Twisting Elements in Homotopy G-algebras

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Abstract

We study the notion of twisting elements $da = a \smile_1 a$ with respect to $\smile_1$ product when it is a part of homotopy Gerstenhaber algebra structure. This allows to bring to one context the two classical concepts, the theory of deformation of algebras of M. Gerstenhaber, and $A(\infty)$-algebras of J. Stasheff.

1 Introduction

A *twisting element* in a differential graded algebra (dga) $(A = \{A^i\}, d : A^n \to A^{n+1}, a^m \cdot b^n \in A^{m+n})$ is defined as an element $t \in A^1$ satisfying the Brown's condition

$$dt = t \cdot t. \tag{1}$$

Denote the set of all twisting elements by $Tw(A)$. An useful consequences of the Brown' condition is the following: let M be a dg module over A, then a twisting element $t \in Tw(A)$ defines on M a new differential $d_t : M \to M$ by $d_t(x) = dx + t \cdot x$, and the condition (1) guarantees that $d_t d_t = 0$.

Twisting elements show up in various problems of algebraic topology and homological algebra. The first appearance was in homology theory of fibre bundles [5]: For a fibre bundle $F \to E \to B$ with structure group G there exists a twisting element $t \in A = C^*(B, C_*(G))$ such that $(M = C_*(B) \otimes C_*(F), d_t)$ (the twisted tensor product) gives homology of the total space E.

Later N. Berikashvili [4] has introduced in $Tw(A)$ an *equivalence relation* induced by the following group action. Let G be the group of invertible elements in A^0, then for $g \in G$ and $t \in Tw(A)$ let

$$g * t = g \cdot t \cdot g^{-1} + dg \cdot g^{-1}, \tag{2}$$

easy to see that $g * t \in Tw(A)$. The factor set $D(A) = Tw(A)/G$, called Berikashvili's functor D, has nice properties and useful applications. In particular if $t \sim t'$ then (M, d_t) and $(M, d_{t'})$ are *isomorphic*.

The notion of *homotopy G-algebra* (hGa in short) was introduced by Gerstenhaber and Voronov in [8] as an additional structure on a dg algebra $(A, d, \cdot)$ that induces a Gerstenhaber algebra structure on homology. The main example is Hochschild cochain complex of an algebra.

Another point of view is that hGa is a particular case of $B(\infty)$-algebra [10]: this is an additional structure on a dg algebra $(A, d, \cdot)$ that induces a dg bialgebra structure on the bar construction BA.

There is the third aspect of hGa [16]: this is a structure which measures the noncommutativity of A. The Steenrod's $\smile_1$ product which is the classical tool which measures the noncommutativity of a dg algebra $(A, d, \cdot)$ satisfies the condition

$$d(a \smile_1 b) = da \smile_1 b + a \smile_1 db + a \cdot b + b \cdot a. \qquad (3)$$

The existence of such $\smile_1$ guarantees the commutativity of $H(A)$, but a $\smile_1$ product satisfying just the condition (3) is too poor for some applications. In many constructions some deeper properties of $\smile_1$ are needed, for example the compatibility with the dot product of A (the Hirsch formula)

$$(a \cdot b) \smile_1 c + a \cdot (b \smile_1 c) + (a \smile_1 c) \cdot b = 0. \qquad (4)$$

A hGa $(A, d, \cdot, \{E_{1,k}\})$ is a dga $(A, d, \cdot)$ equipped additionally with a sequence of operations (some authors call them *braces*)

$$\{E_{1,k} : A \otimes A^{\otimes k} \to A,\ k = 1, 2, ...\}$$

satisfying some coherence conditions (see bellow). The starting operation $E_{1,1}$ is a kind of $\smile_1$ product: it satisfies the conditions (3) and (4). As for the symmetric expression

$$a \smile_1 (b \cdot c) + b \cdot (a \smile_1 c) + (a \smile_1 b) \cdot c,$$

it is just *homotopical to zero* and the appropriate chain homotopy is the operation $E_{1,2}$. So we can say that a hGa is a dga with a "good" $\smile_1$ product.

There is one more aspect of hGa: the operation $E_{1,1} = \smile_1$ is not associative but the commutator $[a, b] = a \smile_1 b - b \smile_1 a$ satisfies the Jacobi identity, so it forms on the desuspension $s^{-1}A$ a structure of dg Lie algebra.

Let us present three remarkable examples of homotopy G-algebras.

The first one is the cochain complex of 1-reduced simplicial set $C^*(X)$. The operations $E_{1,k}$ here are dual to cooperations defined by Baues in [2], and the starting operation $E_{1,1}$ is the classical Steenrod's $\smile_1$ product.

The second example is the Hochschild cochain complex $C^*(U,U)$ of an associative algebra U. The operations $E_{1,k}$ here were defined in [14] with the purpose to describe $A(\infty)$-algebras in terms of Hochschild cochains although the properties of those operations which where used as defining ones for the notion of homotopy G-algebra in [8] did not appear there. These operations where defined also in [10]. Again the starting operation $E_{1,1}$ is the classical Gerstenhaber's circle product which is sort of $\smile_1$-product.

The third example is the the cobar construction ΩC of a dg *bialgebra* C. The operations $E_{1,k}$ are constructed in [17]. And again the starting operation $E_{1,1}$ is classical: it is Adams's $\smile_1$-product defined for ΩC in [1] using the *multiplication* of C.

The main task of this paper is to introduce the notion of a twisting element and their transformation in a hGa. Shortly a twisting element in a hGa $(A, d, \cdot, \{E_{1,k}\})$ is an element $a \in A$ such that $da = a \smile_1 a$ and two twisting elements $a, \overline{a} \in A$ we call equivalent if there exists $g \in A$ such that

$$\overline{a} = a + dg + g \cdot g + g \smile_1 a + \overline{a} \smile_1 g + E_{1,2}(\overline{a}; g, g) + E_{1,3}(\overline{a}; g, g, g) + ... \; .$$

As we see in the definition of a twisting element participates just the operation $E_{1,1} = \smile_1$ but in the definition of equivalence participates the whole hGa structure. We remark that such a twisting element $a \in A$ is a Lie twisting element in the dg Lie algebra $(s^{-1}A, d, [\ ,\ ])$, i.e. satisfies $da = \frac{1}{2}[a, a]$. But it is unclear wether the equivalence can be formulated in terms the bracket $[\ ,\ ]$.

In this paper we present the following application of the notion of twisting element in a hGa: it allows to unify two classical concepts, namely the theory of deformation of algebras of M. Gerstenhaber, and J. Stasheff's $A(\infty)$-algebras.

Namely, a Gerstenhaber's deformation of an associative algebra U (see [7], and bellow)

$$a \star b = a \cdot b + B_1(a \otimes b)t + B_2(a \otimes b)t^2 + B_3(a \otimes b)t^3 + ... \in U[[t]],$$

can be considered as a twisting element $B = B_1 + B_2 + ... \in C^2(U, U)$ in the Hochschild cochain complex of U with coefficients in itself: the defining condition of deformation means exactly $dB = B \smile_1 B$. Furthermore, two deformations are equivalent if and only if the corresponding twisting elements are equivalent in the above sense.

