bigraded* homotopy G-algebra, we mean the following:

$$d(C^{m,n}) \subset C^{m+1,n}; \quad C^{m,n} \cdot C^{p,q} \subset C^{m+p,n+q};$$
$$E_{1,k}(C^{m,n}|C^{p_1,q_1} \otimes ... \otimes C^{p_k,q_k}) \subset C^{m+p_1+...+p_k-k,n+q_1+...+q_k}.$$

Bellow we introduce two versions of the notion of *twisting elements* in a homotopy G-algebra and the appropriate group actions. The first one controls the degeneracy of A_∞-algebra structures and the second controls deformations of algebras.

Version1. A *twisting element* in $C^{*,*}$ we define as

$$m = m^3 + m^4 + ... + m^p + ... \;;\; m^p \in C^{p,2-p}$$

satisfying the condition $dm = E_{1,1}(m|m) = m \cup_1 m$. This condition can be rewritten as

$$dm^p = \sum_{i=3}^{p-1} m^i \cup_1 m^{p-i+2}. \qquad (16)$$

Particularly $dm^3 = 0, dm^4 = m^3 \cup_1 m^3, dm^5 = m^3 \cup_1 m^4 + m^4 \cup_1 m^3, ...$ The set of all twisting elements we denote by $Tw(C^{*,*})$.

Consider the set $G = \{g = g^2 + g^3 + ... + g^p + ...;\; g^p \in C^{p,1-p}\}$. This set is a group with respect to the following operation

$$\overline{g} * g = \overline{g} + g + \sum_{k=1}^{\infty} E_{1,k}(\overline{g}|g \otimes ... \otimes g), \qquad (17)$$

particularly

$$(\overline{g} * g)^2 = \overline{g} + g^2; \quad (\overline{g} * g)^3 = \overline{g}^3 + g^3 + \overline{g}^2 \cup_1 g^3;$$
$$(\overline{g} * g)^3 = \overline{g}^4 + g^3 + \overline{g}^2 \cup_1 g^3 + \overline{g}^3 \cup_1 g^2 + E_{1,2}(\overline{g}^2|g^2 \otimes g^2).$$

This operation is associative, has the unit $e = 0 + 0 + ...$ and the opposite g^{-1} can be solved inductively from the equation $g * g^{-1} = e$.

The group G acts on the set $Tw(C^{*,*})$ by the rule $g * m = \overline{m}$ where

$$\overline{m} = m + dg + g \cdot g + E_{1,1}(g|m) + \sum_{k=1}^{\infty} E_{1,k}(\overline{m}|g \otimes ... \otimes g), \tag{18}$$

particularly

$$\overline{m}^3 = m^3 + dg^2; \quad \overline{m}^4 = m^4 + dg^3 + g^2 \cdot g^2 + g^2 \cup_1 m^3 + m^3 \cup_1 g^2;$$
$$\overline{m}^5 = m^5 + dg^4 + g^2 \cdot g^3 + g^3 \cdot g^2 + g^2 \cup_1 m^4 + g^3 \cup_1 m^3 +$$
$$\overline{m}^3 \cup_1 g^3 + \overline{m}^4 \cup_1 g^2 + E_{1,2}(\overline{m}^3|g^2 \otimes g^2).$$

note, that although in the right hand side of this formula participates $\overline{m}$ but it has less dimension then the left hand side $\overline{m}$, thus the components of $\overline{m}$ can be solved from this equation inductively. The resulting $\overline{m}$ is a twisting element too. By $D(C^{*,*})$ we denote the set of orbits $Tw(C^{*,*})/G$.

This group action allows us to perturb twisting elements. Let $g^n \in C^{n,1-n}$ be an arbitrary element, then for

$$g = 0 + ... + 0 + g^n + 0 + ...$$

the twisting element $\overline{m} = g * m$ looks as

$$\overline{m} = m_3 + ... + m_n + (m_{n+1} + dg_n) + \overline{m}^{n+2} + \overline{m}^{n+3} + ... \quad .$$

For an arbitrary twisting element $m = m^3 + m^4 + ...$ the first component $m^4 \in C^{3,-1}$ is a cocycle. If it's class in the cohomology module $H^{3,-1}(C^{*,*})$ is zero, then $m^3 = dg^2$ for some $g^2 \in C^{2,-1}$. Perturbing m by $g = g^2 + 0 + 0 + ...$ we can kill the first component m^3, i.e. we get the twisting element $\overline{m} \sim m$, which looks as

$$\overline{m} = 0 + \overline{m}^4 + \overline{m}^5 + ...,$$

now the component $\overline{m}^4$ becomes a cocycle. If it's class is zero, then we can kill it, etc. Finally we get the

Proposition 1 *If for a bigraded homotopy G-algebra $C^{*,*}$ all homology modules $H^{n,2-n}(C^{*,*})$ are trivial for $n \geq 3$, then $D(C^{*,*}) = 0$, i.e. each twisting element is equivalent to trivial one.*

Version 2. A twisting element in this case we define as

$$b = b_1 + b_2 + ... + b_n + ... \ ; \ b_n \in C^{2,n}$$

satisfying the condition

$$db_n = \sum_{i=2}^{n-1} b_i \cup_1 b_{n-i}.$$

Particularly

$$db_1 = 0; \quad db_2 = b_1 \cup_1 b_1; \quad db_3 = b_1 \cup_1 b_2 + b_2 \cup_1 b_1; \quad ... \quad .$$

The set of all twisting elements we denote by $Tw'(C)$.

