On the Differentials of the Spectral Sequence of a Fibre Bundle

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The theorem of W. Shi, [1], which in its turn generalizes the theorem of Faddell-Hurewicz [2], states the following: if the structure group G of a fibre bundle $F \to E \to B$ is $(n-1)$-connected, then in the spectral sequence of this bundle the differentials $d^r = 0$ for $r < n$ and for $n \leq r \leq 2n-2$ are determined by certain characteristic classes of the associated principal bundle. The above mentioned Faddell-Hurewicz theorem calculates only the first nontrivial differential d^n.

For simplicity, to avoid difficulties with signees, all homologies bellow we consider over Z_2 although it is enough to assume them free.

In this paper we generalize the Shis result for higher dimensions: we construct some cochains from $C^*(B, H_*(G))$ and certain *polylinear operations* which define all differentials d^r. In dimensions $n \leq r \leq 2n-2$ these cochains are cocycles form the classes from Shis theorem and the above mentioned polylinear operations is just $\smile$ product.

In papers [3], [4] N.A. Berikashvili has constructed cochain $h = h^2 + h^3 + ...$, $h^r \in C^r(B, Hom(H_*(F), H_*(F))$ (representative of the *predifferential* of the fibration) which determines all differentials of the spectral sequence. From this point of view the Shis theorem states that for $(n-1)$-connected structure group one has $h^r = 0$ for $r < n$ and for $n \leq r \leq 2n-2$ the components h^r can be expressed in terms of some cocycles from $C^r(B, H_{r-1}(G))$.

Let $\xi = (X, p, B, G)$ be a principal G-bundle, F be a G space and $\eta = (E, p, B, F)$ be the associated bundle with the fiber F.

Let $C^* = C^*(H_*(G), H_*(G))$ be the Hochschil cochain complex of the ring $H_*(G)$ with coefficients in itself: cochains in C^i are homomorphisms $f^i : H_*(G) \otimes ...(i-times)... \otimes H_*(G) \to H_*(G)$ and the coboundary operator is given by $\delta f^i(a_1, ..., a_{i+1}) = a_1 \cdot f^i(a_2, ..., a_{i+1}) + \sum_k f^i(a_1, ..., a_k \cdot a_{k+1}, ..., a_{i+1}) + f^i(a_1, ..., a_i) \cdot a_{i+1}$.

There are $\smile$ and $\smile_1$ products in the Hochschild complex C^* defined as follows: for $f \in C^i$ and $g \in C^j$ let

$$f \smile g(a_1, ..., a_{i+j}) = f(a_1, ..., a_i) \cdot g(a_{i+1}, ..., a_{i+j});$$
$$f \smile_1 g(a_1, ..., a_{i+j-1}) = \sum_k f(a_1, ..., a_k, g(a_{k+1}, ..., a_{k+j}), a_{k+j+1}, ..., a_{i+j-1}).$$

The standard formulas $\delta(f \smile g) = \delta f \smile g + f \smile \delta g$, $\delta(f \smile_1 g) = \delta f \smile_1 g + f \smile_1 \delta g + f \smile g + g \smile f$ are valid for so defined $\smile$ and $\smile_1$.

Now let $\bar{C^*} = C^*(C^*(B, H_*(G)), C^*(B, H_*(G))$ be the Hochschild complex of the ring $C^*(B, H_*(G))$. There exists a map $\mu : C^* \to \bar{C^*}$ which assignees to $f^i \in C^i$ the following homomorphism: for $b^k \in C^*(B, H_*(G))$ let $(\mu f)(b_1, ..., b_i)$ be the i fold $\smile$ product of elements $b_1, ..., b_i$ when the coefficients are multiplied by f^i.

Let us define a *Hochschild twisting cochain* as an element $f = f^3 + f^4 + ...,\ f^k \in C^k$ satisfying the condition $\delta f = f \smile_1 f$. The set of all such twisting cochains we denote by N.

Let us define the subset $L \subset N \times C^*(B, H_*(G))$ as $L = \{(f, h_0),\ f \in N,\ h_0 \in C^*(B, H_*(G)),\ s.t.\ \delta h_0 = h_0 \cdot h_0 + \sum_k (\mu f^k)(h_0, ..., h_0)\}$.

Definition of the map $\alpha : L \to D^*(B, H_*(G))$. First recall [3], [4] that $D^*(B, H_*(G))$ is defined as the factorset of the set of twisting cochains $M(B, H_*(G)) = \{h = h^2 + h^3 + ...,\ h^k \in C^k(B, Hom(H_*(G), H_*(G)),\ \delta h = h \smile h\}$. Note that since the modules $H_i(G)$ are assumed free, an element h is determined by a collection $\{h(a) \in C^*(B, H_*(G)),\ a \in I\}$ where I is the set of free generators of $H_*(G)$. Now for an element $(f, h_0) \in L$ we define $h \in M(B, H_*(G))$ as $h = \{h(a),\ a \in I,\ h(a) = h_0 \cdot a + \sum_i (\mu f^i)(h_0, ..., h_0, a)$. Inspection shows that $\delta h = h \smile h$. Finally we define $\alpha(f, h_0)$ as the class of h in $D(B, H_*(G))$.

