THE TWISTED CARTESIAN MODEL FOR THE DOUBLE PATH FIBRATION

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Abstract. In the paper the notion of truncating twisting function from a cubical set to a permutahedral set and the corresponding notion of twisted Cartesian product of these sets are introduced. The latter becomes a permutocubical set that models in particular the path fibration on a loop space. The chain complex of this twisted Cartesian product in fact is a comultiplicative twisted tensor product of cubical chains of base and permutahedral chains of fibre. This construction is formalized as a theory of twisted tensor products for Hirsch algebras.

1. introduction

The paper continues [13] in which a combinatorial model for a fibration was constructed based on the notion of a *truncating twisting function* from a simplicial set to a cubical set and on the corresponding notion of twisted Cartesian product of these sets being a cubical set. Applying the cochain functor we obtained a *multiplicative* twisted tensor product modeling the corresponding fibration.

There arises a need to iterate this construction for fibrations over loop or path spaces the bases of which are modeled by cubical sets. A cubical base naturally requires a permutahedral fibre; this really agrees with the first usage of the permutahedra (the Zilchgons) as modeling polytopes for loops on the standard cube due to R.J. Milgram [17] (see also [8]).

For this we proceed almost parallel to [13]. Namely, let Q be a 1-reduced cubical set, $\mathcal{Z}$ a monoidal permutahedral set, and $\mathcal{L}$ a permutahedral $\mathcal{Z}$-module, i.e., $\mathcal{Z}$ and $\mathcal{L}$ are permutahedral sets with given associative permutahedral maps $\mathcal{Z}\times\mathcal{Z}\to\mathcal{Z}$ and $\mathcal{Z}\times\mathcal{L}\to\mathcal{L}$ (see [19] or Section 2 below). We introduce the notion of *truncating twisting function* $\vartheta : Q_* \to \mathcal{Z}_{*-1}$ from a cubical set to a monoidal permutahedral set (the term *truncating* comes from the universal example $\vartheta_U : I^n \to P_n$ of such functions obtained by the standard truncation procedure, see Section 4 below). Such a twisting function ϑ defines the twisted Cartesian product $Q\times_\vartheta \mathcal{L}$ as a *permutocubical set*. The *permutocube* is defined as a polytope which is obtained from the standard cube by a certain truncation procedure due to N. Berikashvili [6], see also bellow. The permutocube can be thought of as a modeling polytope for paths on the cube.

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We construct a functor which assigns to a cubical set Q a monoidal permutahedral set $\mathbf{\Omega}Q$ and present a truncating twisting function $\vartheta_U : Q \to \mathbf{\Omega}Q$ of degree -1 which is universal in the following sense: Given an arbitrary truncating function $\vartheta : Q_* \to \mathcal{Z}_{*-1}$, there is a monoidal permutahedral map $f_\vartheta : Q \to \mathcal{Z}$ such that $\vartheta = f_\vartheta \vartheta_U$. The twisted Cartesian product $\mathbf{P}Q = Q \times_{\vartheta_U} \mathbf{\Omega}Q$ is a permutocubical set functorially depending on Q. Note that $\mathbf{\Omega}Q$ models the loop space $\Omega|Q|$ and $\mathbf{P}Q$ models the path fibration on $|Q|$.

The chain complex $C_*^{\diamondsuit}(\mathbf{\Omega}Q)$ coincides with the cobar construction $\Omega C_*^{\square}(Q)$. Similarly, the chain complex $C_*^{\boxminus}(Q \times_{\vartheta_U} \mathbf{\Omega}Q)$ coincides with the acyclic cobar construction $\Omega\left(C_*^{\square}(Q)\,;C_*^{\square}(Q)\right)$; furthermore, $\vartheta_* = C_*(\vartheta) : C_*^{\square}(Q) \to C_{*-1}^{\diamondsuit}(\mathcal{Z})$ is a twisting cochain and $C_*^{\boxminus}(Q \times_\vartheta \mathcal{L})$ coincides with the twisted tensor product $C_*^{\square}(Q) \otimes_{\vartheta_*} C_*^{\diamondsuit}(\mathcal{L})$.