On the other side, the same concept of twisting elements in hGa works in the following problem. Suppose $(H, \mu : H \otimes H \to H)$ is a graded algebra. Let us define it's *Stasheff's deformation* as an $A(\infty)$ algebra structure $(H, \{m_i\})$ with $m_1 = 0$ and $m_2 = \mu$, i.e. which extends the given algebra structure. Then each deformation can be considered as a twisting element $m = m_3 + m_4 + ...,\ m_i \in C^i(H, H)$ in the Hochschild cochain complex of H with coefficients in itself: the Stasheffs defining condition of $A(\infty)$-algebra means exactly $dm = m \smile_1 m$. Furthermore, to isomorphic (as $A(\infty)$-algebras) deformations correspond equivalent twisting elements in the above sense.

In both cases we present the obstruction theory for the existence of suitable deformations. The obstructions live in suitable Hochschild cohomologies: in $H^2(U, U)$ in Gerstenhaber's deformation case and in $H^i(H, H),\ i = 3, 4, ...$ in Stasheff's deformation case.

The structure of the paper is following. In the section 2 necessary definitions are given. In the section 3 the definition Homotopy G-algebra is presented. In the section 4 the notion of twisted element in a homotopy G-algebra is studied. In the last two sections 5 and 6 the above mentioned applications of this notion are given.

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2 Notation and Preliminaries

We work over $R = Z_2$. For a graded module M we denote by sM the suspension of M, i.e. $(sM)^i = M^{i-1}$. Respectively $(s^{-1}M)^i = M^{i+1}$.

2.1 Differential Graded Algebras and Coalgebras

A *differential graded algebra* (dg algebra, or dga) is a graded R-module $C = \{C^i,\ i \in Z\}$ with an associative *multiplication* $\mu : C^i \otimes C^j \to C^{i+j}$ and a *differential* $d : C^i \to C^{i+1}$ satisfying $dd = 0$ and the *derivation condition*

$d(x \cdot y) = dx \cdot y + x \cdot dy$, where $x \cdot y = \mu(x \otimes y)$. A dga C is *connected* if $C^{<0} = 0$ and $C^0 = R$. A connected dga C is *n-reduced* if $C^i = 0$ for $1 \leq i \leq n$.

A *differential graded coalgebra* (dg coalgebra, or dgc) is a graded R-module $C = \{C^i,\ i \in Z\}$ with a coassociative *comultiplication* $\Delta : C \to C \otimes C$ and a *differential* $d : C^i \to C^{i+1}$ satisfying $dd = 0$ and the *coderivation condition*$\Delta d = (d \otimes id + id \otimes d)\Delta$. A dgc C is *connected* if $C_{<0} = 0$ and $C_0 = R$. A connected dgc C is *n-reduced* if $C_i = 0$ for $1 \leq i \leq n$.

A *differential graded bialgebra* (dg bialgebra) (C, d, μ, Δ) is a dg coalgebra (C, d, Δ) with a morphism of dg coalgebras $\mu : C \otimes C \to C$ turning (C, d, μ) into a dg algebra.

2.2 Cobar and Bar Constructions

Let M be a graded R-module with $M^{i \leq 0} = 0$ and let $T(M)$ be the tensor algebra of M, i.e. $T(M) = \oplus_{i=0}^{\infty} M^{\otimes i}$. Tensor algebra $T(M)$ is a free graded algebra generated by M: for a graded algebra A and a homomorphism $\alpha : M \to A$ of degree zero there exists a unique morphism of graded algebras $f_\alpha : T(M) \to A$ (called *multiplicative extension* of α)such that $f_\alpha(a) = \alpha(a)$. The map f_α is given by $f_\alpha(a_1 \otimes ... \otimes a_n) = \alpha(a_1) \cdot ... \cdot \alpha(a_n)$.

Dually, let $T^c(M)$ be the tensor coalgebra of M, i.e. $T^c(M) = \oplus_{i=0}^{\infty} M^{\otimes i}$, and the comultiplication $\nabla : T^c(M) \to T^c(M) \otimes T^c(M)$ is given by

$$\nabla(a_1 \otimes ... \otimes a_n) = \sum_{k=0}^{n} (a_1 \otimes ... \otimes a_k) \otimes (a_{k+1} \otimes ... \otimes a_n).$$

The tensor coalgebra $(T^c(M), \nabla)$ is a cofree graded coalgebra: for a graded coalgebra C and a homomorphism $\beta : C \to M$ of degree zero there exists a unique morphism of graded coalgebras $g_\beta : C \to T^c(M)$ (called *comultiplicative coextension* of β) such that $p_1 g_\beta = \beta$, here $p_1 : T^c(M) \to M$ is the clear projection. The map g_β is given by

$$g_\beta = \sum_{n=0}^{\infty} (\beta \otimes ... \otimes \beta)\Delta^n,$$

where $\Delta^n : C \to C^{\otimes n}$ is n-th iteration of the diagonal $\Delta : C \to C \otimes C$, i.e. $\Delta^1 = id,\ \Delta^2 = \Delta,\ \Delta^n = (\Delta^{n-1} \otimes id)\Delta$.

Let (C, d_C, Δ) be a connected dg coalgebra and $\Delta(c) = c \otimes 1_R + 1_R \otimes c + \Delta'(c)$. The (reduced) *cobar construction* ΩC on C is a dg algebra whose underlying graded algebra is $T(sC^{>0})$. An element $(sc_1\otimes ...\otimes sc_n) \in (sC)^{\otimes n} \subset T(sC^{>0})$ is denoted by $[c_1, ..., c_n] \in \Omega C$. The differential d_Ω of ΩC for a generator $[c] \in \Omega C$ is defined by $d_\Omega[c] = [d_C(c)] + \sum[c', c'']$ where $\Delta'(c) = \sum c' \otimes c''$, and is extended as a derivation.

Let (A, d_A, μ) be a 1-reduced dg algebra. The (reduced) *bar construction* BA on A is a dg coalgebra whose underlying graded coalgebra is $T^c(s^{-1}A^{>0})$. Again an element $(s^{-1}a_1 \otimes ... \otimes s^{-1}a_n) \in (s^{-1}A)^{\otimes n} \subset T^c(s^{-1}A^{>0})$ we denote as $[a_1, ..., a_n] \in BA$. The differential d_B of BA is defined by

$$d_B[a_1, ..., a_n] = \sum_{i=1}^{n}[a_1, ..., d_A a_i, ..., a_n] + \sum_{i=1}^{n-1}[a_1, ..., a_i \cdot a_{i+1}, ..., a_n].$$

2.3 Twisting Cochains

Let (C, d, Δ) be a dgc and (A, d, μ) be a dga. A twisting cochain [5] is a homomorphism $\tau : C \to A$ of degree +1 satisfying the Brown's condition

$$d\tau + \tau d = \tau \smile \tau, \tag{5}$$

where $\tau \smile \tau' = \mu_A(\tau \otimes \tau')\Delta$. We denote by $Tw(C, A)$ the set of all twisting cochains $\tau : C \to A$.

There are universal twisting cochains $\tau_C : C \to \Omega C$ and $\tau_A : BA \to A$ being clear inclusion and projection respectively.

