Structure of $A(\infty)$-algebra and Hochschild and Harrison cohomology

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In [6] for an arbitrary differential algebra C (with free homology modules) in the homology algebra $H(C)$ we have constructed a sequence of operations $\{m_i : \otimes^i H(C) \to H(C),\ i = 3, 4, ...\}$, which, together with ordinary multiplication $m_2 : H(C) \otimes H(C) \to H(C)$, turns $H(C)$ into an $A(\infty)$-algebra in the sense of Stasheff [8]. If a differential algebra C is commutative then on $H(C)$ arises an $A(\infty)$-algebra structure of special type which we call commutative. Particularly $A(\infty)$-algebra structure arises on the cohomology algebra $H^*(B, \Lambda)$ of a topological space, and a commutative $A(\infty)$-algebra structure arises on the rational cohomology algebra $H^*(B, Q)$. Clearly the $A(\infty)$-algebra $(H^*(B, \Lambda), \{m_i\})$ carries more information than algebra $H^*(B, \Lambda)$. Particularly cohomology $A(\infty)$-algebra $(H^*(B, \Lambda), \{m_i\})$ determines cohomology groups of the loop space ΩB, and commutative $A(\infty)$-algebra $(H^*(B, Q), \{m_i\})$ determines the rational homotopy type of B. Naturally arises a question when these structures are degenerate, that is when for an $A(\infty)$-algebra $(H(C), \{m_i\})$ the operations $m_i,\ i \geq 3$ are trivial?

In this paper we study the connection between $A(\infty)$-structures and Hochschild (Harrison in commutative case) cohomology of the algebra $H(C)$. Particularly we show that if Hochschild cohomology $Hoch^{n,2-n}(H(C), H(C)) = 0$ for $n \geq 3$ then any $A(\infty)$-algebra structure on $H(C)$ is degenerate. Respectively, in commutative case, any commutative $A(\infty)$-algebra structure is degenerate whenever Harrison cohomology $Harr^{n,2-n}(H(C), H(C)) = 0$ for $n \geq 3$

Bellow Λ denotes a field. If $M = \sum M_q$ is a graded Λ-module and $a \in M_p$ then $\hat{a}$ denotes $(-1)^{dima}$. To the permutation of elements $a \in M_p,\ b \in M_q$ corresponds the sign $(-1)^{pq}$, this rule assignees to an arbitrary permutation σ of graded elements $a_1, ..., a_n$ the sign denoted by $\epsilon(\sigma)$ (the Koszul sign).

1 Products

For an arbitrary graded module $M = \sum M_q$ the tensor coalgebra $T(M)$ is defined as

$$T(M) = \Lambda + M + M \otimes M + ... = \sum_{i=0}^{\infty} \otimes^i M;$$

the grading in $T(M)$ is defined by $dim(a_1 \otimes ... \otimes a_n) = \sum dim a_i - n$, and the comultipliciation $\nabla : T(M) \to T(M) \otimes T(M)$ looks as

$$\nabla(a_1 \otimes ... \otimes a_n) = \sum_{k=0}^{n} (a_1 \otimes ... \otimes a_k) \otimes (a_{k+1} \otimes ... \otimes a_n)$$

(here the empty bracket () means $1 \in \Lambda$). Iterating the comuultiplication ∇ we obtain a sequence of homomorphisms

$$\{\nabla^i : T(M) \to \otimes^i T(M),\ i = 1, 2, ...\}$$

where

$$\nabla^1 = id,\ \nabla^2 = \nabla,\ \nabla^n = (id \otimes \nabla^{n-1})\nabla.$$

There is also a product $\mu : T(M) \otimes T(M) \to T(M)$ which, together with ∇, determines on $T(M)$ a structure of Hopf algebra, this is the shuffle product defined by Eilenberg and MacLane in[4]. This product is defined as

$$\mu((a_1 \otimes ... \otimes a_n) \otimes (a_{n+1} \otimes ... \otimes a_{n+m})) = \sum \epsilon(\sigma) a_{\sigma(1)} \otimes ... \otimes a_{\sigma(n+m)},$$

where summation is taken over all permutations of the set $(1, 2, ..., n + m)$ which satisfy the condition: $i < j$ if $1 \leq \sigma(i) < \sigma(j) \leq n$ or $n + 1 \leq \sigma(i) < \sigma(j) \leq n + m$.

