On the homology theory of fiber spaces

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In this paper the homology theory of fibre spaces is studied by introducing additional algebraic structure in homology and cohomology.

All modules are assumed to be over an arbitrary associative ring Λ with unit; by a differential algebra, coalgebra, module, or comodule we mean these objects graded by non-negative integers; $\hat{a}$ denotes $(-1)^{dega}$.

The category $A(\infty)$**.** An $A(\infty)$-*algebra in the sense of Stasheff* [1] is defined to be a graded Λ-module M, endowed with a set of operations $\{m_i : \otimes^i M \to M, i = 1, 2, ...\}$ satisfying the conditions $m_i((\otimes^i M)q) \subset M_{q+i-2}$ and

$$\sum_{k=0}^{i-1}\sum_{j=1}^{i-k}(-1)^k m_{i-j+1}(\hat{a}_1 \otimes ... \otimes \hat{a}_k \otimes m_j(a_{k+1} \otimes ... \otimes a_{k+j}) \otimes a_{k+j+1} \otimes ... \otimes a_i) = 0$$

for any $a_i \in M$ and $i \geq 1$. A *morphism of* $A(\infty)$-*algebras* $(M, \{m_i\}) \to (M', \{m'_i\})$ is a set of homomorphisms $\{f_i : \otimes^i M \to M', i = 1, 2, ...\}$ satisfying the conditions $f_i((\otimes^i M)q) \subset M'_{q+i-1}$ and

$$\sum_{k=0}^{i-1}\sum_{j=1}^{i-k}(-1)^k f_{i-j+1}(\hat{a}_1 \otimes ... \otimes \hat{a}_k \otimes m_j(a_{k+1} \otimes ... \otimes a_{k+j}) \otimes ... \otimes a_i) =$$
$$\sum_{t=1}^{i}\sum_{S(t,i)} m'_t(f_{k_1}(a_1 \otimes ... \otimes a_{k_1}) \otimes ... \otimes f_{k_t}(a_{i-k_t+1} \otimes ... \otimes a_i))$$

where $S(t,i) = \{k_1, ..., k_t \in N, \sum k_p = i\}$. The $A(\infty)$-algebras together with these morphisms form a category, which we denote by $A(\infty)$.

The specification on M of an $A(\infty)$-algebra structure $(M, \{m_i\})$ is equivalent to the specification on the tensor coalgebra $T^c(M) = \Lambda + M + M \otimes M + ...$ with the grading $dim(a_1 \otimes ... \otimes a_n) = \sum dima_i + n$ and comultiplication

$$\Delta(a_1 \otimes ... \otimes a_n) = \sum_{i=0}^{n}(a_1 \otimes ... \otimes a_i) \otimes (a_{i+1} \otimes ... \otimes a_n)$$

of a differential $d_m : T^c(M) \to T^c(M)$ that turns $T^c(M)$ into a differential coalgebra; this set $\{m_i\}$ determines the differential d_m by

$$d_m(a_1 \otimes ... \otimes a_n) = \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} (-1)^k \hat{a}_1 \otimes ... \otimes \hat{a}_k \otimes m_j(a_{k+1} \otimes ... \otimes a_{k+j}) \otimes ... \otimes a_n,$$

and the differential coalgebra $(T^c(M), d_m)$ is called the $\tilde{B}$-construction of the $A(\infty)$-algera $(M, \{m_i\})$ (Stasheff [1]) and is denoted by $\tilde{B}(M, \{m_i\})$. The specification of an $A(\infty)$-algebra morphism $\{f_i\} : (M, \{m_i\}) \to (M', \{m'_i\})$ is equivalent to that of a differential coalgebra mapping $f : \tilde{B}(M, \{m_i\}) \to \tilde{B}(M', \{m'_i\})$; the morphism $\{f_i\}$ determines the mapping f by

$$f(a_1 \otimes ... \otimes a_n) = \sum_{t=1}^{n} \sum_{S(t,n)} f_{k_1}(a_1 \otimes ... \otimes a_{k_1}) \otimes ... \otimes f_{k_t}(a_{n-k_t+1} \otimes ... \otimes a_i).$$

Thus the category $A(\infty)$ can be identified with a full subcategory of the category of differential coalgebras.