Here we consider the group

$$G' = \{g = g_1 + g_2 + ... + g_p + ... \ ; \quad g_p \in C^{1,p}\}$$

with operation

$$g' * g = g' + g + \sum_{k=1}^{\infty} E_{1,k}(g'|g \otimes .. \otimes g).$$

Particularly

$$(g' * g)_1 = g'_1 + g_1; \quad (g' * g)_2 = g'_2 + g_2 + g'_1 \cup_1 g_1;$$
$$(g' * g)_3 = g'_3 + g_3 + g'_1 \cup_1 g_2 + g'_2 \cup_1 g_1 + E_{1,2}(g'_1|g_1 \otimes g_1).$$

As above this operation is associative, has the unit $e = 0 + 0 + ...$ and the opposite g^{-1} can be solved inductively from the equation $g * g^{-1} = e$.

The group G' acts on the set $Tw'(C^{*,*})$ by the rule $g * b = b'$ where

$$b' = b + dg + g \cdot g + E_{1,1}(g|b) + \sum_{k=1}^{\infty} E_{1,k}(b'|g \otimes ... \otimes g), \qquad (19)$$

particularly

$$b'_1 = b_1 + dg_1; \quad b'_2 = b_2 + dg_2 + g_1 \cdot g_1 + g_1 \cup_1 b_2 + b'_1 \cup_1 g_1;$$
$$b'_3 = b_3 + dg_3 + g_1 \cdot g_2 + g_2 \cdot g_1 + g_1 \cup_1 b_2 + g_2 \cup_1 b_1 +$$
$$b'_1 \cup_1 g_2 + b'_2 \cup_1 g_1 + E_{1,2}(b'_1|g_1 \otimes g_1).$$

The components of b' can be solved from this equation inductively. The resulting b' is a twisting element too. By $D'(C^{*,*})$ we denote the set of orbits $Tw'(C^{*,*})/G'$.

As above this group action allows to perturb twisting elements and we have the

Proposition 2 *If for a bigraded homotopy G-algebra $C^{*,*}$ all homology modules $H^{2,n}(C^{*,*})$ are trivial for $n \geq 1$, then $D'(C^{*,*}) = 0$, i.e. each twisting element is equivalent to trivial one.*

2.2 Twisting elements in a DG-Lie algebra

There is a modified notion of twisting element in a DG-Lie algebra $(L, d, [\ ,\ ])$. This is an element $a \in L^1$ such that $da = \frac{1}{2}[a, a]$ (this equation in literature is called Maurer-Carton equation, or, master equation). The systematic steady of this notion is done in [10].

As it is described above for a homotopy G-algebra $(C, \cdot, d\{E_{1k}\})$ in the desuspension $s^{-1}A$ there appears the structure of DG-Lie algebra with the bracket $[a, b] = a \cup_1 b - b \cup_1 a$. Note that if $C^{*,*}$ is a bigraded homotopy G-algebra, then $s^{-1}C^{*,*}$ where $(s^{-1}C^{*,*})^{p,q} = C^{p-1,q}$ is a bigraded DG-Lie algebra.

Suppose $m = m^3 + m^4 + ... + m^p + ...\ ;\ m^p \in C^{p,2-p}$ is a twisting element in A of version 1. The defining equation $dm = m \cup_1 m$ can be rewritten in terms of bracket as $dm = \frac{1}{2}[m, m]$, so the same m can be regarded as a Lie twisting element.

Exactly the same is true for a twisting element $b = b_1 + b_2 + ... + b_n + ...\ ;\ b_n \in C^{2,n}$ of version 2: the condition $db = b \cup_1 b$ in terms of bracket looks as $db = \frac{1}{2}[b, b]$.

As an open question of J. Huebschmann remains to rewrite the formulae (18) and (19) of transformation of twisting elements it terms of bracket.

3 Examples of Hirsch algebras

3.1 Cochain algebra of a simplicial set

An example of Hirsch algebra is the cochain complex $C^*(S)$ of a 1-reduced simplicial set S. In [2] Baues has constructed the strictly associative product in BA where $A = C^*(S)$. Examining the appropriate twisting cochain, one can discover that it is "one line', of level 3, thus it forms a structure of Hirsch algebra.

3.2 Hochschild cochain complex

Let A be an algebra and M be a two sided module on A. The Hochschild cochain complex $C^*(A;M)$ of A with coefficients in M is defined by $C^n(A;M) = Hom(\otimes^n A, M)$ with differential $\delta : C^{n-1}(A;M) \to C^n(A;M)$ given by

$$\delta f(a_1 \otimes ... \otimes a_n) = a_1 f(a_2 \otimes ... \otimes a_n) +$$
$$\sum_{k=1}^{n-1} f(a_1 \otimes ... \otimes a_{k-1} \otimes a_k a_{k+1} \otimes .. \otimes a_n) + f(a_1 \otimes ... \otimes a_{n-1}) a_n.$$

If M is an algebra over A then in the Hochschild complex there appears the $\cup$ product

$$f \cup g(a_1 \otimes ... \otimes a_{n+m}) = f(a_1 \otimes ... \otimes a_n) \cdot g(a_{n+1} \otimes ... \otimes a_{n+m})$$

which turns $C^*(A;M)$ into a cochain algebra.