In [7] it is shown that this product is uniquely characterized by the following axioms:

(1) μ turns $T(M)$ into a Hopf algebra;

(2) $p_0(\mu(a_+ \otimes b_+)) = p_1(\mu(a_+ \otimes b_+)) = 0$ for arbitrary $a_+, b_+ \in T(M)_+$ (here $T(M)_+ = \sum_{n \geq 1} \otimes^n M$ and $p_i : T(M) \to \otimes^i M$ is the clear projection).

Let us denote

$$Sh^n(M) = \textstyle\sum_{k=1}^{n-1} \mu((\otimes^k M) \otimes (\otimes^{n-k} M)) \subset \otimes^n M;$$
$$Sh(M) = \sum Sh^n(M);\ Ch^n(M) = \otimes^n M / Sh^n(M);\ Ch(M) = \sum Ch^n(M).$$

It is clear that

$$Sh(M) = \mu(T(M)_+ \otimes T(M)_+);\ Ch(M) = T(M)/Sh(M);$$
$$Sh^0(M) = Sh^1(M) = 0;\ Ch^0(M) = \Lambda; Ch^1(M) = M.$$

Suppose now that together with $T(M)$ there is given a graded Λ-module $N = \sum N_q$. Denote by $Hom^k(\otimes^n M, N)$ a set of Λ-homomorphisms $f : \otimes^n M \to N$ of degree k (that is $f(a_1 \otimes ... \otimes a_n) \subset M_{q+k}$ where $q = \sum dim a_i$). We also use the notation

$$Hom(\otimes^n M, N) = \sum_k Hom^k(\otimes^n M, N),\ Hom(T(M), N) = \sum_n Hom(\otimes^n M, N).$$

In $Hom(T(M), M)$ Gerstenhaber [5] has introduced a product which we denote by $\smile_1$, defined as follows. For $f \in Hom^k(\otimes^m M, N),\ g \in Hom^t(\otimes^n M, N)$ the product $f \smile_1 g \in Hom^{k+t}(\otimes^{m+n-1} M, N)$ looks as

$$f \smile_1 g(a_1 \otimes ... \otimes a_{m+n-1}) = \sum_{k=0}^{m-1} \pm f(a_1 \otimes ... \otimes a_k \otimes g(a_{k+1} \otimes ... \otimes a_{k+n}) \otimes ... \otimes a_{m+n-1}).$$

We shall use also more general product of an element $f \in Hom(T(M), M)$ and a sequence $(g_1, ..., g_k)$, $g_i \in Hom(T(M), M)$ defined as

$$
\begin{aligned}
& f \smile_1 (g_1, ..., g_k)(a_1 \otimes ... \otimes a_n) = \\
& \sum \pm f(a_1 \otimes ... \otimes a_{k_1} \otimes g(a_{k_1+1} \otimes ... \otimes a_{k_1+p_1}) \otimes a_{k_1+p_1+1} \otimes ... \otimes a_{k_1+t_1} \otimes \\
& g_2(a_{k_1+t_1+1} \otimes ... \otimes a_{k_1+t_1+p_2}) \otimes ... \\
& \otimes a_s \otimes g_k(a_{s+1} \otimes ... \otimes a_{s+p_k}) \otimes a_{s+p_k+1} \otimes ... \otimes a_n).
\end{aligned}
$$

The product $f \smile_1 g$ can be written as $f \smile_1 g = f(id \otimes g \otimes id)\nabla^3$. Similarly $f \smile_1 (g_1, ..., g_k) = f(id \otimes g_1 \otimes id \otimes g_2 \otimes ... \otimes id \otimes g_k \otimes id)\nabla^{2k+1}$. We remark also that these products are defined if $f \in Hom(T(M), N)$ and $g_i \in Hom(T(M), M)$.

The product $f \smile_1 g$ is not associative generally but easy to see that

$$f \smile_1 (g \smile_1 h) - (f \smile_1 g) \smile_1 h = f \smile_1 (g, h) - f \smile_1 (h, g).$$

Let us consider $Hom(Ch(M), M) = \sum_n Hom(Ch^n(M), M)$. Clearly

$$Hom(Ch^n(M), M) = \{f \in Hom(\otimes^n(M), M),\ \ f|Sh^n(M) = 0\},$$

hence $Hom(Ch^n(M), M)$ is also graded by degree of homomorphisms, i.e.

$$Hom(Ch^n(M), M) = \sum_k Hom^k(Ch^n(M), M)$$

where

$$Hom^k(Ch^n(M), M) = \{f \in Hom^k(\otimes^n M, M),\ \ f|Sh^n(M) = 0\},$$

thus $Hom(Ch(M), M) \subset Hom(T(M), M)$. It is possible to show that if $f, g \in Hom(Ch(M), M)$ then $f \smile_1 g \in Hom(Ch(M), M)$. Moreover, $f \smile_1 (g, ..., g) \in Hom(Ch(M), M)$ too.