Here are essential consequences of the condition (5):
(i) The multiplicative extension $f_\tau : \Omega C \to A$ is a map of dg algebras, so there is a bijection $Tw(C, A) \leftrightarrow Hom_{dg-Alg}(\Omega C, A)$;
(ii) The comultiplicative coextension $g_\tau : C \to BA$ is a map of dg coalgebras, so there is a bijection $Tw(C, A) \leftrightarrow Hom_{dg-Coalg}(C, BA)$.

3 Homotopy G-algebras

A *homotopy G-algebra* (hGa in short) is a dg algebra with "good" $\smile_1$ product. The general notion was introduced in [8], see also [24].

Definition 1 *A homotopy G-algebra is defined as a dg algebra* $(A, d, \cdot)$ *with a given sequence of operations*

$$E_{1,k} : A \otimes (A^{\otimes k}) \to A, \quad k = 0, 1, 2, 3, ...$$

(the value of the operation $E_{1,k}$ on $a \otimes b_1 \otimes ... \otimes b_k \in A \otimes (A \otimes ... \otimes A)$ we write as $E_{1,k}(a; b_1, ..., b_k)$) which satisfies the conditions

$$E_{1,0} = id, \tag{6}$$

$$\begin{aligned} &dE_{1,k}(a; b_1, ..., b_k) + E_{1,k}(da; b_1, ..., b_k) + \textstyle\sum_i E_{1,k}(a; b_1, ..., db_i, ..., b_k) \\ &= b_1 \cdot E_{1,k-1}(a; b_2, ..., b_k) + E_{1,k-1}(a; b_1, ..., b_{k-1}) \cdot b_k + \\ &\quad \textstyle\sum_i E_{1,k-1}(a; b_1, ..., b_i \cdot b_{i+1}, ..., b_k), \end{aligned} \tag{7}$$

$$\begin{aligned} &E_{1,k}(a_1 \cdot a_2; b_1, .., b_k) \\ &= a_1 \cdot E_{1,k}(a_2; b_1, ..., b_k) + E_{1,k}(a_1; b_1, ..., b_k) \cdot a_2 + \\ &\quad \textstyle\sum_{p=1}^{k-1} E_{1,p}(a_1; b_1, ..., b_p) \cdot E_{1,m-p}(a_2; b_{p+1}, ..., b_k), \end{aligned} \tag{8}$$

$$\begin{aligned} &E_{1,n}(E_{1,m}(a; b_1, ..., b_m); c_1, ..., c_n) \\ &= \textstyle\sum_{0 \le i_1 \le j_1 \le ... \le i_m \le j_m \le n} \\ &\quad E_{1,n-(j_1+...+j_m)+(i_1+...+i_m)+m}(a; c_1, ..., c_{i_1}, E_{1,j_1-i_1}(b_1; c_{i_1+1}, ..., c_{j_1}), \\ &\quad c_{j_1+1}, ..., c_{i_2}, E_{1,j_2-i_2}(b_2; c_{i_2+1}, ..., c_{j_2}), c_{j_2+1}, ..., c_{i_m}, \\ &\quad E_{1,j_m-i_m}(b_m; c_{i_m+1}, ..., c_{j_m}), c_{j_m+1}, ..., c_n). \end{aligned} \tag{9}$$

Let us present these conditions in low dimensions.

The condition (7) for $k = 1$ looks as

$$dE_{1,1}(a; b) + E_{1,1}(da; b) + E_{1,1}(a; db) = a \cdot b + b \cdot a. \tag{10}$$

So the operation $E_{1,1}$ is a sort of $\smile_1$ product: it is the chain homotopy which which measures the noncommutativity of A, c.f. the condition (3). Below we denote $a \smile_1 b = E_{1,1}(a; b)$.

The condition (8) for $k = 1$ looks as

$$(a \cdot b) \smile_1 c + a \cdot (b \smile_1 c) + (a \smile_1 c) \cdot b = 0, \tag{11}$$

this means, that the operation $E_{1,1} = \smile_1$ satisfies the *left Hirsch formula* (4).

The condition (7) for $k = 2$ looks as

$$\begin{aligned} &dE_{1,2}(a; b, c) + E_{1,2}(da; b, c) + E_{1,2}(a; db, c) + E_{1,2}(a; b, dc) \\ &= a \smile_1 (b \cdot c) + (a \smile_1 b) \cdot c + b \cdot (a \smile_1 c), \end{aligned} \tag{12}$$

this means, that this $\smile_1$ satisfies the *right Hirsch formula* just up to homotopy and the appropriate homotopy is the operation $E_{1,2}$.

The condition (9) for $n = m = 2$ looks as

$$(a \smile_1 b) \smile_1 c + a \smile_1 (b \smile_1 c) = E_{1,2}(a; b, c) + E_{1,2}(a; c, b), \tag{13}$$

this means that the same operation $E_{1,2}$ measures also the deviation from the associativity of the operation $E_{1,1} = \smile_1$.

3.1 hGa as a $B(\infty)$-algebra

The notion of $B_\infty-$algebra was introduced in [10] as an additional structure on a dg module (A, d) which turns the tensor coalgebra $T^c(s^{-1}A)$ into a dg bialgebra. So it requires a differential

$$\widetilde{d} : T^c(s^{-1}A) \to T^c(s^{-1}A)$$

which is a coderivation (that is an $A(\infty)$-algebra structure on A, see bellow) and a an associative multiplication

$$\widetilde{\mu} : T^c(s^{-1}A) \otimes T^c(s^{-1}A) \to T^c(s^{-1}A)$$

which is a map of dg coalgebras.

Here we show that a hGa structure on A is a particular $B(\infty)$-algebra structure: it induces on $B(A) = (T^c(s^{-1}A), d_B)$ a multiplication but does not change the differential d_B (see [10], [16], [17], [18] for more details).

Let us extend our sequence $\{E_{1,k},\ k = 0, 1, 2, ...\}$ to the sequence $\{E_{p,q} : (A^{\otimes p}) \otimes (A^{\otimes q}) \to A,\ p, q = 0, 1, ...\}$ adding

$$E_{0,1} = id,\ E_{0,q>1} = 0,\ E_{1,0} = id,\ E_{p>1,0} = 0, \tag{14}$$

and $E_{p>1,q} = 0$.

This sequence defines a map $E : B(A) \otimes B(A) \to A$ by $E([a_1, ..., a_m] \otimes [b_1, ..., b_n]) = E_{p,q}(a_1, ..., a_m; b_1, ..., b_n)$. The conditions (7) and (8) mean exactly $dE + E(d_B \otimes id + id \otimes d_B) = E \smile E$, i.e. E is a twisting cochain. Thus according to the section 2.3 it's coextesionis a dg coalgebra map

$$\mu_E : B(A) \otimes B(A) \to B(A).$$

The condition (9) can be rewritten as $E(\mu_E \otimes id - id \otimes \mu_E) = 0$, so this condition means that the multiplication μ_E is associative. And the condition (14) means that $[\ ] \in B(A)$ is the unit for this multiplication.