We focus on the case $M = A$. In [13] the explicit formulae for operations, which specify on the Hochschild cochain complex $C^*(A;A)$ a structure of homotopy G-algebra. Bellow we describe this structure. Note, that the same operations were described in [7] when constructing the $B(\infty)$-algebra structure on $C^*(A;A)$, and in [6], [25].

In [5] Gerstenhaber has defined a product $f \circ g$ in the Hochschild complex $C^*(A,A)$, given by

$$f \circ g(a_1 \otimes ... \otimes a_{n+m-1}) =$$
$$\sum_{k=0}^{n-1} f(a_1 \otimes ... \otimes a_k \otimes g(a_{k+1} \otimes .. \otimes a_{k+m}) \otimes a_{k+m+1} \otimes ... \otimes a_{n+m-1}).$$

The Gerstenhaber's product has the following properties:

$$\delta(f \circ g) = \delta f \circ g + f \circ \delta g + f \cup g - g \cup f,$$

and

$$(f \cup g) \circ h = f \cup (g \circ h) + (f \circ h) \cup g,$$

this means, that the product $f \circ g$ has the properties of $\cup_1$product: if we use the notation $f \circ g = f \cup_1 g$, then the first condition gives the standard condition on the $\cup_1$ product

$$\delta(f \cup_1 g) = \delta f \cup_1 g + f \cup_1 \delta g + f \cup g - g \cup f$$

and the second gives the left Hirsch formula

$$(f \cup g) \cup_1 h = f \cup (g \cup_1 h) + (f \cup_1 h) \cup g.$$

As for right Hirsch formula, there is the different kind of $\cup_1$ product of a cochain and a *couple of cochains*: for $f \in C^p(A;A), g \in C^q(A;A), h \in C^r(A;A)$ we define $f \cup_1 (g,h) \in C^{p+q+r-2}(A;A)$ by

$$\begin{gathered}(f \cup_1 (g,h))(a_1 \otimes ... \otimes a_{p+q+r-2}) = \\ \sum_{k,l} f(a_1 \otimes ... \otimes a_k \otimes g(a_{k+1} \otimes ... \otimes a_{k+q}) \otimes a_{k+m+1} \otimes ... \\ \otimes a_l \otimes h(a_{l+1} \otimes ... \otimes a_{l+r}) \otimes a_{l+r+1} \otimes ... \otimes a_{p+q+r-2}).\end{gathered}$$

The straightforward verification shows, that the $\cup_1$ product in $C^p(A;A)$ satisfies the right Hirsch formula up to homotopy and the appropriate homotopy is $f \cup_1 (g,h)$, i.e. the following condition is satisfied

$$\begin{gathered}\delta(f \cup_1 (g,h)) + \delta f \cup_1 (g,h) + f \cup_1 (\delta g, h) + f \cup_1 (g, \delta h) = \\ f \cup_1 (g \cup h) + g \cup (f \cup_1 h) + (f \cup_1 g) \cup h.\end{gathered}$$

Let us mention also the following property of the introduced product: the product $f \cup_1 (g,h)$ measures the nonassociativity of $\cup_1$ product:

$$f \cup_1 (g \cup_1 h) - (f \cup_1 g) \cup_1 h = f \cup_1 (g,h) + f \cup_1 (h,g). \tag{20}$$

Remark. In [5], see also [23], in the desuspension of Hochschild complex $s^{-1}C^*(A;A)$ a DG-Lie algebra structure was introduced. Actually the Lie bracket $[f,g]$ is the commutator of $\cup_1$ product: $[f,g] = f \cup_1 g - g \cup_1 f$. Although the $\cup_1$ product is not associative, the condition (20) allows to check, that the Jacobi identity is satisfied.

In [13] we have defined the generalized $\cup_1$ products of a hochschild cochain and a sequence of cochains:

$$\begin{gathered}(f \cup_1 (g_1, ..., g_i))(a_1 \otimes ... \otimes a_n) = \\ \sum f(a_1 \otimes ... \otimes a_{k_1} \otimes g_1(a_{k_1+1} \otimes ..a_{k_1+n_1}) \otimes ... \otimes a_{k_i} \otimes \\ g_i(a_{k_i+1} \otimes ...a_{k_i+n_i}) \otimes ... \otimes a_n).\end{gathered}$$

The straightforward verification shows that the collection $\{E_{1,k}\}$ given by

$$E_{1,k}(f|g_1 \otimes ... \otimes g_k) = f \cup_1 (g_1, ..., g_k)$$

satisfies the conditions (5), (13), (14) and (15), thus it forms on the Hochschild complex $C^*(A;A)$ is a structure of homotopy G-algebra.

3.3 Cobar construction of a Hopf algebra

As the third example of Hirsch algebra we present the cobar construction of a Hopf algebra.