2 Hochshild cohomology

Let A be a graded algebra and M be a graded bimodule over A. Hochschild cochain complex is defined as

$$C^*(A, M) = \sum C^n(A, M), \quad C^n(A, M) = Hom(\otimes^n A, M),$$

the coboundary operator $\delta : C^n(A, M) \to C^{n+1}(A, M)$ is given by

$$
\begin{gathered}
\delta f(a_1 \otimes ... \otimes a_{n+1}) = a_1 f((a_2 \otimes ... \otimes a_{n+1}) + \\
\sum_k \pm f(a_1 \otimes ... \otimes a_k a_{k+1} \otimes ... \otimes a_{n+1}) \pm f(a_1 \otimes ... \otimes a_n) a_{n+1}.
\end{gathered}
$$

Hochschild cohomology of A with coefficients in M is defined as homology of this cochain complex and is denoted by $Hoch^*(A, M)$. Since A and M are graded, each $C^n(A,M)$ is graded too: $C^n(A,M)=\sum_k C^{n,k}(A,M)$ where $C^{n,k}(A,M)=Hom^k(\otimes^n A,M)$. It is easy to see that $\delta : C^{n,k}(A,M)\to C^{n+1,k}(A,M)$, so $(C^{*,k}(A,M),\delta)$ is a direct summand in $(C^*(A,M),\delta)$, thus Hochschild cohomology in this case is bigraded: $Hoch^n(A,M)=\sum_k Hoch^{n,k}(A,M)$ where $Hoch^{n,k}(A,M)$ is the n-th homology module of $(C^{*,k}(A,M),\delta)$.

Instead of M we can take the algebra A itself. The complex $C^{*,*}(A,A)$ is a differential algebra with respect to the following product: for $f\in C^{m,k}(A,A)$ and $g\in C^{n,t}(A,A)$ the product $f\smile g\in C^{m+n,k+t}(A,A)$ is defined by

$$f\smile g(a_1\otimes ...\otimes a_{m+n})=f(a_1\otimes ...\otimes a_m)\cdot g(a_{n+1}\otimes ...\otimes a_{m+n}).$$

Besides this product in $C^{*,*}(A,A)$ we have also the product $f\smile_1 g$ (see the previous section). In [5] it is shown that these products satisfy the standard conditions

$$\delta(f\smile g)=\delta f\smile g\pm f\smile \delta g; \quad (1)$$

$$\delta(f\smile_1 g)=\delta f\smile_1 g\pm f\smile_1 \delta g\pm f\smile g\pm g\smile f. \quad (2)$$

3 Harrison cohomology

Suppose now A is a commutative graded algebra and M is a module over A. The Harrison cochain complex $\bar{C}^*(A,M)$ is defined as a subcomplex of the Hochschild complex

$$\bar{C}^*(A,M)=\{f\in C^*(A,M)|\ f|Sh(A)=0\},$$

i.e. $\bar{C}^*(A,M)=Hom(Ch(A),M)=\sum_n Hom(Ch^n(A),M)$. In [1] it is shown that $\bar{C}^*(A,M)$ is closed with respect to the differential δ (that is if $f|Sh(A)=0$ then $\delta f|Sh(A)=0$). Harrison cohomology of a commutative algebra A with coefficients in an A-module M is defined as homology of the cochain complex $(\bar{C}^*(A,M),\delta)$. These cohomologies we denote by $Harr^*(A,M)$. As above each module $\bar{C}^n(A,M)=Hom(Ch^n(A),M)$ is graded by degrees of homomorphisms: $\bar{C}^n(A,M)=\sum_k \bar{C}^{n,k}(A,M)$ where $\bar{C}^{n,k}(A,M)=Hom^k(Ch^n(A),M)$. Besides $\delta : \bar{C}^{n,k}(A,M)\to \bar{C}^{n+1,k}(A,M)$, so $(\bar{C}^{*,k}(A,M),\delta)$ is a direct summand in the Harrison complex $(\bar{C}^n(A,M),\delta)$. Thus $Harr^n(A,M)=\sum_k Harr^{n,k}(A,M)$ where $Harr^{n,k}(A,M)$ is the n-th homology module of $(\bar{C}^{*,k}(A,M),\delta)$. So Harrison cohomology in is bigraded for graded A and M.