An arbitrary object in $A(\infty)$ of the form $(M, \{m_1, m_2, 0, 0, ...\})$ is identified with the differential algebra $(M, \partial, \cdot)$ where $\partial = m_1$ and $a_1 \cdot a_2 = -m_2(\tilde{a}_1 \otimes a_2)$. For such an object the $\tilde{B}$-construction coincides with the usual B-construction, any morphism of such objects of the form $\{f_1, 0, 0, ...\}$ is identified with the differential algebra mapping $f_1 : (M, \partial, \cdot) \to (M, \partial, \cdot)$. Thus the category of differential algebras is a subcategory of $A(\infty)$, while the category $DASH$ (see [2]) is the full subcategory of $A(\infty)$ generated by differential algebras, and the functor $\tilde{B}$ is an extension of B from this subcategory to $A(\infty)$.

Theorem 1 *For any differential algebra C with free $H_i(C), i \geq 0$ it is possible to introduce on $H(C)$ an $A(\infty)$-algebra structure*

$$(H(C), \{X_i\}),\ X_i : \otimes^i H(C) \to H(C),\ i = 1, 2, 3, ...$$

such that $X_1 = 0,\ X_2(a_1 \otimes a_2) = -\tilde{a}_1 \cdot a_2$ and there exists an $A(\infty)$-morphism

$$\{f_i\} : (H(C), \{X_i\}) \to (C, \{m_1, m_2, 0, 0, ...\})$$

for which $f_1 : H(C) \to C$ induces an identical isomorphis in homology.

Proof. We need to construct two sets of homomorphisms

$$\{X_i : \otimes^i H(C) \to H(C),\ i = 1, 2, 3, ...\},\ \{f_i : \otimes^i H(C) \to C),\ i = 1, 2, 3, ...\},$$

satisfying the conditions in the definition of the category $A(\infty)$:

$$\sum_{k=0}^{i-1} \sum_{j=1}^{i-k} (-1)^k X_{i-j+1}(\hat{a}_1 \otimes ... \otimes \hat{a}_k \otimes X_j(a_{k+1} \otimes ... \otimes a_{k+j}) \otimes a_{k+j+1} \otimes ... \otimes a_i) = 0, \tag{1}$$

$\sum_{k=0}^{i-1}\sum_{j=1}^{i-k}(-1)^k f_{i-j+1}(\hat{a}_1\otimes ...\otimes \hat{a}_k\otimes X_j(a_{k+1}\otimes ...\otimes a_{k+j})\otimes ...\otimes a_i)=$
$m_1 f_i(a_1\otimes ...\otimes a_i)+$
$\sum_{s=1}^{i-1} m_2(f_s(a_1\otimes ...\otimes a_s)\otimes f_{i-s}(a_{s+1}\otimes ...\otimes a_i))$ (2)

for arbitrary $a_k \in H(C)$ and $i \geq 1$. Fir $i = 1$ we take $X_1 = 0$, and, using the fact that $H_i(C)$ is free, we define $f_1 : H(C) \to C$ to be a cycle-choosing homomorphis; the conditions (1) and (2), as well as the initial condition on f_1 are thereby satisfied. Suppose now that X_i and f_i have been constructed for $i < n$ in such a way that the conditions (1) and (2) hold. Let

$$U_n(a_1\otimes ...\otimes a_n)=\sum_{s=1}^{n-1} m_2(f_s(a_1\otimes ...\otimes a_s)\otimes f_{n-s}(a_{s+1}\otimes ...\otimes a_n))+ \sum_{k=0}^{n-2}\sum_{j=2}^{n-1}(-1)^{k+1} f_{n-j+1}(\hat{a}_1\otimes ...\otimes \hat{a}_k\otimes X_j(a_{k+1}\otimes ...\otimes a_{k+j})\otimes ...\otimes a_n)$$