The cobar construction ΩA of a coalgebra $(A, \nabla : A \to A \otimes A)$ is a DG-algebra

$$\Omega A = T(A) = \Lambda + A + A \otimes A + A \otimes A \otimes A + ...$$

with the product

$$(a_1 \otimes ... \otimes a_p) \cdot (a_{p+1} \otimes ... \otimes a_{p+q}) = a_1 \otimes ... \otimes a_{p+q}$$

(i.e. it is a free graded algebra, generated by A), with differential

$$d_\Omega(a_1 \otimes ... \otimes a_n) = \sum_i a_1 \otimes ... \otimes \nabla a_i \otimes ... \otimes a_n.$$

What additional structure appears on ΩA if A is a *Hopf algebra*, i.e. if it is equipped additionally with a product $A \otimes A \to A$, which is a coalgebra map? It is shown in [1], that if the ground ring is Z_2 then in ΩA there exists a $\cup_1$ product, given by

$$(a_1 \otimes ... \otimes a_p) \cup_1 (b_1 \otimes ... \otimes b_q) = \sum_i a_1 \otimes ... \otimes a_{i-1} \otimes a_i^{(1)} \cdot b_1 \otimes .. \otimes a_i^{(q)} \cdot b_q \otimes a_{i+1} \otimes ... \otimes a_p,$$

where $\nabla^q(a_i) = a_i^{(1)} \otimes ... \otimes a_i^{(q)}$ is the q-fold iteration of ∇ and $a \cdot b$ is the product in A. It is clear that this $\cup_1$ product is functorial on the category of Hopf algebras.

Let's introduce the following notation. For $a \in A$ and for $b_1 \otimes .. \otimes b_q \in \otimes^q A$ we define $a \vee (b_1 \otimes ... \otimes b_q) \in \otimes^q A$ as

$$a^{(1)} \cdot b_1 \otimes ... \otimes a^{(q)} \cdot b_q.$$

Thus the definition of Adams's $\cup_1$ product now looks as

$$(a_1 \otimes ... \otimes a_p) \cup_1 (b_1 \otimes ... \otimes b_q) = \sum_i a_1 \otimes ... \otimes a_{i-1} \otimes a_i \vee (b_1 \otimes .. \otimes b_q) \otimes a_{i+1} \otimes ... \otimes a_p.$$

Bellow we show that there exist functorial operations

$$E_{1,k} : (\Omega A) \otimes (\otimes^k \Omega A) \to \Omega A,$$

with $E_{1,1} = \cup_1$ and which satisfy the conditions (13), (14) and (4) i.e. which form on ΩA a structure of homotopy G- algebra.

Here is the formula for operation $E_{1,k}$. Let $\alpha = a_1 \otimes .. \otimes a_n \in \Omega A$ and $\beta_1,\ \beta_2,\ ...\ \beta_k \in \Omega A$, then define

$$E_{1,k}(\alpha|\beta_1 \otimes ... \otimes \beta_k) = \sum a_1 \otimes .. \otimes a_{i_1-1} \otimes a_{i_1} \vee \beta_1 \otimes a_{i_1+1} \otimes .. \otimes \\ a_{i_k-1} \otimes a_{i_k} \vee \beta_k \otimes a_{i_k+1} \otimes ... \otimes a_n, \tag{21}$$

where the summation is taken over all $1 \leq i_1 < ... < i_k \leq n$. It is clear that $E_{1,k}(\alpha|\beta_1 \otimes ... \otimes \beta_k) = 0$ if $n < k$.

Remark. The way, how this formula is obtained is following. We take the starting condition $E_{1,k}(a_1|\beta_1 \otimes ... \otimes \beta_k) = 0$ and extend the products $E_{1,k}(a_1 \otimes a_2|\beta_1 \otimes ... \otimes \beta_k)$ using the condition (14).

Theorem. The operations $E_{1,k} : (\Omega A) \otimes (\otimes^k \Omega A) \to \Omega A$, given by (21) are functorial on the category of Hopf algebras and satisfy the conditions (13), (14) and (4), thus they form on ΩA a structure of Hirsch algebra.

4 Applications

4.1 Multiplicative twisted tensor product

In this section we present the result from [15]: the construction of *multiplicative version* of Brown's [4] twisted tensor product.

4.1.1 Twisting cochains

Let $(C, d, \nabla : C \to C \otimes C)$ be a DG-coalgebra and $(A, d, \mu : A \otimes A \to A)$ be a DG-algebra (both differentials $d : C \to C$ and $d : A \to A$ are assumed of degree +1). A twisting cochain is a homomorphism

$$\varphi : C \to A$$

of degree 1, satisfying the condition

$$d\varphi + \varphi d = \varphi \cup \varphi. \tag{22}$$

A given twisting cochain $\varphi : C \to A$ determines the following three important maps:

1. A DG-coalgebra map $f_\varphi : C \to BA$ from C to the bar construction BA, given by

$$f_\varphi = \sum_{n=0}^{\infty} (\varphi \otimes ... \otimes \varphi)\nabla^n,$$

where $\nabla^0 = \epsilon : C \to \Lambda$ is the coaugmentation, $\nabla^1 = id$ and $\nabla^n = (\nabla^{n-1} \otimes id)\nabla$ is the iteration of coproduct ∇.

2. A DG-algebra map $g_\varphi : \Omega C \to A$ from the cobar construction ΩC to A, given by

$$g_\varphi|_{\otimes^n C} = \mu^n(\varphi \otimes ... \otimes \varphi),$$

where $\mu^0 : \Lambda \to A$ is the unit of A, $\mu^1 = id$ and $\mu^n = \mu(\mu^{n-1} \otimes id)$ is the iteration of the product μ.

Remark. Let's denote by $T(C, A)$ the set of all twisting cochains $\varphi : C \to A$. Then the assignments $\varphi \longmapsto f_\varphi$ and $\varphi \longmapsto g_\varphi$ form the *bijections*

$$Hom_{DG-a\lg}(\Omega C, A) \longleftrightarrow T(C, A) \longleftrightarrow Hom_{DG-Coa\lg}(C, BA)$$

which realize the adjunction of functors B and Ω.