Finally we obtained that $(B(A), d_B, \Delta, \mu_E)$ is a dg bialgebra thus a hGa is a $B(\infty)$-algebra.

Let us mention, that a twisting cochain E satisfying just the starting condition (14) was constructed in [19] using acyclic models for $A = C^*(X)$, the singular cochain complex of a topological space. The condition (14) determines this twisting cochain E uniquely up to equivalence of twisting cochains (2).

3.2 Homology of a hGa is a Gerstenhaber algebra

A structure of a hGa on A induces on the homology $H(A)$ a structure of Gerstenhaber algebra (G-algebra).

Gerstenhaber algebra (see [6], [8], [24]) is defined as a commutative graded algebra $(H, \cdot)$ together with a Lie bracket of degree -1

$$[\ ,\ ] : H^p \otimes H^q \to H^{p+q-1}$$

(i.e. a graded Lie algebra structure on the desuspension $s^{-1}H$) which is a biderivation: $[a, b \cdot c] = [a, b] \cdot c + b \cdot [a, c]$. Main example of Gerstenhaber algebra is Hochschild cohomology of an associative algebra.

The following argument shows the existence of this structure on the homology $H(A)$ of a hGa.

First, there appears on the desuspension $s^{-1}A$ a structure of dg Lie algebra: although the $\smile_1 = E_{1,1}$ is not associative, the condition (13) implies the pre-Jacobi identity

$$a \smile_1 (b \smile_1 c) + (a \smile_1 b) \smile_1 c = a \smile_1 (c \smile_1 b) + (a \smile_1 c) \smile_1 b,$$

this condition guarantees that the commutator $[a, b] = a \smile_1 b + b \smile_1 a$ satisfies the Jacobi identity, besides the condition (10) implies that $[\ ,\ ] : s^{-1}A \otimes s^{-1}A \to s^{-1}A$ is a chain map. Consequently there appears on $s^{-1}H(A)$ a structure of graded Lie algebra. The Hirsch formulae (11) and (12) imply that the induced Lie bracket is a biderivation.

3.3 Operadic Description

The operations $E_{1,k}$ forming hGa have nice description in terms of the *surjection operad*, see [20], [21], [3] for definition. Namely, to the dot product corresponds the element $(1,2) \in \chi_0(2)$; to $E_{1,1} = \smile_1$ product corresponds $(1,2,1) \in \chi_1(2)$, and generally to the operation $E_{1,k}$ corresponds the element

$$E_{1,k} = (1,2,1,3,...,1,k,1,k+1,1) \in \chi_k(k+1). \tag{15}$$

We remark here that the defining conditions of a hGa (6), (7), (8), (9) can be expressed in terms of operadic structure (differential, symmetric group action and composition product) and the elements (15) satisfy these conditions *already in the operad* χ.

Note that the elements (15) together with the element $(1,2)$ generate the suboperad $F_2\chi$ which is equivalent to the little square operad ([20], [21], [3]). This in particular implies that a cochain complex (A,d) is a hGa if and only if it is an algebra over the operad $F_2\chi$.

This fact and the hGa structure on the Hochschild cochain complex $C^*(U,U)$ of an algebra U [14] were used by some authors to prove the *Deligne conjecture* about the action of the little square operad on on the Hochschild cochain complex $C^*(U,U)$.

3.4 Hochschild Cochain Complex as a hGa

Let A be an algebra and M be a two sided module on A. The Hochschild cochain complex $C^*(A;M)$ is defined as $C^n(A;M) = Hom(A^{\otimes^n}, M)$ with differential $\delta : C^{n-1}(A;M) \to C^n(A;M)$ given by

$$\begin{aligned}\delta f(a_1 \otimes ... \otimes a_n) = \;& a_1 \cdot f(a_2 \otimes ... \otimes a_n)\\ & + \textstyle\sum_{k=1}^{n-1} f(a_1 \otimes ... \otimes a_{k-1} \otimes a_k \cdot a_{k+1} \otimes .. \otimes a_n)\\ & + f(a_1 \otimes ... \otimes a_{n-1}) \cdot a_n.\end{aligned}$$

We focus on the case $M = A$.

In this case the Hochschild complex becomes a dg algebra with respect to the $\smile$ product defined in [6] by

$$f \smile g(a_1 \otimes ... \otimes a_{n+m}) = f(a_1 \otimes ... \otimes a_n) \cdot g(a_{n+1} \otimes ... \otimes a_{n+m}).$$

In [14] (see also [10], [8]) there are defined the operations

$$E_{1,i} : C^n(A;A) \otimes C^{n_1}(A;A) \otimes ... \otimes C^{n_i}(A;A) \to C^{n+n_1+...+n_i-i}(A;A)$$

given by $E_{1,i}(f;g_1,...,g_i)=0$ for $i>n$ and

$$\begin{aligned}&E_{1,i}(f;g_1,...,g_i)(a_1\otimes...\otimes a_{n+n_1+...+n_i-i})\\&=\sum_{k_1,...,k_i} f(a_1\otimes...\otimes a_{k_1}\otimes g_1(a_{k_1+1}\otimes...\otimes a_{k_1+n_1})\otimes a_{k_1+n_1+1}\otimes...\\&\quad\otimes a_{k_2}\otimes g_2(a_{k_2+1}\otimes...\otimes a_{k_2+n_2})\otimes a_{k_2+n_2+1}\otimes...\\&\quad\otimes a_{k_i}\otimes g_i(a_{k_i+1}\otimes...\otimes a_{k_i+n_i})\otimes a_{k_i+n_i+1}\otimes...\otimes a_{n+n_1+...+n_i-i}).\end{aligned} \tag{16}$$

The straightforward verification shows that the collection $\{E_{1,k}\}$ satisfies the conditions (6), (7), (8) and (9), thus it forms on the Hochschild complex $C^*(A;A)$ a structure of homotopy G-algebra.

We note that the operation $E_{1,1}$ coincides with the circle product defined by Gerstenhaber in [6], note also that the operation $E_{1,2}$ satisfying (12) and (13) also is defined there.

4 Twisting Elements

In this section we present an analog of the notion of twisting element (see the introduction) in a homotopy G-algebra replacing in the defining equation $da=a\cdot a$ the dot product by the $\smile_1$ product. The appropriate notion of equivalence also will be introduced.

Let $(C^{*,*},d,\cdot,\{E_{1,k}\})$ be a *bigraded* homotopy G-algebra. That is $(C^{*,*},\cdot)$ is a bigraded algebra $C^{m,n}\cdot C^{p,q}\subset C^{m+p,n+q}$, and we require the existence of a differential (derivation) $d(C^{m,n})\subset C^{m+1,n}$ and of a sequence of operations

$$E_{1,k}:C^{m,n}\otimes C^{p_1,q_1}\otimes...\otimes C^{p_k,q_k}\longrightarrow C^{m+p_1+...+p_k-k,n+q_1+...+q_k}$$

so that the *total complex* (the total degree of $C^{p,q}$ is $p+q$) is a hGa.

Bellow we introduce two versions of the notion of *twisting elements* in a bigraded homotopy G-algebra. Although it is possible to reduce them to each other by changing gradings, we prefer to consider them separately in order to emphasize different areas of their applications. The first one controls Stasheff's A_∞-deformation of graded algebras and the second controls Gerstenhaber's deformation of associative algebras, see the next two sections.