We construct an explicit diagonal for the permutocube B_n which agrees with that of P_n [19] by means of the natural embedding $P_n \to B_n$. The equalities $C_*^{\diamondsuit}(\mathbf{\Omega}Q) = \Omega C_*^{\square}(Q)$ and $C_*^{\boxminus}(Q \times_\vartheta \mathcal{L}) = C_*^{\square}(Q) \otimes_{\vartheta_*} C_*^{\diamondsuit}(\mathcal{L})$ allow us to transport these diagonals to the cobar construction $\Omega C_*^{\square}(Q)$ and the twisted tensor product $C_*^{\square}(Q) \otimes_{\vartheta_*} C_*^{\diamondsuit}(\mathcal{L})$ respectively. Dually, we immediately obtain a multiplication on $C^*_{\square}(Q) \otimes_{\vartheta^*} C^*_{\diamondsuit}(\mathcal{L}) \subset C^*_{\boxminus}(Q \times_\vartheta \mathcal{L})$ (which is an equality if the graded sets are of finite type). Note that this (co)multiplication is not strictly (co)associative but could be extended to an A_∞-(co)algebra structure.

Next we express the resulting comultiplication on $C_*^{\square}(Q) \otimes_{\vartheta_*} C_*^{\diamondsuit}(\mathcal{L})$ in terms of certain chain operations of degree $p+q-1$:

$$\{E^{p,q} : C_*^{\square}(Q) \to C_*^{\square}(Q)^{\otimes p} \otimes C_*^{\square}(Q)^{\otimes q}\}_{p+q>0},$$

which give $C_*^{\square}(Q)$ a structure what we call a *Hircsh coalgebra structure.* This structure is a consequence of the permutahedral diagonal on $C_*^{\diamondsuit}(\mathbf{\Omega}Q) = \Omega C_*^{\square}(Q)$: The permutahedral diagonal from [19] induces the diagonal $\Omega C_*^{\square}(Q) \to \Omega C_*^{\square}(Q) \otimes \Omega C_*^{\square}(Q)$ being a multiplicative map, thus it extends a certain homomorphism $C_*^{\square}(Q) \to \Omega C_*^{\square}(Q) \otimes \Omega C_*^{\square}(Q)$, which itself consists of components $E^{p,q} : C_*^{\square}(Q) \to C_*^{\square}(Q)^{\otimes p} \otimes C_*^{\square}(Q)^{\otimes q}$, $p,q \geq 0$. The operation $E^{1,1}$ is dual to the cubical version of Steenrod's $\smile_1$-cochain operation; thus when $E^{1,1} = 0$ a Hirsch coalgebra specializes to a cocommutative dg coalgebra (and dually for *Hirsch algebras*).

Towards the end of the paper we develop the theory of multiplicative twisted tensor products for Hirsch algebras, which provides a general algebraic framework for our multiplicative model of a fibration. A Hirsch algebra we define as an object $(A, d, \cdot, \{E_{p,q} : A^{\otimes p} \otimes A^{\otimes q} \to A\}_{p+q>0})$, i.e., $(A, d, \cdot)$ is an associative dga and the sequence of operations $\{E_{p,q}\}$ determines a product on the bar construction BA turning it into a dg Hopf algebra (this multiplication can be viewed as a perturbation of the shuffle product and is not necessarily associative). In particular $E_{1,1}$ has properties similar to $\smile_1$ product, so that a Hirsch algebra can be considered as to have a structure measuring the lack of commutativity of A. Let C be a dg Hopf algebra and M be a dga and a dg C-comodule simultaneously. We say that a twisting cochain $\phi : C \to A$ is *multiplicative* if the induced map $C \to BA$ is a dg Hopf algebra map. We introduce on $A \otimes_\phi M$ a twisted multiplication μ_ϕ in terms of ϕ and the Hirsch algebra structure of A by the same formulas as in the case $A = C^*_{\square}(Q)$, $C = C^*_{\diamondsuit}(\mathcal{Z})$ and $M = C^*_{\diamondsuit}(\mathcal{L})$; then $\phi = \vartheta^* : C^*_{\diamondsuit}(\mathcal{Z}) \to C^{*+1}_{\square}(Q)$ provides a basic example of a multiplicative twisting cochain.