3. A *twisted differential* $d_\varphi : A \otimes C \to A \otimes C$ given by

$$d_\varphi(a \otimes c) = da \otimes c + a \otimes dc + \varphi \cap (a \otimes c),$$

where $\varphi \cap (a \otimes c) = (\mu \otimes id)(id \otimes \varphi \otimes id)(id \otimes \nabla)$. The tensor product $A \otimes C$ equipped with the differential d_φ is called a *twisted tensor product* and is denoted by $A \otimes_\varphi C$ (the notion belongs to E. Brown [4]). This construction has the essential applications in the homology theory of fibrations.

4.1.2 Multiplicative twisting cochains (commutative case)

Suppose now that (C, d, ∇, μ) is a *DG-Hopf algebra* and (A, d, μ) is a *commutative* DG-algebra. Then the bar construction BA is a DG-Hopf algebra with respect to shuffle product $\mu_{sh} : BA \otimes BA \to BA$.

A twisting cochain $\varphi : C \to A$ is called *multiplicative* if in addition to the standard Brown condition (22) the following condition is satisfied:

$$\varphi(ab) = \eta a \cdot \varphi(b) + \varphi(a) \cdot \eta(b),$$

(this notion was introduced by Prute in [18]). This condition is equivalent to the condition of $f_\varphi : C \to BA$ being a map of DG-Hopf algebras. Note that this condition can be reformulated in the following form: φ factors through indecomposables $QC = C/C_+ \cdot C_+$, i.e. there exists ψ for which commutes the diagram (see [14])

$$\begin{array}{ccc} C & \stackrel{\varphi}{\to} & A \\ \psi\downarrow & \nearrow & \\ QC & & \end{array} .$$

On the other hand the tensor product $A \otimes C$ is a graded algebra (since C and A both are algebras). When the twisted differential d_φ is compatible with this product? As it is shown in [18] it happens when φ is multiplicative.

4.1.3 Multiplicative twisting cochains (noncommutative case)

The result of this subsection was announced in [15].

Suppose now, that (C, d, ∇, μ) is a *DG-Hopf algebra* and $(A, d, \mu, \{E_{1,k}\})$ is a *homotopy G-algebra*. Then, as we know, in the bar construction BA there appears the product $\mu_E : BA \otimes BA \to BA$.

A twisting cochain $\varphi : C \to A$ we call *multiplicative* if if, in addition to the standard Brown condition (22), the following condition is satisfied:

$$\begin{gathered} \varphi(ab) = \eta a \cdot \varphi(b) + \varphi(a) \cdot \eta(b) + E_{1,1,}(\varphi(a)|\varphi(b)) + \\ E_{1,2}(\varphi(a)|(\varphi \otimes \varphi)\nabla^2(b)) + E_{1,3}(\varphi(a)|(\varphi \otimes \varphi \otimes \varphi)\nabla^3(b)) + \dots . \end{gathered} \tag{23}$$

This condition is equivalent to the condition of $f_\varphi : C \to BA$ being multiplicative, i.e. a map of DG-Hopf algebras.

Generally, even if φ is multiplicative in this sense, the twisted differential d_φ is not a derivation with respect to the standard multiplication of tensor product $A \otimes C$. There appears the need *to twist* the multiplication in $A \otimes C$ too. Here is the formula for this twisted multiplication:

$$\mu_\varphi = (\mu_A \otimes \mu_C)(1 \otimes E_{1,*} \otimes 1 \otimes 1)(1 \otimes 1 \otimes f_\varphi \otimes 1 \otimes 1)(1 \otimes 1 \otimes \nabla \otimes 1)(1 \otimes T \otimes 1).$$

Direct inspections proves the following

Theorem 2 *Let (C, d, ∇, μ) be a DG-Hopf algebra, $(A, d, \mu, \{E_{1,k}\})$ be a homotopy G-algebra and $\varphi : C \to A$ be a multiplicative twisting cochain (i.e. satisfies (22) and (23)), then the twisted differential $d_\varphi : A \otimes C \to A \otimes C$ is a derivation with respect to the twisted multiplication $\mu_\varphi : (A \otimes C) \otimes (A \otimes C) \to A \otimes C$, i.e. the twisted tensor product $(A \otimes C, d_\varphi, \mu_\varphi)$ is a DG-algebra in this case.*

4.2 Deformation of algebras

This is just illustrative application. Using the homotopy G-algebra structure, the notion of twisting element and gauge transformation we obtain the well known result of Gerstenhaber from [5].

Let $(A,\cdot)$ be an algebra over a field k, $k[[t]]$ be the algebra of formal power series in variable t and $A[[t]]=A\otimes k[[t]]$ be the algebra of formal power series with coefficients from A.

Deformation of an algebra $(A,\cdot)$ is defined as a sequence of homomorphisms

$$B_i: A\otimes A\to A,\quad i=0,1,2,...;\quad B_0(a\otimes b)=a\cdot b$$

satisfying the *associativity* condition

$$\sum_{i+j=n} B_i(a\otimes B_j(b\otimes c))=\sum_{i+j=n} B_i(B_j(a\otimes b)\otimes c) \tag{24}$$

for all $n\geq 1$.