As it is mentioned in the section 1 the Harrison complex $(\bar{C}^{*,k}(A,M)$ is closed with respect to products $f\smile_1 g$ and $f\smile_1 (g,...,g)$. The formulae (1) and (2) are valid in this subcomplex too.

4 Twisting cochains in the Hochschild and Harrison complexes

In [2], [3] N. Berikashvilihas has defined a functor D from the category of differential algebras to the category of pointed sets, which have applications in the homology theory of fibrations. We recall shortly its definition. Let (C,d) be a differential graded algebra. A twisting cochain is defined as an element $a = a^2 + a^3 + ...,\ a^i \in C^i$ satisfying the condition $da = \pm a \cdot a$. Let $Tw(C)$ be the set of all twisting cochains. In this set by Berikashvili was introduced the following equivalence relation: $a \sim a$ if there exists an element $p = p^1 + p^2 + ...,\ p^i \in C^i$ such that

$$a - a` = p \cdot a \pm a \cdot p \pm dp.$$

The factorset of the set $Tw(C)$ by this equivalence is denoted by $D(C)$. We are going to introduce the similar definition in the Hochschild and Harrison complexes but with respect to $\smile_1$ product. Note that Hochschild and Hasrrison complexes are not differential algebras with respect to the product $f \smile_1 g$, besides this product is not associative, hence in order to define the functor D some modification is needed.

Let us define a twisting cochain in the Hochschild complex $C^{*,*}(A,A)$ as an element $a = a^{3,-1} + a^{4,-2} + ... + a^{i,2-i} + ...,\ a^{i,2-i} \in C^{i,2-i}(A,A)$ satisfying the condition $\delta a = a \smile_1 a$. Let $Tw(A,A)$ be the set of all twisting cochains. Now we introduce in this set the following equivalence relation: $a \sim a'$ if there exists an element $p = p^{2,-1} + p^{3,-2} + ... + p^{i,1-i} + ...,\ ,\ p^{i,1-i} \in C^{i,1-i}(A,A)$ such that

$$a - a' = \delta p \pm p \smile_1 a \pm a' \smile_1 p \pm a' \smile_1 (p,p) \pm a' \smile_1 (p,p,p) \pm ... \qquad (3)$$

(the sum is finite in each dimension). This is an equivalence relation; we denote by $D(A,A)$ the factorset $Tw(A,A)/\sim$. The set $D(A,A)$ is a pointed set: a distinguished point is the class of $a = 0$, which we denote by $0 \in D(A,A)$.

There is a possibility to perturb twisting cochains without changing their equivalence classes in $D(A,A)$. Indeed, let $A \in Tw(A,A)$ and $p \in C^{n,1-n}(A,A)$ be an arbitrary cochain, then there exists a twisting cochain $\bar{a} \in Tw(A,A)$ such that $a^i = \bar{a}^i$ for $i \leq n$, $\bar{a}^{n+1} = a^{n+1} + \delta p$ and $\bar{a} \sim a$. The twisting cochain $\bar{a}$ can be solved inductively from the equation (3).

Theorem 1 *If $Hoch^{n,2-n}(A,A) = 0$ for $n \geq 3$ then $D(A,A) = 0$.*

Proof. We have to show that in this case an arbitrary twisting cochain is equivalent to zero. From the equality $\delta a = a \smile_1 a$ in dimension $n = 3$ we obtain $\delta a^3 = 0$ that is $a^3 \in C^{3,-1}(A,A)$ is a cocycle. Since $Hoch^{3,-1}(A,A) = 0$ there exists $p^{2,-1} \in C^{2,-1}(A,A)$ such that $a^3 = \delta p^{2,-1}$. Perturbing our twisting cochain $a = a^3 + a^4 + ...$ by $p^{2,-1}$ we we obtain new twisting cochain $\bar{a} = \bar{a}^3 + \bar{a}^4 + ...$ equivalent to a and with $\bar{a}^3 = 0$. Now the component $\bar{a}^4$ becomes a cocycle, which, which can be killed using $Hoch^{4,-2}(A,A) = 0$ etc. This completes the proof.