Now let us consider the set of triples $\bar{L} = \{(f, h_0, \bar{f})\}$, where $(f, h_0) \in L$ and $\bar{f} = \bar{f}^3 + \bar{f}^4 + ...,\ \bar{f}^i : H_*(G) \otimes ...((i-1)-times)... \otimes H_*(G) \otimes H_*(F) \to$

$H(F)$ such that the following condition is satisfied

$$\begin{array}{l}\sum_{s+t=i+1} \bar{f}^s(a_1, ..., a_k, f^t(a_{k+1}, ..., a_{k+t}), ..., a_{i-1}, x)+ \\ \sum_{s+t=i+1} \bar{f}^s(a_1, ..., a_{i-t}, \bar{f}^t(a_{i-t+1}, ..., a_{i-1}, x)) = \delta \bar{f}^i(a_, ..., a_{i-1}, x).\end{array} \quad (1)$$

As above each polylinear map $\bar{f}^i$ induces the map

$$(\bar{\mu}\bar{f}^i) : C^*(B, H_*(G)) \otimes ... \otimes C^*(B, H_*(G)) \otimes C^*(B, H_*(F)) \to C^*(B, H_*(F)).$$

Definition of the map $\beta : \bar{L} \to D^*(B, H_*(F))$. We define $\beta(f, f_0, \bar{f})$ as the class of the twisting cochain $\bar{h} \in M(B, H_*(F)) = \{\bar{h} = \bar{h}^2 + \bar{h}^3 + ..., \ \bar{h}^k \in C^k(B, Hom(H_*(F), H_*(F)), \ \delta h = h \smile h\}$, defined as follows: $\bar{h} = \{\bar{h}(x); \ x \in J\}$ (here J is the set of free generators of $H_*(F)$), where $\bar{h}(x) = h_0 \cdot x + \sum_i (\bar{\mu}\bar{f}^i)(h_0, ..., h_0, x)$. The condition (1) allows to check that $\delta\bar{h} = \bar{h} \smile \bar{h}$.

The group G and, respectively, the action $G \times F \to F$, define (non uniquely) certain Hochschild twisting cochain f, and respectively the element $\bar{f}$, for which the condition (1) is satisfied. This can be done as follows.

Since the groups $H_i(G)$ are assumed free, it is possible to fix a *cycle choosing* homomorphism $g : H_*(G) \to Z_*(G)$. Besides let us fix also a homomorphism $\delta^{-1} : B_*(G) \to C_*(G)$ which satisfies $\delta\delta^{-1} = id$. Let us define also a homomorphism $\phi : Z_*(G) \to C_*(G)$ by $\phi(z) = \delta^{-1}(z - g(cl(z)))$, where $cl(z)$ is the homology class of z..

We construct by induction a sequences of multioperations $f^i = f^3 + f^4 + ...$ and homomorphisms $A_i : H_*(G) \otimes ... \otimes H_*(G) \to C_*(G)$ using the following conditions:

1) $A_2(a_1, a_2) = g(a_1) \cdot g(a_2)$;
2) $A_i(a_1, ..., a_i) \in Z_*(G)$;
3) $f^i(a_1, ..., a_i) = cl(A_i(a_1, ..., a_i)) \in H_*(G)$;
4) $A_{i+1}(a_1, ..., a_{i+1}) = g(a_1) \cdot \phi A_i(a_2, ..., a_{i+1}) + \phi A_i(a_1, ..., a_i) \cdot g(a_{i+1}) +$
$\sum_{s,t,k} \phi A_s(a_1, ..., a_k, f^t(a_{k+1}, ..., a_{i+t}), ..., a_{i+1}) +$
$\sum_{s,t} \phi A_s(a_1, ..., a_s) \cdot \phi A_t(a_{s+1}, ..., a_{i+1})$.

Now we construct $\bar{f}$. Let $\bar{g} : H_*(F) \to Z_*(F)$ be a *cycle choosing* homomorphism; $\delta^{-1} : B_*(F) \to C_*(F)$, $\delta\delta^{-1} = id$; and $\psi : Z_*(F) \to C_*(F)$

is given by $\psi(z) = \delta^{-1}(z - \bar{g}(cl(z)))$. As above we construct by induction a sequences of multioperations $\bar{f}^i = \bar{f}^3 + \bar{f}^4 + ...$ and homomorphisms $\bar{A}_i : H_*(G) \otimes ... \otimes H_*(G) \otimes H_*(F) \to C_*(F)$ using the following conditions:

1) $\bar{A}_2(a, x) = g(a) \cdot \bar{g}(x)$;
2) $\bar{A}_i(a_1, ..., a_{i-1}, x) \in Z_*(F)$;
3) $\bar{f}^i(a_1, ..., a_{i-1}, x) = class(\bar{A}^i(a_1, ..., a_{i-1}, x)) \in H_*(F)$;
4) $\bar{A}_{i+1}(a_1, ..., a_i, x) = g(a_1) \cdot \psi\bar{A}^i(a_2, ..., a_{i-1}, x) + \phi A_i(a_1, ..., a_i) \cdot \bar{g}(x) +$
$\sum_{s,t,k} \psi\bar{A}_s(a_1, ..., a_k, f^t(a_{k+1}, ..., a_{i+t}), ..., a_i, x) +$
$\sum_{s,t} \psi\bar{A}_s(a_1, ..., a_{i-t}, \bar{f}^t(a_{i-t+1}, ..., a_i, x)) +$
$\sum_{s,t} \phi A_s(a_1, ..., a_s) \cdot \psi\bar{A}_t(a_{s+1}, ..., a_i, x)$.

Theorem. For the above constructed f and $\bar{f}$ there exists a cochain $h_0 = h_0^2 + h_0^3 + ...,\ h_0^i \in C^i(B, H_{i-1}(G))$ such that $(f, h_0, \bar{f}) \in \bar{L}$, $\alpha(f, h_0)$ is the predifferential of of the principal bundle ξ, and $\beta(f, h_0, \bar{f})$ is the predifferential of of the associated bundle η.

In particular if the group G is $(n-1)$-connected, we obtain the result of Shih: $h_0^r = 0$ for $r < n$, and consequently $h^r = 0$; as for $n \leq r \leq 2n-2$ we have $h^r(a) = h_0^r \cdot a$, and h_0^r are cocycles.

References

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