Applying our machinery to a fibration $F \to E \to Y$ on a 1-connected space Y and an associated principal G-fibration $G \to P \to Y$ with action $G \times F \to F$ we obtain the following permutocubical model (Theorem 8.1): Let $Q = \mathrm{Sing}^{1^I} Y \subset \mathrm{Sing}^I Y$ be the Eilenberg 1-subcomplex generated by singular cubes that send the 1-skeleton of the standard n-cube I^n to the base point of Y. Let $\mathcal{Z} = \mathrm{Sing}^M G$ and $Y = \mathrm{Sing}^M F$ be the singular multipermutahedral sets (see [19] and Section 2). We construct the Adams-Milgram map

$$\omega_* : \Omega C_*^{\square}(Q) \to C_*^{\diamondsuit}(\Omega Y)$$

which is realized by a monoidal multipermutahedral map $\omega : \mathbf{\Omega} Q \to \mathrm{Sing}^M \Omega Y$. Composing ω with a map of monoidal multipermutahedral sets $\mathrm{Sing}^M \Omega Y \to \mathrm{Sing}^M G = \mathcal{Z}$ induced by the canonical map $\Omega Y \to G$ of monoids we immediately obtain a truncating twisting function $\vartheta : Q \to \mathcal{Z}$. The resulting twisted Cartesian product $\mathrm{Sing}^{1^I} Y \times_\vartheta \mathrm{Sing}^M F$ provides the required permutocubical model of E; and there exists a permutocubical weak equivalence $\mathrm{Sing}^{1^I} Y \times_\vartheta \mathrm{Sing}^M F \to \mathrm{Sing}^B E$, where Sing^B denotes the singular permutocubical complex of a space. Applying the cochain functor we obtain a certain multiplicative twisted tensor product for the fibration.

In particular, we can obtain a permutocubical model for the path fibration $\Omega^2 Y' \to P\Omega Y' \to \Omega Y'$ in the following way. Taking for the base $Y = \Omega Y'$ the cubical model $Q = \mathbf{\Omega}\, \mathrm{Sing}^2 Y'$ from [13] the above machinery yields the twisted Cartesian model $\mathbf{\Omega}\, \mathrm{Sing}^2 Y' \times_{\vartheta_U} \mathbf{\Omega\Omega}\, \mathrm{Sing}^2 Y'$ being a *permutocubical* set.

Consequently, we introduce the multiplication on the acyclic bar construction $B\,(BC^*(Y); BC^*(Y))$ whose restriction to the double bar construction $BBC^*(Y)$ is just the one constructed in [19].

To summarize we observe the following. In [13] it is indicated the homotopy G-algebra structure on $C^*(Y)$ consisting of cochain operations

$$\{E_{k,1} : C^*(Y)^{\otimes k} \otimes C^*(Y) \to C^*(Y)\}_{k \geq 1},$$

defining a multiplication on $BC^*(Y)$. Here we extend this multiplication to the structure of Hirsch algebra on $BC^*(Y)$, i.e., to operations

$$\{E_{p,q} : (BC^*(Y))^{\otimes p} \otimes (BC^*(Y))^{\otimes q} \to BC^*(Y)\}_{p+q>0},$$

which actually are cochain operations of type $C^*(Y)^{\otimes m} \to C^*(Y)^{\otimes n}$. This two sets of operations including in particular $\smile$, $\smile_1$ and $\smile_2$ operations, allow us to construct multiplicative models for ΩY, $\Omega^2 Y$ and multiplicative twisted tensor products for path fibrations on Y and ΩY as well as for fibrations associated with them.

As an example we present fibrations with the base being the loop space on a double suspension (in this case the Hirsch algebra structure consists just of $E_{1,1} = \smile_1$ and all other operations $E_{p,q}$ are trivial) and for which the formula for the multiplication in the twisted tensor product has a very simple form. Moreover, in this case we present small multiplicative model being the twisted tensor product of cohomologies of base and fiber with the multiplicative structure purely defined by the $\smile$, $\smile_1$ and $\smile_2$ operations.

Finally, we mention that the geometric realization $|\mathbf{\Omega\Omega}\, \mathrm{Sing}^2 Y|$ of $\mathbf{\Omega\Omega}\, \mathrm{Sing}^2 Y$ is homeomorphic to the cellular model for the double loop space due to G. Carlsson and R. J. Milgram [8] and is homotopically equivalent to the cellular model due to H.-J. Baues [3].