Such a sequence determines the *star product*

$$a\star b=a\cdot b+B_1(a\otimes b)t+B_2(a\otimes b)t^2+B_3(a\otimes b)t^3+...\in A[[t]],$$

which can be naturally extended to a $k[[t]]-$bilinear product

$$\star : A[[t]]\otimes A[[t]]\to A[[t]]$$

and the condition 24 guarantees, that this product will be associative.

Two deformations $\{B_i\}$ and $\{B'_i\}$ are called *equivalent* if there exists a sequence of homomorphisms

$$\{g_i: A\to A;\quad i=0,1,2,...;\quad g_0=id\}$$

such that

$$\sum_{r+s=n} g_r(B_s(a\otimes B)=\sum_{i+j+k=n} B'_i(g_j(a)\otimes g_k(b)). \tag{25}$$

In this case the sequence $\{g_i\}$ determines the power series

$$g=id+g_1t+g_2t^2+...=\sum g_it^i: A\to A[[t]],$$

so, that the appropriate natural $k[[t]]-$linear map $(A[[t]],\star)\to (A[[t]],\star')$ is multiplicative isomorphism.

A deformation $\{B_i\}$ is called *trivial*, if $\{B_i\}$ is equivalent to $\{B_0, 0, 0, ...\}$. In this case the deformed algebra $(A[[t]], \star)$ is isomorphic to $A[[t]]$. An algebra A is called *rigid*, if each deformation is trivial.

As it is mentioned above the Hochschild complex $C^*(A, A)$ for an algebra A is a homotopy G-algebra. Then the tensor product

$$C^*(A, A)[[t]] = C^*(A, A) \otimes k[[t]]$$

is a *bigraded* Hirsch algebra:

$$C^{p,q} = C^p(A, A) \cdot t^q, \quad d(f \cdot t^q) = \delta f \cdot t^q, \quad f \cdot t^p \cup g \cdot t^q = (f \cup g) \cdot t^{p+q},$$
$$E_{1,k}(f \cdot t^p | g_1 \cdot t^{q_1} \otimes ... \otimes g_k \cdot t^{q_k}) = E_{1,k}(f | g_1 \otimes ... \otimes g_k) \cdot t^{p+q_1+...+q_k}.$$

Each deformation $\{B_i : \otimes^i A \to A, \ i = 1, 2, 3, ...\}$ can be interpreted as a twisting element $B = B_1 \cdot t + B_2 \cdot t^2 + ... \in C^{2,*}$: the associativity condition 24 can be rewritten as

$$\delta B_n \cdot t^n = \sum_{i+j=n} B_i \cdot t^i \cup_1 B_j \cdot t^j.$$

If two deformations are equivalent, then the appropriate Hochschild twisting elements B and B' are equivalent too, the condition 25 can be rewritten as

$$B' = B + \delta g + g \cup g + g \cup_1 B + \sum_{k=1}^{\infty} E_{1,k}(B' | g \otimes ... \otimes g).$$

Thus the set of equivalence classes of deformations is bijective to $D'(C^{*,*})$. It is clear that $H^{p,q}(C^{*,*}) = Hoch^p(A, A) \cdot t^q$, then from the Proposition 2 follows the result of Gerstenhaber: if $Hoch^2(A, A) = 0$, then A is rigid.

4.3 Degeneracy of $A(\infty)-$algebras

In this section, using the homotopy G-algebra structure in Hochschild complex, we study the problem of degeneracy of $A(\infty)$-algebra structure. Actually these results are given in [13], [14].

4.3.1 $A(\infty)$-algebras

The notion of $A(\infty)$-algebra was introduced by J.D. Stasheff in [22]. This notion generalizes the notion of DG-algebra.

An $A(\infty)$-algebra is a graded module M with a given sequence of operations

$$\{m_i : (\otimes^i M) \to M, \quad i = 1, 2, ..., \quad \deg m_i = 2 - i\}$$

which satisfies the following conditions

$$\sum_{i+j=n+1} \sum_{k=0}^{n-j} m_i(a_1 \otimes ... \otimes a_k \otimes m_j(a_{k+1} \otimes ... \otimes a_{k+j}) \otimes ... \otimes a_n) = 0. \tag{26}$$

Particularly, for the operation $m_1 : M \to M$ we have $\deg m_1 = 1$ and $m_1 m_1 = 0$, this m_1 can be regarded as a differential on M. The operation $m_2 : M \otimes M \to M$ is of degree 0 and satisfies

$$m_1 m_2(a_1 \otimes a_2) + m_2(m_1 a_1 \otimes a_2) + m_2(a_1 \otimes m_1 a_2) = 0,$$

i.e. m_2 can be regarded as a multiplication on M and m_1 is a derivation with respect to it. Thus (M, m_1, m_2) is a sort of (maybe nonassociative) DG-algebra. For the operation m_3 we have $\deg m_3 = -1$ and

$$m_1 m_3(a_1 \otimes a_2 \otimes a_3) + m_3(m_1 a_1 \otimes a_2 \otimes a_3) + m_3(a_1 \otimes m_1 a_2 \otimes a_3) +$$
$$m_3(a_1 \otimes a_2 \otimes m_1 a_3) + m_2(m_2(a_1 \otimes a_2) \otimes a_3) + m_2(a_1 \otimes m_2(a_2 \otimes a_3)) = 0,$$

thus the product m_2 is *homotopy associative* and the appropriate chain homotopy is m_3 (some authors call $A(\infty)$-algebras *strong homotopy associative DG-algebras*).