4.1 Twisting Elements in a Bigraded Homotopy G-algebra (version 1)

A *twisting element* in $C^{*,*}$ we define as

$$m = m^3 + m^4 + ... + m^p + ... \ , \ m^p \in C^{p,2-p}$$

satisfying the condition $dm = E_{1,1}(m;m)$ or changing the notation $dm = m \smile_1 m$. This condition can be rewritten in terms of components as

$$dm^p = \sum_{i=3}^{p-1} m^i \smile_1 m^{p-i+2}. \tag{17}$$

Particularly $dm^3 = 0, \ dm^4 = m^3 \smile_1 m^3, \ dm^5 = m^3 \smile_1 m^4 + m^4 \smile_1 m^3, ...$. The set of all twisting elements we denote by $Tw(C^{*,*})$.

Consider the set $G = \{g = g^2 + g^3 + ... + g^p + ...; \ g^p \in C^{p,1-p}\}$, and let us introduce on G the following operation

$$\overline{g} * g = \overline{g} + g + \sum_{k=1}^{\infty} E_{1,k}(\overline{g}; g, ..., g), \tag{18}$$

particularly

$$\begin{aligned}
&(\overline{g} * g)^2 = \overline{g} + g^2;\\
&(\overline{g} * g)^3 = \overline{g}^3 + g^3 + \overline{g}^2 \smile_1 g^3;\\
&(\overline{g} * g)^4 = \overline{g}^4 + g^3 + \overline{g}^2 \smile_1 g^3 + \overline{g}^3 \smile_1 g^2 + E_{1,2}(\overline{g}^2; g^2, g^2).
\end{aligned}$$

It is possible to check, using the defining conditions of a hGa (6), (7), (8), (9) that this operation is associative, has the unit $e = 0 + 0 + ...$ and the opposite g^{-1} can be solved inductively from the equation $g * g^{-1} = e$. Thus G is a group.

The group G acts on the set $Tw(C^{*,*})$ by the rule $g * m = \overline{m}$ where

$$\overline{m} = m + dg + g \cdot g + E_{1,1}(g; m) + \sum_{k=1}^{\infty} E_{1,k}(\overline{m}; g, ..., g), \tag{19}$$

particularly

$$\begin{aligned}
&\overline{m}^3 = m^3 + dg^2;\\
&\overline{m}^4 = m^4 + dg^3 + g^2 \cdot g^2 + g^2 \smile_1 m^3 + \overline{m^3} \smile_1 g^2;\\
&\overline{m}^5 = m^5 + dg^4 + g^2 \cdot g^3 + g^3 \cdot g^2 + g^2 \smile_1 m^4 + g^3 \smile_1 m^3 +\\
&\qquad \overline{m}^3 \smile_1 g^3 + \overline{m}^4 \smile_1 g^2 + E_{1,2}(\overline{m}^3; g^2, g^2).
\end{aligned}$$

Note that although in the right hand side of the formula (19) participates $\overline{m}$ but it has less dimension then the left hand side $\overline{m}$, thus this action is well defined: the components of $\overline{m}$ can be solved from this equation inductively. It is possible to check that the resulting $\overline{m}$ is a twisting element. By $D(C^{*,*})$ we denote the set of orbits $Tw(C^{*,*})/G$.

This group action allows us to *perturb* twisting elements in the following sense. Let $g^n \in C^{n,1-n}$ be an arbitrary element, then for $g = 0 + ... + 0 + g^n + 0 + ...$ the twisting element $\overline{m} = g * m$ looks as

$$\overline{m} = m^3 + ... + m^n + (m^{n+1} + dg^n) + \overline{m}^{n+2} + \overline{m}^{n+3} + ... \quad , \qquad (20)$$

so the components $m^3, ..., m^n$ remain unchanged and $\overline{m}^{n+1} = m^{n+1} + dg^n$.

The perturbations allow to consider the following two problems.

Quantization. Let us first mention that for a twisting element $m = \sum m^k$ the first component $m^3 \in C^{3,-1}$ is a cycle and any perturbation does not change it's homology class $[m^3] \in H^{3,-1}(C^{*,*})$. Thus we have the correct map $\phi : D(C^{*,*}) \to H^{3,-1}(C^{*,*})$.

A *quantization* of a homology class $\alpha \in H^{3,-1}(C^{*,*})$ we define as a twisting element $m = m^3 + m^4 + ...$ such that $[m^3] = \alpha$. Thus α is quantizable if it belongs to the image of ϕ.

The obstructions for quatizability lay in homologies $H^{n,3-n}(C^{*,*}),\ n \geq 5$. Indeed, let $m^3 \in C^{3,-1}$ be a cycle from α. The first step to quantize α is to construct m^4 such that $dm^4 = m^3 \smile_1 m^3$. The necessary and sufficient condition for this is $[m^3 \smile_1 m^3] = 0 \in H^{5,-2}(C^{*,*})$, so this homology class is the first obstruction $O(m^3)$. Suppose it vanishes; so there exists m^4. Then it is easy to see that $m^3 \smile_1 m^4 + m^4 \smile_1 m^3$ is a cycle and its class $O(m^3, m^4) \in H^{6,-3}(C^{*,*})$ is the second obstruction. If $O(m^3, m^4) = 0$ then there exists m^5 such that $dm^5 = m^3 \smile_1 m^4 + m^4 \smile_1 m^3$. If not then we take another m^4 and try new second obstruction (we remark that changing of m^3 makes no sense). Generally the obstruction is

$$O(m^3, m^4, ..., m^{n-2}) = [\sum_{k=3}^{n-2} m^k \smile_1 m^{n-k+1}] \in H^{n,3-n}(C^{*,*}).$$

Rigidity. A twisting element $m = m^3 + m^4 + ... + m^p + ...$ is called *trivial* if it is equivalent to 0. A bigraded hGa $C^{*,*}$ is *rigid* if each twisting element is trivial, i.e. if $D(C^{*,*}) = \{0\}$. The obstructions to triviality of a twisting element lay in homologies $H^{n,2-n}(C^{*,*}),\ n \geq 3$. Indeed, for a twisting element

$m = m^3 + m^4 + ... + m^p + ...$ the first component m^3 is a cycle and by (19) each perturbation of m leaves the class $[m^3] \in H^{3,-1}(C^{*,*})$ unchanged and this class is the first obstruction for triviality. If this class is zero, then we choose $g^2 \in C^{2,-1}$ such that $dg^2 = m^3$. Perturbing m by $g = g^2 + 0 + 0 + ...$ we kill the first component m^3, i.e. we get the twisting element $\overline{m} \sim m$, which looks as $\overline{m} = 0 + \overline{m}^4 + \overline{m}^5 + ...$. Now, since of (17), the component $\overline{m}^4$ becomes a cycle and it's homology class is the second obstruction. If this class is zero then we can kill $\overline{m}^4$. If it is not then we take another g^2 and try new second obstruction. Generally after killing first components, for $m = 0 + 0 + ... + 0 + m^n + m^{n+1} + ...$ the obstruction is the homology class $[m^n] \in H^{n,2-n}(C^{*,*})$.