(here the X_i and f_i already defined are involved). Then the condition (2) takes the form

$$m_1 f_n(a_1\otimes ...\otimes a_n) = (f_1X_n - U_n)(a_1\otimes ...\otimes a_n). \quad (3)$$

Direct calculations show that $\partial U_n = 0$, that is, $U_n(a_1 \otimes ... \otimes a_n)$ is a cycle in C for arbitrary $a_i \in H(C)$, and we define $X_n(a_1 \otimes ... \otimes a_n)$ to be the class of this cycle, that is, $X_n = \{U_n\}$. Since f_1 is a cycle-choosing homomorphism, the difference $f_1X_n - U_n$ is homological to zero. Assuming that $a_i \in H(C)$ are free generators we define $f_n(a_1 \otimes ... \otimes a_n)$ as an element of C boundaring this difference and extend by linearity. For the X_n and f_n thus defined the condition (3) is automatically satisfied. The remanning condition (1) can be proved by a straightforward check.

We remark that the theorem is true also when an arbitrary $A(\infty)$-algebra is taken instead of C, and $H(M)$ is understood to be the homology of M with respect to the differential m_1.

The $A(\infty)$-algebra $(H(C), \{X_i\})$ we call the *homology $A(\infty)$-algebra of the differential algebra C*. As is clear from the proof, this structure is not uniquely determined on $H(C)$ (there is an arbitrariness in the choice of the f_i). We show later that the structure of the homology $A(\infty)$-algebra on $H(C)$ is unique up to isomorphism in $A(\infty)$.

We mention that if $a_1 \cdot a_2 = a_2 \cdot a_3 = 0$ for $a_1, a_2, a_3 \in H(C)$, then $X_3(a_1 \otimes a_2 \otimes a_3)$ is an element of the Massey product $< a_1, a_2, a_3 >$, and this fact provides us with examples in which the operation X_3 is non-trivial. The next result follows from the Theorem 1.

Corollary 1 *The mapping of differential coalgebras*

$$f : \tilde{B}(H(C), \{X_i\}) \to B(C)$$

induces an isomorphism in homology.

The category $M(\infty)$**.** An $A(\infty)$-module over an $A(\infty)$-algebra $(M, \{m_i\})$ we define to be a graded Λ-module P, endowed with a set of operations $\{p_i :$

$(\otimes^{i-1}M)\otimes P\to P,\ i=1,2,3,...\}$ satisfying the conditions $p_i(((\otimes^{i-1}M)\otimes P)_q)\subset M_{q+i-2}$ and

$$\sum_{k=0}^{i-2}\sum_{j=1}^{i-k-1}(-1)^k p_{i-j+1}(\hat{a}_1\otimes...\otimes\hat{a}_k\otimes m_j(a_{k+1}\otimes...\otimes a_{k+j})\otimes...\otimes a_{i-1}\otimes b)+\sum_{k=0}^{i-1}(-1)^k p_{k+1}(\hat{a}_1\otimes...\otimes\hat{a}_k\otimes p_{i-k}(a_{k+1}\otimes...\otimes a_{i-1}\otimes b))=0.$$

The specification on P of an $A(\infty)$-module structure over $(M,\{m_i\})$ is equivalent to the specification on $\tilde{B}(M,\{m_i\})\otimes P$ of a differential that turns it into a differential comodule over $\tilde{B}(M,\{m_i\})$. The objects of the category $M(\infty)$ are defined to be the pairs $((M,\{m_i\}),(P,\{p_i\}))$, where $(M,\{m_i\})$ is an $A(\infty)$-algebra, and $(P,\{p_i\})$ is an $A(\infty)$-module over it. A morphism is defined to be a pair of sets of homomorphisms $\{f_i\},\{g_i\}$ where $\{f_i\}:(M,\{m_i\})\to(M',\{m'_i\})$ is a morphism of $A(\infty)$-algebras and