Now turn to the Harrison complex $\bar{C}(A,A)$ of a commutative algebra A. Since $\bar{C}(A,A)$ is closed with respect to $\smile_1$ product, here we also can define twisting cochains and the set $\bar{D}(A,A)$ and prove the similar

Theorem 2 *If $Harr^{n,2-n}(A,A) = 0$ for $n \geq 3$ then $D(A,A) = 0$.*

5 Structure of $A(\infty)$-algebra and Hochschild and Harrison cohomology

$A(\infty)$-algebra was defined by Stasheff in[8]. It is a graded Λ-module $M = \sum M_q$ equipped with a sequence of operations - Λ-homomorphisms $\{m_i : \otimes^i M \to M,\ i = 1, 2, ...\}$ satisfying the following conditions

$$m_i(\otimes^i M)_q \subset M_{q-i+2},\ \ i.e.\ \ deg m_i = 2 - i; \tag{4}$$

$$\sum_{j=1}^{n}\sum_{k=0}^{n-j} \pm m_i(a_1 \otimes ... \otimes a_k \otimes m_j(a_{k+1} \otimes ... \otimes a_{k+j}) \otimes ... \otimes a_n) = 0. \tag{5}$$

A morphism of $A(\infty)$-algebras $f : (M, \{m_i\}) \to (M', \{m'_i\})$ is a sequence of homomorphisms $\{f_i : \otimes^i M \to M',\ i = 1, 2, ...\}$ satisfying the following conditions

$$f_i(\otimes^i M)_q \subset M'_{q-i+1},\ \ \ i.e.\ \ deg f_i = 1 - i; \tag{6}$$

$$\begin{array}{c}\sum_{j=1}^{n}\sum_{k=0}^{n-j} \pm f_i(a_1 \otimes ... \otimes a_k \otimes m_j(a_{k+1} \otimes ... \otimes a_{k+j}) \otimes ... \otimes a_n) = \\ \sum_{t=1}^{n}\sum_{k_1+...+k_t=n} \pm m'_t(f_{k_1}(a_1 \otimes ... \otimes a_{k_1}) \otimes ... \otimes f_{k_t}(a_{n-k_t+1} \otimes ... \otimes a_n)).\end{array} \tag{7}$$

The obtained category is denoted by $A(\infty)$.

For an arbitrary $A(\infty)$-algebra $(M, \{m_i\})$ the sequence of operations $\{m_i\}$ defines on the tensor coalgebra $T(M)$ a differential $d : T(M) \to T(M)$ given by

$$d(a_1 \otimes ... \otimes a_n) = \sum_{k,j} \pm a_1 \otimes ... \otimes a_k \otimes m_j(a_{k+1} \otimes ... \otimes a_{k+j}) \otimes ... \otimes a_n$$

which fits with the coproduct $\nabla : T(M) \to T(M) \otimes T(M)$, i.e. turns $T(M)$ into a differential coalgebra. This differential coalgebra $(T(M), d)$ is called $\tilde{B}$-construction of $A(\infty)$-algebra $(M, \{m_i\})$ and is denoted by $\tilde{B}(M, \{m_i\})$ ([8]).

An arbitrary morphism of $A(\infty)$-algebras $\{f_i\} : (M, \{m_i\}) \to (M', \{m'_i\})$ induces a DG-coalgebra morphism $\tilde{B}(f) : \tilde{B}(M, \{m_i\}) \to \tilde{B}(M', \{m'_i\})$ by

$$\tilde{B}(f)(a_1 \otimes ... \otimes a_n) = \sum_{t=1}^{n} \sum_{k_1+...+k_t=n} f_{k_1}(a_1 \otimes ... \otimes a_{k_1}) \otimes ... \otimes f_{k_t}(a_{n-k_t+1} \otimes ... \otimes a_n).$$

Thus $\tilde{B}$ is a functor from the category of $A(\infty)$-algebras to the category of DG-coalgebras.

We are interested in $A(\infty)$-algebras of type $(M, \{m_1 = 0, m_2, m_3, ...\})$, i.e. with $m_1 = 0$. The full subcategory of $A(\infty)$ which objects are such $A(\infty)$-algebras we denote by $A^0(\infty)$.