This in particular implies that if for a bigraded homotopy G-algebra $C^{*,*}$ all homology modules $H^{n,2-n}(C^{*,*})$ are trivial for $n \geq 3$, then $D(C^{*,*}) = 0$, thus $C^{*,*}$ is rigid.

4.2 Twisting Elements in a Bigraded Homotopy G-algebra (version 2)

This version can be obtained from the previous one by changing grading: take new bigraded module $\overline{C}^{p,q} = C^{p+q,-q}$. The same operations turn $\overline{C}^{*,*}$ into a bigraded hGa.

A twisting element $m \in C^{*,2-*}$ in this case looks as $b = b_1 + b_2 + ... + b_n + ...$, $b_n \in \overline{C}^{2,n}$ where $b_k = m^{k-2}$ and satisfies the condition $db = b \smile_1 b$, or equivalently $db_n = \sum_{i=2}^{n-1} b_i \smile_1 b_{n-i}$.

Here we have the group $G' = \{g = g_1 + g_2 + ... + g_p + ... ; \quad g_p \in B^{1,p}\}$ with operation $g' * g = g' + g + \sum_{k=1}^{\infty} E_{1,k}(g'; g, .., g)$. This group acts on the set $Tw'(\overline{C}^{*,*})$ by the rule $g * b = b'$ where

$$b' = b + dg + g \cdot g + E_{1,1}(g; b) + \sum_{k=1}^{\infty} E_{1,k}(b'; g, ..., g). \tag{21}$$

By $D'(\overline{C}^{*,*})$ we denote the set of orbits $Tw'(\overline{C}^{*,*})/G'$.

We consider the following two problems.

Quantization. The first component $b_1 \in \overline{C}^{2,1}$ of a twisting element $b = \sum b_i$ is a cycle and any perturbation does not change it's homology class $\alpha = [b_1] \in H^{2,1}(\overline{C}^{*,*})$. Thus we have a correct map $\psi : D'(\overline{C}^{*,*}) \to H^{2,1}(\overline{C}^{*,*})$.

A *quantization* of a homology class $\alpha \in H^{2,1}(\overline{C}^{*,*})$ we define as a twisting element $b = b_1 + b_2 + ...$ such that $[b_1] = \alpha$. Thus α is quantizable if $\alpha \in Im\ \psi$.

The argument similar to above shows that the obstructions to quatizability lay in homologies $H^{3,n}(\overline{C}^{*,*}),\ n \geq 2$.

Rigidity. A twisting element $b = b_1 + b_2 + ...$ is called *trivial* if it is equivalent to 0. A bigraded hGa $\overline{C}^{*,*}$ is *rigid* if each twisting element is trivial, i.e. if $D'(\overline{C}^{*,*}) = \{0\}$. The obstructions to triviality of a twisting element lay in homologies $H^{2,n}(\overline{C}^{*,*}),\ n \geq 1$.

This in particular implies that if for a bigraded hGa $\overline{C}^{*,*}$ we have $H^{2,n}(\overline{C}^{*,*}) = 0,\ n \geq 1$ then $D'(\overline{C}^{*,*}) = 0$ thus $\overline{C}^{*,*}$ is rigid.

4.3 Twisting Elements in a dg Lie Algebra corresponding to a hGa

As it is described above for a homotopy G-algebra $(C, \cdot, d, \{E_{1k}\})$ the desuspension $s^{-1}A$ is a dg Lie algebra with the bracket $[a, b] = a \smile_1 b - b \smile_1 a$. Note that if $C^{*,*}$ is a bigraded homotopy G-algebra, then $L^{*,*} = s^{-1}C^{*,*}C^{*-1,*}$ is a bigraded dg Lie algebra.

Suppose $m \in C^{*,2-*}$ is a twisting element in $C^{*,*}$. The defining equation $dm = m \smile_1 m$ can be rewritten in terms of bracket as $dm = \frac{1}{2}[m, m]$, so the same m can be regarded as a Lie twisting element in the bigraded dg Lie algebra $L^{*,*}$.

So the notion of the twisting element in a hGa, which involves just the operation $E_{1,1} = \smile_1$ in fact can be expressed in terms of Lie bracket [,].

But it is unclear whether the group action formulas (19) and (21), which involve all the operations $\{E_{1,k},\ k = 1, 2, ...\}$ can be expressed just in terms of bracket.

5 Deformation of Algebras

This is just illustrative application. Using the homotopy G-algebra structure, the notions of twisting element and their transformation one can obtain the well known results of Gerstenhaber from [7].

Let $(A, \cdot)$ be an algebra over a field k, $k[[t]]$ be the algebra of formal power series in variable t and $A[[t]] = A \otimes k[[t]]$ be the algebra of formal power series with coefficients from A.

Gerstenhaber's deformation of an algebra $(A, \cdot)$ is defined as a sequence of homomorphisms

$$B_i : A \otimes A \to A, \quad i = 0, 1, 2, ...; \quad B_0(a \otimes b) = a \cdot b$$

satisfying the *associativity* condition

$$\sum_{i+j=n} B_i(a \otimes B_j(b \otimes c)) = \sum_{i+j=n} B_i(B_j(a \otimes b) \otimes c). \tag{22}$$

Such a sequence determines the *star product*

$$a \star b = a \cdot b + B_1(a \otimes b)t + B_2(a \otimes b)t^2 + B_3(a \otimes b)t^3 + ... \in A[[t]],$$

which can be naturally extended to a $k[[t]]-$bilinear product $\star : A[[t]] \otimes A[[t]] \to A[[t]]$ and the condition (22) guarantees it's associativity.

Two deformations $\{B_i\}$ and $\{B_i'\}$ are called *equivalent* if there exists a sequence of homomorphisms $\{G_i : A \to A; \quad i = 0, 1, 2, ...; \quad G_0 = id\}$ such that

$$\sum_{r+s=n} G_r(B_s(a \otimes b)) = \sum_{i+j+k=n} B_i'(G_j(a) \otimes G_k(b)). \tag{23}$$

The sequence $\{G_i\}$ determines homomorphism $G = \sum G_i t^i : A \to A[[t]]$. On it's turn this G naturally extends to a $k[[t]]-$linear bijection $(A[[t]], \star) \to (A[[t]], \star')$ and the condition (23) guarantees that this extension is multiplicative.

A deformation $\{B_i\}$ is called *trivial*, if $\{B_i\}$ is equivalent to $\{B_0, 0, 0, ...\}$. An algebra A is called *rigid*, if each it's deformation is trivial.

Now we present the interpretation of deformations and their equivalence in terms of twisting elements of version 2 type and their equivalence in hGa of Hochschild cochains.