$$\{g_i:(\otimes^{i-1}M)\otimes P\to P',\ i=1,2,3,...\}$$

is a set satisfying the conditions $g_i(((\otimes^{i-1}M)\otimes P)_q)\subset P'_{q+i-1}$ and

$$\sum_{k=0}^{i-2}\sum_{j=1}^{i-k-1}(-1)^k g_{i-j+1}(\hat{a}_1\otimes...\otimes\hat{a}_k\otimes m_j(a_{k+1}\otimes...\otimes a_{k+j})\otimes...\otimes a_{i-1}\otimes b)=\sum_{t=1}^{i}\sum_{S(t,i)}p_t(f_{k_1}(a_1\otimes...\otimes a_{k_1})\otimes f_{k_2}(a_{k_1+1}\otimes...\otimes a_{k_1+k_2})\otimes...\otimes f_{k_{t-1}}(a_{k_1+...+k_{t-2}+1}\otimes...\otimes a_{k_1+...+k_{t-1}})\otimes g_{k_t}(a_{i-k_t+1}\otimes...\otimes a_{i-1}\otimes b));$$

These conditions ensure that the mapping

$$g:\tilde{B}(M,\{m_i\})\otimes P\to\tilde{B}(M',\{m'_i\})\otimes P'$$

given by

$$g(a_1\otimes...\otimes a_{i-1}\otimes b)=\sum_{t=1}^{i}\sum_{S(t,i)}f_{k_1}(a_1\otimes...\otimes a_{k_1})\otimes...\otimes f_{k_{t-1}}(a_{k_1+...+k_{t-2}+1}\otimes...\otimes a_{k_1+...+k_{t-1}})\otimes g_{k_t}(a_{k_1+...+k_{t-1}+1}\otimes...\otimes a_{i-1}\otimes b)$$

is a differential comodule mapping compatible with f. With the obvious morphisms the category of pairs (C,D), where C is a differential algebra and D is a differential module over it, forms a subcategory of $M(\infty)$.

Theorem 2 *If C is a differential algebra and D is a differential module over it such that $H_i(C)$ and $H_i(D)$ are free, then on $H(D)$ it is possible to introduce the structure of an $A(\infty)$-module $(H(D),\{Y_i\}),\ Y_i:(\otimes^{i-1}H(C))\otimes H(D)\to H(D),\ p=1,2,3,...$ over the homology $A(\infty)$-algebra $(H(C),\{X_i\})$ such that $Y_1=0,\ Y_2(a\otimes b)=-\tilde{a}\cdot b$ and there exists a morphism $(\{f_i\},\{g_i\}):((H(C),\{X_i\}),(H(D),\{Y_i\}))\to(C,D)$ of $M(\infty)$ for which $f_1:H(C)\to C$ and $g_1:H(D)\to D$ induce identical isomorphisms in homology.*

Proof. The sets $\{g_i\}$ and $\{Y_i\}$ are constructed by induction on i just as in the proof of Theorem 1. Using the fact that $H_i(D)$ is free, we define g_1 to be a cycle-choosing homomorphism, while $Y_1 = 0$, and the conditions of the category $M(\infty)$ are satisfied for $i = 1$. Let

$$V_n(a_1 \otimes ... \otimes a_{n-1} \otimes b)) = \\ \sum_{s=1}^{n-1} p_2(f_s(a_1 \otimes ... \otimes a_s) \otimes g_{n-s}(a_{s+1} \otimes ... \otimes a_{n-1} \otimes b))+ \\ \sum_{k=0}^{n-3} \sum_{j=2}^{n-1} (-1)^{k+1} g_{i-j+1}(\hat{a}_1 \otimes ... \otimes \hat{a}_k \otimes X_j(a_{k+1} \otimes ... \otimes a_{k+j}) \otimes ... \otimes \\ a_{n-1} \otimes b) + \sum_{k=1}^{n-2} (-1)^k g_{k+1}(\hat{a}_1 \otimes ... \otimes \hat{a}_k \otimes Y_{n-k}(a_{k+1} \otimes ... \otimes a_{n-1} \otimes b)),$$