The main meaning of defining condition 26 of an $A(\infty)$-algebra $(M, \{m_i\})$ is the following. The sequence of operations $\{m_i\}$ determines on the bar construction

$$BM = T^c(s^{-1}M) = \Lambda + s^{-1}M + s^{-1}M \otimes s^{-1}M + s^{-1}M \otimes s^{-1}M \otimes s^{-1}M + ...$$

a coderivation

$$d_m(a_1 \otimes ... \otimes a_n) = \sum_{k,j} a_1 \otimes ... \otimes a_k \otimes m_j(a_{k+1} \otimes ... \otimes a_{k+j}) \otimes ... \otimes a_n),$$

and the Stasheffs condition 26 is equivalent to $d_m d_m = 0$, thus (BM, d_m) is a DG-coalgebra, which is called *bar construction* of $A(\infty)$-algebra $(M, \{m_i\})$.

A morphism of $A(\infty)$-algebras $f : (M, \{m_i\}) \to (M', \{m'_i\})$ is defined as a DG-coalgebra map of the bar constructions

$$f : B(M, \{m_i\}) \to B(M', \{m'_i\}),$$

which, since of cofreeness of the tensor coalgebra $T^c(s^{-1}M)$, is uniquely determined by the projection

$$f : B(M, \{m_i\}) \to B(M', \{m'_i\}) \to M',$$

which, in fact is a collection of homomorphisms

$$\{f_i : (\otimes^i M) \to M', \quad i = 1, 2, ..., \quad \deg f_i = 1 - i\},$$

subject of some conditions, see for example [12], [14]. Particularly $f_1 m_1 = m_1 f_1$, i.e. $f_1 : (M, m_1) \to (M', m'_1)$ is a chain map. We define a weak equivalence of $A(\infty)$-algebras as a morphism $\{f_i\}$ where f_1 is homology isomorphism.

An $A(\infty)$-algebra $(M, \{m_i\})$ we call *minimal* if $m_1 = 0$, in this case (M, m_2) is *strictly* associative graded algebra. Suppose

$$f : (M, \{m_i\}) \to (M', \{m'_i\})$$

is a weak equivalence of minimal $A(\infty)$-algebras, then $f_1 : (M, m_1 = 0) \to (M', m'_1 = 0)$, which by definition should be a weak equivalence, is an isomprphism. It is not hard to check, that in this case f is an isomorphism of $A(\infty)$-algebras, thus a weak equivalence of minimal $A(\infty)$-algebras is an isomorphism. This fact motivates the word *minimal* in this notion.

Suppose now that $(H, \{m_i\})$ be a *minimal* ($m_1 = 0$) $A(\infty)$-algebra. Such an $A(\infty)$-algebra we call *degenerate*, if it is isomorphic to the $A(\infty)$-algebra $(M, \{0, m_2, 0, 0, ...\})$, i.e. to the ordinary associative graded algebra (M, m_2). Bellow we are discussing the question of degeneracy of such $A(\infty)$-algebras. The similar question is considered also in [21].

4.3.2 Hochschild cohomology and $A(\infty)-$algebra structures

Suppose $(H, \mu : H \otimes H \to H)$ is a graded algebra. We shall consider Hochschild cochain complex of H with coefficients in itself, which is bigraded in this case: $C^{m,n}(H, H) = Hom^n(\otimes^m H, H)$. It is clear that the coboundary operator δ maps $C^{m,n}(H, H)$ to $C^{m+1,n}(H, H)$. Let us denote the n-the homology module of the complex $(C^{*,k}(H, H), \delta)$ by $Hoch^{n,k}(H, H)$.

Besides, for $f \in C^{m,n}(H, H)$ and $g \in C^{p,q}(H, H)$ one has $f \cup g \in C^{m+p,n+q}(H, H)$ and $f \cup_1 g \in C^{m+p-1,n+q}(H, H)$. Moreover, the above constructed operations $\{E_{1,k}\}$, which form on the Hochschild complex a structure of homotopy G-algebra behave with bigrading by the following manner:

$$E_{1,k}(f|g_1 \otimes ... \otimes g_k) \in C^{m+p_1+...+p_k-k, n+q_1+...+q_k}(H,H),$$

thus the Hochschild complex $C^{*,*}(H,H)$ is a *bigraded* homotopy G-algebra in this case.

Suppose now that $(H, \{m_i\})$ is a *minimal* ($m_1 = 0$) $A(\infty)$-algebra with $m_2 = \mu$. Each operation $m_i : (\otimes^i H) \to H$ can be regarded as a Hochschild cochain from $C^{i,2-i}(H,H)$. The condition 26 can be rewritten as

$$\delta m_k = \sum_{i=3}^{k-1} m_i \cup_1 m_{k-i+2},$$

thus $m = m_3 + m_4 + ...$ is a *twisting element* in $C^{*,*}(H,H)$. Thus each minimal $A(\infty)$-algebra structure on H can be regarded as a Hochscild twisting element and vice versa.