As it is mentioned in 3.4 the Hochschild complex $C^*(A, A)$ for an algebra A is a homotopy G-algebra. Then the tensor product $C^{*,*} = C^*(A, A) \otimes k[[t]]$ is a *bigraded* Hirsch algebra with the following structure

$$C^{p,q} = C^p(A, A) \cdot t^q, \ \delta(f \cdot t^q) = \delta f \cdot t^q, \quad f \cdot t^p \smile g \cdot t^q = (f \smile g) \cdot t^{p+q},$$
$$E_{1,k}(f \cdot t^p; g_1 \cdot t^{q_1}, ..., g_k \cdot t^{q_k}) = E_{1,k}(f; g_1, ..., g_k) \cdot t^{p+q_1+...+q_k},$$

here we use the notation $f \otimes t^p = f \cdot t^p$.

Then each deformation $\{B_i : A^{\otimes^2} \to A,\ i = 1, 2, 3, ...\}$ can be interpreted as a version 2 type twisting element $b = b_1+b_2+...+b_k+...,\ b_k = B_k \cdot t^k \in C^{2,k}$: the associativity condition (22) can be rewritten as

$$\delta B_n \cdot t^n = \sum_{i+j=n} B_i \cdot t^i \smile_1 B_j \cdot t^j .$$

Suppose now two deformations $\{B_i\}$ and $\{B'_i\}$ are equivalent, i.e. there exists $\{G_i\}$ such that the condition (23) satisfied. In terms of Hochschild cochsins this condition looks as

$$b' = b + \delta g + g \smile g + g \smile_1 b + E_{1,1}(b'; g) + E_{1,2}(b'; g, g),$$

where $g = g_1 + ... + g_k + ...,\ g_k = G_k \cdot t^k \in C^{1,k}$. This equality slightly differs from (21), but since $g \in C^1(A, A)$ and $b' \in C^2(A, A))$, we have $E_{1,k}(b'; g, ..., g) = 0$ for $k \geq 3$ (see 3.4), thus they in fact coincide.

So we obtain that deformations are equivalent if and only if the corresponding Hochschild twisting elements b and b' are equivalent. Consequently the set of equivalence classes of deformations is bijective to $D'(C^{*,*})$.

It is clear that $H^{p,q}(C^{*,*}) = HH^p(A, A) \cdot t^q$. Then from the section 4.2 follow the classical results of Gerstenhaber: obstructions for quantization of a homomorphism $b_1 : A \otimes A \to A$ lay in $HH^3(A, A)$, and if $HH^3(A, A) = 0$ then each b_1 is quantizable (or *integrable* as it is called in [7]). Furthermore, the obstructions for triviality of a deformation lay in $HH^2(A, A)$, and if $HH^2(A, A) = 0$ then A is rigid.

Remark 1 *As we see in the definition of equivalence of deformations participate just the operations $E_{1,1}$ and $E_{1,2}$, the higher operations $E_{1,k},\ k > 2$ disappear since of (16). So observing just deformation problem it is impossible to establish general formula (21) for transformation of twisting elements.*

6 $A(\infty)$-deformation of Graded Algebras

In this section we give the similar description of $A(\infty)$-deformation of graded algebras in terms of twisting elements in the hGa of Hochschild cochains. So this two types of deformation will be unified by the notion of twisting element in hGa. Partially these results are given in [14], [15].

6.1 $A(\infty)$-algebras

The notion of $A(\infty)$-algebra was introduced by J.D. Stasheff in [23]. This notion generalizes the notion of dg algebra.

An $A(\infty)$-algebra is a graded module M with a given sequence of operations

$$\{m_i : M^{\otimes i} \to M, \quad i = 1, 2, ..., \quad \deg m_i = 2 - i\}$$

which satisfies the following conditions

$$\sum_{i+j=n+1} \sum_{k=0}^{n-j} m_i(a_1 \otimes ... \otimes a_k \otimes m_j(a_{k+1} \otimes ... \otimes a_{k+j}) \otimes ... \otimes a_n) = 0. \quad (24)$$

Particularly, for the operation $m_1 : M \to M$ we have $\deg m_1 = 1$ and $m_1 m_1 = 0$, this m_1 can be regarded as a differential on M. The operation $m_2 : M \otimes M \to M$ is of degree 0 and satisfies

$$m_1 m_2(a_1 \otimes a_2) + m_2(m_1 a_1 \otimes a_2) + m_2(a_1 \otimes m_1 a_2) = 0,$$

i.e. m_2 can be regarded as a multiplication on M and m_1 is a derivation. Thus (M, m_1, m_2) is a sort of (maybe nonassociative) dg algebra. For the operation m_3 we have $\deg m_3 = -1$ and

$$\begin{aligned} &m_1 m_3(a_1 \otimes a_2 \otimes a_3) + m_3(m_1 a_1 \otimes a_2 \otimes a_3) + m_3(a_1 \otimes m_1 a_2 \otimes a_3) \\ &+ m_3(a_1 \otimes a_2 \otimes m_1 a_3) + m_2(m_2(a_1 \otimes a_2) \otimes a_3) + m_2(a_1 \otimes m_2(a_2 \otimes a_3)) = 0, \end{aligned}$$

thus the multiplication m_2 is *homotopy associative* and the appropriate chain homotopy is m_3.

The sequence of operations $\{m_i\}$ determines on the tensor coalgebra

$$T^c(s^{-1}M) = R + s^{-1}M + s^{-1}M \otimes s^{-1}M + s^{-1}M \otimes s^{-1}M \otimes s^{-1}M + ...$$

a coderivation

$$d_m(a_1 \otimes ... \otimes a_n) = \sum_{k,j} a_1 \otimes ... \otimes a_k \otimes m_j(a_{k+1} \otimes ... \otimes a_{k+j}) \otimes ... \otimes a_n,$$

and the condition (24) is equivalent to $d_m d_m = 0$. The obtained dg coalgebra $(T^c(s^{-1}M), d_m)$ is called *bar construction* and is denoted as $B(M, \{m_i\})$.

A morphism of $A(\infty)$-algebras $(M,\{m_i\}) \to (M',\{m'_i\})$ is defined as a sequence of homomorphisms

$$\{f_i : M^{\otimes^i} \to M', \quad i=1,2,..., \quad \deg f_i = 1-i\},$$

which satisfy the following condition

$$\begin{array}{l}\sum_{i+j=n+1}\sum_{k=0}^{n-j} f_i(a_1\otimes ...\otimes a_k \otimes m_j(a_{k+1}\otimes ...\otimes a_{k+j})\otimes ...\otimes a_n)\\ =\sum_{k_1+...+k_t=n} m'_t(f_{k_1}(a_1\otimes ...\otimes a_{k_1})\otimes f_{k_2}(a_{k_1+1}\otimes ...\otimes a_{k_1+k_2})\\ \quad \otimes ... \otimes f_{k_t}(a_{k_1+...+k_{t-1}+1}\otimes ...\otimes a_n)).\end{array} \tag{25}$$

In particular for $n=1$ this condition gives $f_1m_1(a) = m'_1f_1(a)$, i.e. $f_1 : (M,m_1) \to (M',m'_1)$ is a chain map; for $n=2$ it gives

$$\begin{array}{l} f_1m_2(a_1\otimes a_2) + m'_2(f_1(a_1)\otimes f_1(a_2))\\ = m'_1f_2(a_1\otimes a_2) + f_2(m_1a_1\otimes a_2) + f_2(a_1\otimes m_1a_2),\end{array}$$

thus $f_1 : (M,m_1,m_2) \to (M',m'_1,m'_2)$ is multiplicative up to homotopy f_2.