then $\partial V_n = 0$, therefore, we define $Y_n = \{V_n\}$. Since g_1 is a cycle-choosing homomorphism, $g_1 Y_n - V_n$ is a cycle in D homologous to zero. Using the fact that $H_i(D)$ is free, we define $g_n : (\otimes^{n-1}) \otimes H(D) \to H(D)$ to be a homomorphism for which $\partial g_n = g_1 V_n - V_n$. The conditions of the category $M(\infty)$ are satisfied for the Y_n and g_n thus defined.

Twisted tensor products. The twisted tensor products of Brown [3] can be generalized from the case of differential algebras and modules to the case of $A(\infty)$-algebras and $A(\infty)$-modules: for an arbitrary differential coalgebra (K, d) and an $A(\infty)$-algebra $(M, \{m_i\})$ a $\sim$-twisting cochain we define to be a homomorphism $\phi : K \to M$ of degree -1 that satisfies the condition

$$\phi d = \sum_{i=1}^{\infty} m_i(\phi \otimes \phi \otimes ... \otimes \phi)\Delta^i,$$

where $\Delta^i : K \to K \otimes K$ is the homomorphism defined by $\Delta^1 = id_K$, $\Delta^2 = \Delta : K \to K \otimes K$, $\Delta^i = (id_K \otimes \Delta^{i-1})\Delta$. The specification of a $\sim$-twisting cochain $\phi : K \to M$ is equivalent to that of a mapping of differential coalgebra $f_\phi : (K, d) \to \tilde{B}(M, \{m_i\})$. For any (K, d) and $((M, \{m_i\}), (P, \{p_i\})) \in M(\infty)$ any $\sim$-twisting cochain $\phi : K \to M$ on the tensor product $K \otimes P$ determines by

$$\partial_\phi = d \otimes id_P + \sum_{i=1}^{\infty} (\hat{id} \otimes p_i)(id_K \otimes \phi \otimes ... \otimes \phi \otimes id_P)(\Delta^i \otimes id_P)$$

a differential, turning $(K \otimes P, \partial_\phi)$ into a differential comodule over (K, d); this differential comodule is called the *$\sim$-twisted tensor product* $K \otimes_\phi P$. If M is an $A(\infty)$-algebra of the form $(M, \{m_1, m_2, 0, 0, ...\})$, and P is an $A(\infty)$-module of the form $(P, \{p_1, p_2, 0, 0, ...\})$, then ϕ is the usual twisting cochain, and $K \otimes_\phi P$ coincides with the usual twisted tensor product $K \otimes_\phi P$.

We need the concept of equivalence of twisting cochains (see [4], [5], [6]). We say that $\phi, \psi : K \to C$ are *equivalent* if there is a homomorphism $c : K \to C$ of degree 0 for which $c_0 = c|C_0 = 0$ and $\psi = (1 + c) \star \phi$ where

$$(1 + c) \star \phi = (1 + \hat{c}) \cdot \phi \cdot (1 + c)^{-1} - (cd + \partial c) \cdot (1 + c)^{-1};$$

$\phi \sim \psi$ if and only if $f_\phi, f_\psi : K \to B$ are homotopic in the sense of [2] (coderivation homotopy): $f_\phi - f_\psi = \partial D + D\partial$ with $(D \otimes f_\psi + \hat{f}_\phi \otimes D)\Delta = \Delta D$.