Suppose now that $(H, \{m_i\})$ and $(H, \{m'_i\})$ are two minimal $A(\infty)$-algebras. Then, it follows from **??** that the appropriate twisting elements m and m' are in the same orbit if and only if $A(\infty)$-algebras $(H, \{m_i\})$ and $(H, \{m'_i\})$ are isomorphic: if $m' = p * m$, then $\{p_i\} : (H, \{m_i\}) \to (H, \{m'_i\})$ with $p_0 = id$ is an *isomorphism* of $A(\infty)$-algebras. Thus, using the Proposition 1 we get the following

Theorem 3 *If for a graded algebra (H, μ) it's Hochschild cohomology modules $Hoch^{n,2-n}(H,H)$ are trivial for $n \geq 3$, then each minimal $A(\infty)$-algebra structure $\{m_i\}$ on H is degenerate, i.e. there exists an isomorphism of $A(\infty)$-algebras*

$$(H, \{m_i\}) \cong (H, \{m_2 = \mu, 0, 0, ...\}).$$

4.3.3 $A(\infty)$-algebra structure in homology of a DG-algebra

Let (A, d, μ) be a DG-algebra and $(H(A), \mu^*)$ be it's homology algebra. Although the product in $H(A)$ is associative, there appears a structure of a (generally nondegenerate) minimal $A(\infty)$-algebra, which extends the usual structure of graded algebra of $H(A)$. Namely, in [12] the following result was proved (see also [20], [8]):

Theorem 4 *If for a DG-algebra all homology Λ-modules $H_i(A)$ are free, then there exist: a structure of minimal $A(\infty)$-algebra $(H(A), \{m_i\})$ on $H(A)$ and* *a weak equivalence of $A(\infty)$-algebras*

$$\{f_i\} : (H(A), \{m_i\}) \rightarrow (A, \{m_1 = d, m_2 = \mu, 0, 0, ...\})$$

such, that $m_1 = 0$, $m_2 = \mu^*$, $f_1^* = id_{H(A)}$, such a structure is unique up to isomorphism in the category of $A(\infty)$-algebras.

Particularly such an $A(\infty)$-algebra structure appears in cohomology of a space or in homology of a topological group or H-space. It is clear, that cohomology algebra (or Pontriagin algebra) equipped with this $A(\infty)$-algebra structure carries more information about the space then cohomology algebra itself. Some applications of this structure are given in [12], [14].

Therefore it is of particular interest the cases, when this additional structure is not needed, that is when $A(\infty)$-algebra $(H(A), \{m_i\})$ is degenerate (in this case a DG-algebra A is called *formal*). The above theorem 3 gives the sufficient condition of formality of A in terms of Hochschild cohomology of $H(A)$.

References

[1] J. Adams, On the non-existence of elements of Hpof invariant one, Ann. Math. 72 (1960), 20-104.

[2] H. Baues, The double bar and cobar construction, Compositio Math.43 (1981), 331-341.

[3] N. Berikashvili, On the differentials of spectral sequence, Proc. Tbil. Math. Inst., 51 (1976), 1-105.

[4] E. Brown, Twisted tensor products. Ann. of Math., 69 (1959), 223-246.

[5] M. Gerstenhaber, On the deformations of rings and algebras, Ann. of Math., 79, 1 (1964), 59-103.

[6] M. Gerstenhaber, A. Voronov, Higher-order operations on the Hochschild complex, Functional Anal. i Prilojen., 29, 1 (1995), 1-6.

[7] E. Getzler and J.D. Jones, Operads, homotopy algebra, and iterated integrals for double loop spaces, preprint.

[8] V.K.A.M. Gugenheim and J.D. Stasheff, On Perturbations and $A(\infty)$-algebras, Bull. soc. Math. de Belg., 38 (1986).

[9] S. Halperin and J.D. Stasheff,

[10] J. Huebschmann,

[11] T. Kadeishvili, On the Homology Theory of Fibrations, Bull. of Georg. Acad. Sci.,77, 2, (1975).

[12] T. Kadeishvili, On the Homology Theory of Fibrations, Russian Math. Surveys, 35, 3 (1980), 231-238.

[13] T. Kadeishvili, The $A(\infty)$-algebra Structure and Cohomology of Hochschild and Harrison, Proc. of Tbil. Math. Inst., 91, (1988), 19-27.

[14] T. Kadeishvili, $A(\infty)$ -algebra Structure in Cohomology and Rational Homotopy Type, Proc. of Tbil. Mat. Inst., 107, (1993), 1-94.

[15] T. Kadeishvili and S. Saneblidze, On a multiplicative model of a fibration, Bull. of Georg. Acad. Sci., 153 (1996), 345-346.

[16] L. Khelaia, On some chain operations, Bull. Georg. Acad. Sci. 96, 3 (1979), 529-531.

[17] H.J. Munkholm, The Eilenberg-Moore spectral sequence and strongly homotopy multiplicative maps, J.P.A.A., 5 (1974), 1-50.

[18] A. Prute, $A(\infty)$-structures, Modele Minimal de Baues-Lemaire et Homologie des Fibrations, Preprint.

[19] S. Saneblidze,

[20] V. Smirnov, Homology of fiber spaces, Russian Math. Surveys, 35, 3 (1980), 294-298.

[21] V. Smirpov, Preprint.

[22] J.D. Stasheff, Homotopy associativity of H-spaces. I, II, Trans. Amer. Math. Soc., 108 (1963), 275-312.

[23] J.D. Stasheff, The intrinsic bracket on defformation complex of an associative algebra, JPAA volume in honor of Alex Heller.

[24] B. Ndombol and J.-C. Thomas, Products in the Hochschild homology of spaces, Preprint, 1998.

[25] A. Voronov, Homotopy Gerstenhaber algebras, QA/9908040