Now let (M, μ) be a graded associative algebra with multiplication $\mu : M \otimes M \to M$. Consider all possible $A(\infty)$ structures $\{m_i\}$ on M with $m_1 = 0,\ m_2 = \mu$. Two such structures we call equivalent if there exists a morphism of $A(\infty)$-algebras $\{p_i\} : (M, \{m_i\}) \to (M, \{m'_i\})$ for which the first component p_1 is the identity map (one can show that such morphisms are isomorphisms in the category $A^0(\infty)$). The obtained factorset we denote by $(M, \mu)(\infty)$. A trivial $A^0(\infty)$ structure we define as $\{m_i\}$ with $m_{i>2} = 0$. It's class we denote as $0 \in (M, \mu)(\infty)$.

Proposition 3 *The sets $(M,\mu)(\infty)$ and $D(M,M)$ are bijective.*

Proof. Let $(M,\{m_i\})$ be an $A(\infty)$-algebra with with $m_1=0,\ m_2=\mu$; we denote $m=m_3+m_4+...$. Each operation $m_i:\otimes^i M\to M$ can be interpreted as a Hochschild cochain from $C^i(M,M)$. It is easy to mention that the condition (5) means exactly $\delta m=m\smile_1 m$, i.e. $m\in Tw(M,M)$. Conversely, each twisting cochain $m=m_3+m_4+...\in Tw(M,M)$ defines on m an $A^0(\infty)$-algebra structure $(M,\{m_1=0,m_2=\mu,m_3,m_4,...\})$. It remains to show that two $A^0(\infty)$-structures $\{m_i\}$ and $\{m_i'\}$ are equivalent if and only if the twisting cochains m and m' are equivalent. Indeed, if $A^0(\infty)$-structures $\{m_i\}$ and $\{m_i'\}$ are equivalent, i.e. there exists an isomorphism of $A^0(\infty)$-algebras $\{id,p_2,p_3,...\}:(M,\{m_i\})\to(M,\{m_i'\})$, then the cochain $p=p_2+p_3+...$ realizes the equivalence of twisting cochains m and m', since the condition (7) is equivalent to (3). Conversely, if $p=p_2+p_3+...$ realizes the equivalence of twisting cochains m and m', then $\{id,p_2,p_3,...\}:(M,\{m_i\})\to(M,\{m_i'\})$ is the needed isomorphism.

From this proposition and the theorem 1 follows the

Corollary 4 *If for a graded algebra (M,μ) all $Hoch^{n,2-n}(M,M)=0$ for $n\geq 3$ then any $A^0(\infty)$-algebra structure $\{m_i\}$ on M (with $m_1=0,\ m_2=\mu$) is equivalent to trivial one.*

Now we turn to the commutative case.

An $A(\infty)$-algebra we call commutative if the sequence of operations $\{m_i\}$ apart of the conditions (4) and (5) satisfies $m_i|Sh^i(M)=0$. In this case the differential $d:T(M)\to T(M)$ defined by $\{m_i\}$ fits with shuffle product, so $\tilde{B}(M,\{m_i\})$ becomes a DG-Hopf algebra. A morphism of commutative $A(\infty)$-algebras $\{f_i\}:(M,\{m_i\})\to(M',\{m_i'\})$ we define as a morphism of $A(\infty)$-algebras which, apart of the conditions (6) and (7) satisfies $f_i|Sh^i(M)=0$. In this case $\tilde{B}(f):\tilde{B}(M,\{m_i\})\to\tilde{B}(M',\{m_i'\})$ becomes a map of DG-Hopf algebras (see [7]). The condition $m_i|Sh^i(M)=0$ for $i=2$ means that $m_2:M\otimes M\to M$ is commutative and all the operations $m_i:\otimes^i M\to M$ are from the Harrison subcomplex $\bar{C}(M,M)$.

Now let (M,μ) be a graded commutative algebra with multiplication $\mu:M\otimes M\to M$. Consider all possible commutative $A(\infty)$ structures $\{m_i\}$ on M with $m_1=0,\ m_2=\mu$; two such structures we call equivalent if there exists a morphism of commutative $A(\infty)$-algebras $\{p_1=id,p_2,p_3,...\}:(M,\{m_i\})\to(M,\{m_i'\})$ (which, as above, is an isomorphism). The obtained factorset we denote by $(M,\mu)(\infty)_c$. Exactly as above we obtain the

Proposition 5 *The sets $(M,\mu)(\infty)_c$ and $\bar{D}(M,M)$ are bijective.*

Corollary 6 *If for a graded commutative algebra (M,μ) all $Harr^{n,2-n}(M,M)=0$ for $n\geq 3$ then any commutative $A^0(\infty)$-algebra structure $\{m_i\}$ on M (with $m_1=0,\ m_2=\mu$)is equivalent to trivial one.*

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