A collection $\{f_i\}$ defines a homomorphism $f : B(M,\{m_1\}) \to M'$. It's *comultiplicative coextension*, see 2.2, is a graded coalgebra map of the bar constructions

$$B(f) : B(M,\{m_i\}) \to B(M',\{m'_i\}),$$

and the condition (25) guarantees that $B(f)$ is a cahin map, i.e. $B(f)$ is a morphism of dg coalgebras. So B is a functor from the category of $A(\infty)$-algebras to the category of dg coalgebras.

A weak equivalence of $A(\infty)$-algebras is defined as a morphism $f=\{f_i\}$ for which $B(f)$ is a weak equivalence of dg coalgebras. It is possible to show (see for example [15]) that
(i) f is a weak equivalence of $A(\infty)$-algebras if and only if f_1 is a weak equivalence of dg modules;
(ii) f is an isomorphism of $A(\infty)$-algebras if and only if f_1 is an isomorphism of dg modules.

An $A(\infty)$-algebra $(M,\{m_i\})$ we call *minimal* if $m_1=0$. In this case (M,m_2) is *strictly* associative graded algebra.

The following proposition is the imediate consequence of (i) and (ii):

Proposition 1 *Each weak equivalence of minimal $A(\infty)$-algebras is an isomorphism.*

6.2 $A(\infty)$ Deformation of Graded Algebras as Twisting Element

Let $(H, \mu : H \otimes H \to H)$ be a graded algebra. It's Stasheff's (or $A(\infty)$) deformation we define as a minimal $A(\infty)$-algebra $(H, \{m_i\})$ with $m_2 = \mu$. Two deformations $(H, \{m_i\})$ and $(H, \{m'_i\})$ we call equivalent if there exists an isomorphism of $A(\infty)$-algebras $\{f_i\} : (H, \{m_i\}) \to (H, \{m'_i\})$ with $f_1 = id$.

A deformation $(H, \{m_i\})$ we call trivial if it is equivalent to $(H, \{m_1 = 0, m_2 = \mu, m_{\geq 3} = 0\})$. An algebra (H, μ) we call *rigid* (or *intrinsically formal*, this term is borrowed from the rational homotopy theory), if each it's deformation is trivial.

Now we present the interpretation of deformations and their equivalence in terms of twisting elements and their equivalence in hGa of Hochschild cochains.

The Hochschild cochain complex of a graded algebra H with coefficients in itself is bigraded: $C^{m,n}(H,H) = Hom^n(H^{\otimes m}, H)$, here Hom^n denotes degree n homomorphisms. The coboundary operator δ maps $C^{m,n}(H,H)$ to $C^{m+1,n}(H,H)$. Besides, for $f \in C^{m,n}(H,H)$ and $g_i \in C^{p_i,q_i}(H,H)$ one has $f \smile g \in C^{m+p,n+q}(H,H)$, $f \smile_1 g \in C^{m+p-1,n+q}(H,H)$ and

$$E_{1,k}(f; g_1, ..., g_k) \in C^{m+p_1+...+p_k-k, n+q_1+...+q_k}(H,H),$$

thus the Hochschild complex $C^{*,*}(H,H)$ is a *bigraded* homotopy G-algebra in this case. Let us denote the n-the homology module of the complex $(C^{*,k}(H,H), \delta)$ by $HH^{n,k}(H,H)$.

Suppose now that $(H, \{m_i\})$ is an $A(\infty)$ deformation of H. Each operation $m_i : H^{\otimes^i} \to H$ can be regarded as a Hochschild cochain from $C^{i,2-i}(H,H)$. The condition (24) can be rewritten as

$$\delta m_k = \sum_{i=3}^{k-1} m_i \smile_1 m_{k-i+2},$$

thus $m = m_3 + m_4 + ...$ is a *twisting element* (version 1) in $C^{*,*}(H,H)$.

Now let $(H, \{m_i\})$ and $(H, \{m'_i\})$ be two $A(\infty)$ deformations of H. Then it follows from (19) that the corresponding twisting elements m and m' are equivalent if and only if these two $A(\infty)$ deformations are equivalent: if $m' = p * m$, then $\{p_i\} : (H, \{m_i\}) \to (H, \{m'_i\})$ with $p_1 = id$ is an *isomorphism* of $A(\infty)$-algebras. So we obtain the

Theorem 1 *The set of isomorphism classes of all $A(\infty)$ deformations of a graded algebra (H,μ) is bijective to the set of equivalence classes of twisting elements $D(C^{*,*}(H,\mu))$.*

Moreover, from 4.1 we get the following

Theorem 2 *If for a graded algebra (H,μ) it's Hochschild cohomology modules $HH^{n,2-n}(H,H)$ are trivial for $n \geq 3$, then (H,μ) is intrinsically formal.*

6.2.1 $A(\infty)$-algebra Structure in Homology of a dg algebra

Let (A,d,μ) be a dg algebra and $(H(A),\mu^*)$ be it's homology algebra. Although the product in $H(A)$ is associative, there appears a structure of a (generally nondegenerate) minimal $A(\infty)$-algebra, which is an $A(\infty)$ deformation of $(H(A),\mu^*)$. Namely, in [12], [13] the following result was proved (see also [22], [11]):

Theorem 3 *If for a dg algebra all homology R-modules $H_i(A)$ are free, then there exist: a structure of minimal $A(\infty)$-algebra $(H(A),\{m_i\})$ on $H(A)$ and a weak equivalence of $A(\infty)$-algebras*

$$\{f_i\} : (H(A),\{m_i\}) \to (A,\{d,\mu,0,0,...\})$$

such, that $m_1 = 0$, $m_2 = \mu^*$, $f_1^* = id_{H(A)}$, such a structure is unique up to isomorphism in the category of $A(\infty)$-algebras.

Particularly an $A(\infty)$-algebra structure appears in cohomology $H^*(X)$ of a topological space X or in homology $H_*(G)$ of a topological group or H-space G. (Co)homology algebra equipped with this additional structure carries more information then just the (co)homology algebra. Some applications of this structure are given in [13] , [15]. For example the cohomology $A(\infty)$-algebra $(H^*(X),\{m_i\})$ determines cohomology of the loop space $H^*(\Omega X)$ when just the algebra $(H^*(X),m_2)$ does not. Similarly, the homology $A(\infty)$-algebra $(H_*(G),\{m_i\})$ determines homology of the classifying space $H_*(B_G)$ when just the Pontriagin algebra $(H_*(G),m_2)$ does not. Furthermore, the rational cohomology $A(\infty)$-algebra $(H^*(X,Q),\{m_i\})$ (which actually is $C(\infty)$ in this case) determines the rational homotopy type of 1-connected X when just the cohomology algebra $(H^*(X,Q),m_2)$ does not.

Therefore it is of particular interest the cases, when this additional structure is vanishes, that is when $A(\infty)$-algebra $(H(A),\{m_i\})$ is degenerate (in this case a dg algebra A is called *formal*). The above theorem 2 gives the sufficient condition of formality of A in terms of Hochschild cohomology of $H(A)$.

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