Theorem 3 *If (K, d) is a differential coalgebra with free K_i, and $\phi : K \to C$ is an arbitrary twisting cochain, then there exists a $\sim$-twisting cochain $\phi^* : K \to H(C)$ such that ϕ and $f^*\phi^* = \sum_{i=1}^{\infty} f_i(\phi * \otimes ... \otimes \phi *)\Delta^i$ are equivalent.*

Proof. To construct $\phi*$ we prove the following inductive assertion: for any $i > 0$ there exists a twisting cochain $\phi^{(i)} : K \to C$ and a homomorphisms $\phi_i^* : K_i \to H_{i-1}(C)$ and $c_i : K_i \to C_i$ such that

$$\begin{array}{ll} (a) & \phi_i^* d = \sum_{t=2}^{i} \sum_{S(t,i)} X_t(\phi_{k_1}^* \otimes ... \otimes \phi_{k_t}^*)\Delta^t; \\ (b) & \phi^{(i)} = (1 + c_i) \star \phi^{(i-1)}; \\ (c) & \phi_i^{(i)} = \sum_{t=1}^{i} \sum_{S(t,i)} f_t(\phi_{k_1}^* \otimes ... \otimes \phi_{k_t}^*)\Delta^t. \end{array}$$

For $i = 1$ we take $\phi_1^* = \{\phi_1\}$. Since the difference $(\phi_1 - f_1\phi_1^*)(k)$ is homologous to zero for each $k \in K_1$ and K_1 is free, we obtain a homomorphism $c_1 : K_1 \to C_1$ for which $-\partial c_1 = \phi_1 - f_1\phi_1^*$. We define $\phi^{(1)} = (1 + c_1) \star \phi$, so $\phi_1^{(1)} = \phi_1 + \partial c_1 = f_1\phi_1^*$. Suppose now that $\phi^{(i)}$, ϕ_i^*, and c_i have already been constructed in such a way that (a), (b), and (c) hold for $i < n$. Let

$$W_n = \phi_n^{(n-1)} - \sum_{t=2}^{n} \sum_{S(t,n)} f_t(\phi_{k_1}^* \otimes ... \otimes \phi_{k_t}^*)\Delta^t;$$

A direct check shows that $\partial W_n = 0$; we define $\phi_n^* = \{W_n\}$. Since the difference $W_n - f_1\phi_n^*$ is homological to zero and K_n is free, we can construct a $c_n : K_n \to C_n$ such that $-\partial c_n = W_n - f_1\phi_n^*$; let $\phi^{(n)} = (1 + c_n) \star \phi^{(n-1)}$. Then

$$\phi_n^{(n)} = \phi_n^{(n-1)} + \partial c_n = f_1\phi_n^* - W_n = \sum_{t=1}^{n} \sum_{S(t,n)} f_t(\phi_{k_1}^* \otimes ... \otimes \phi_{k_t}^*)\Delta^t,$$

consequently, (b) and (c) hold for $\phi^{(n)}$, ϕ_n^*, and c_n, and the validity of (a) can be checked directly. From (a) we see that $\phi^* = \sum_i \phi_i^*$ is a $\sim$-twisting cochain, and from (b) and (c) we deduce that $f^*\phi^* = \phi^\infty$, where $\phi^\infty = \Pi_i(1 + c_i) \star \phi \sim \phi$.

It follows from Theorem 3 that for any differential coalgebra mapping $g : K \to B$ there exists a $G^* : K \to \tilde{B}(H(C), \{X_i\})$ for which g and fg^* are homotopic in the sense of [2]. This assertion implies uniqueness mentioned above for the structure of homology $A(\infty)$-algebra: if $(H(C), \{X_i\})$ and $(H(C), \{X_i\})$ are two structures of homology $A(\infty)$-algebra on $H(C)$, then by taking $K = \tilde{B}(H(C), \{X_i\})$ and $g = f : \tilde{B}(H(C), \{X_i\}) \to B(C)$, we obtain a

$$g^* : \tilde{B}(H(C), \{X_i\}) \to \tilde{B}(H(C), \{X_i\})$$

for which $fg^* \sim g$. Then the first component of the $A(\infty)$algebra morphism $\{g_i^*\} : (H(C), \{X_i\}) \to (H(C), \{X_i\})$ induced by g^* is $g_1^* = id_{H(C)}$, and this implies that $\{g_i^*\}$ is an isomorphism in $A(\infty)$.

The next result follows from Theorems 1, 2 and 3.

Corollary 2 *$K \otimes_\phi D$ and $K \otimes_{\phi^*} H(D)$ have isomorphic homology under the conditions of Theorems 1, 2 and 3.*

The results obtained have the following applications.

The first proposition is obtained from Corollary 1 by taking $C = \bar{C}_*(G)$, where G is a connected topological group such that the $H_i(G)$ are free, bearing in mind that $H(B(C)) = H_*(B_G)$.

Proposition 1 *The homology of the $\tilde{B}$-construction $\tilde{B}(\bar{H}_*(G), \{X_i\})$ are isomorphic to that of classifying space B_G.*

The next proposition is obtained from Corollary 1 by taking $C = C^*(B, b_0)$, where B is a simply connected space with free groups $H^i(B, b_0)$, and bearing in mind that $H(B(C)) = H^*(\Omega C)$.

Proposition 2 *The homology of the $\tilde{B}$-construction $\tilde{B}(\bar{H}^*(B, b_0), \{X_i\})$ are isomorphic to the cohomology of the loop space ΩB.*

Let $\xi = (X, p, B, G)$ be a principal G-fibration with paracompact base and connected G, let F be a G-space, and $\xi[F] = (E, p, B, F, G)$ the associated fiber bundle, with the $H_i(G)$ and $H({}_i(F)$ free. The final proposition is obtained from Corollary 2 ty taking $C = C_*(G),\ D = C_*(F)$, and ϕ a twisting cochain of the fibration ξ ([3]).

Proposition 3 *The homology of the $\sim$-twisted tensor product $C_*(B) \otimes_{\phi^*} H_*(F)$ is isomorphic to that of E.*

This proposition generalize a result of Shih [7]: if G is $(n-1)$-connected, then the components $\phi_i^* \in C^i(B, H_{i-1}(G))$ vanish for $0 < i < n+1$, therefore, the differentials d^i of the spectral sequence of $\xi[F]$ are trivial for $1 < i < n+1$, and the components are cocycles for $n < i < 2n+1$, consequently, d^j can be expressed for $n < i < 2n+1$ in terms of certain characteristic classes of ξ and the operation $Y_2 : H_*(G) \otimes H_*(F) \to H_*(F)$; we remark that higher operations Y_i are needed for computing $d^j,\ j > 2n$, in terms of ϕ^*.

Theorems 1, 2 and Proposition 3 were announced in [8] and Proposition 2 in [9].

References

[1] J.D. Stasheff, Homotopy associativity of H-spaces, I, II, Trans. Amer. Math. Soc. 108 (1963), 27-313. MR 28 1623.

[2] H. Munkholm, The Eilenberg-More spectral sequence and strongly homotopy multiplicative maps, J. Pure and Appl. Algebra 5 (1974), 1-50. MR 50 3227.

[3] E. H. Brown, Twisted tensor products, I, Ann. Of Math. (2) 69 (1959), 223-246. MR 21 4423.

[4] N. A. Berikashvili, The differentials of a spectral sequence, Bull. Of Georg. Acad. Sci., 51 (1968), 9-14. MR 41 9258.

[5] N. A. Berikashvili, Homology theory of spaces, Bull. Of Georg. Acad. Sci., 86 (1977), 529-532. MR 57 13949.

[6] V. A. Smirnov, The functor D for twisted tensor products, Mat. Zametki 20 (1976), 465-472. MR 55 4172.

[7] W. Shih, Homologie des espaces fibres, Inst. Hautes Etudes Sci. Publ. Math. 1962, no. 13, 88. MR 26 1893.

[8] T. Kadeishvili, On the differentials of spectral sequence of a fiber bundle, Bull. Of Georg. Acad. Sci., 82 (1976), 285-288. MR 55 6430.

[9] T. Kadeishvili, On the homology of classifying spaces, Proc. 7th All Union Topology Conference, Minsk 1977.