The paper is organized as follows. We adopt the notions and the terminology from [13]; note that here a (co)algebra need not have a (co)associative (co)multiplication if it is not specially emphasized. In Section 2 we construct the functor $\mathbf{\Omega}$ from the category of cubical sets to the category of permutahedral sets; Section 3 introduces the permutocubes; in Section 4 we introduce the notion of a permutocubical set; Section 5 introduces the notion of a truncating twisting function and the resulting twisted Cartesian product; in Section 6 we define an explicit diagonal on the permutocubes; in Section 7 we build the permutocubical set model for the double path fibration; in Section 8 a permutocubical model and the corresponding multiplicative twisted tensor product for a fibration are constructed, and, finally, in Section 9 the twisted tensor product theory for Hirsch algebras is developed.

2. The permutahedral set functor $\mathbf{\Omega}Q$

For completeness we first recall some basic facts about permutahedral sets from [19] (compare, [14]).

2.1. **Permutahedral sets.** This subsection introduces the notion of a permutahedral set $\mathcal{Z}$, which is a combinatorial object generated by permutahedra and equipped with appropriate face and degeneracy operators. We construct the generating category $\mathbf{P}$ and show how to lift the diagonal on the permutahedra P constructed above to a diagonal on $\mathcal{Z}$. Naturally occurring examples of permutahedral sets include the double cobar construction, i.e., Adams' cobar construction [1] on the cobar with coassociative coproduct [3], [8], [13]. Permutahedral sets are distinguished from simplicial or cubical sets by their higher order structure relations. While our construction of $\mathbf{P}$ follows the analogous (but not equivalent) construction for polyhedral sets given by D.W. Jones in [12], there is no mention of structure relations in [12].

Let S_n be the symmetric group on $\underline{n} = \{1, 2, \ldots, n\}$. Recall that the permutahedron P_n is the convex hull of $n!$ vertices $(\sigma(1), \ldots, \sigma(n)) \in \mathbb{R}^n$, $\sigma \in S_n$ [9], [17]. As a cellular complex, P_n is an $(n-1)$-dimensional convex polytope whose $(n-p)$-faces are indexed by (ordered) partitions $U_1|\cdots|U_p$ of $\underline{n}$. We shall define the permutahedra inductively as subdivisions of the standard n-cube I^n. With this representation the combinatorial connection between faces and partitions is immediately clear.

Assign the label $\underline{1}$ to the single point P_1. If P_{n-1} has been constructed and $u = U_1|\cdots|U_p$ is one of its faces, form the sequence $u_* = \{u_0 = 0, u_1, \ldots, u_{p-1}, u_p = \infty\}$ where $u_j = \#\left(U_{p-j+1} \cup \cdots \cup U_p\right)$, $1 \leq j \leq p-1$ and $\#$ denotes cardinality. Define the *subdivision of I relative to u* to be

$$I/u_* = I_1 \cup I_2 \cup \cdots \cup I_p,$$

where $I_j = \left[1 - \frac{1}{2^{u_{j-1}}}, 1 - \frac{1}{2^{u_j}}\right]$ and $\frac{1}{2^\infty} = 0$. Then

$$P_n = \bigcup_{u \in P_{n-1}} u \times I/u_*$$

with faces labeled as follows (see Figures 1 and 2):

Face of $u \times I/u_*$	**Partition of** $\underline{n}$
$u \times 0$	$U_1 \vert \cdots \vert U_p \vert n$
$u \times (I_j \cap I_{j+1})$	$U_1 \vert \cdots \vert U_{p-j} \vert n \vert U_{p-j+1} \vert \cdots \vert U_p, \quad 1 \leq j \leq p-1$
$u \times 1$	$n \vert U_1 \vert \cdots \vert U_p$
$u \times I_j$	$U_1 \vert \cdots \vert U_{p-j+1} \cup n \vert \cdots \vert U_p.$

A *cubical* vertex of P_n is a vertex common to both P_n and I^{n-1}. Note that u is a cubical vertex of P_{n-1} if and only if $u|n$ and $n|u$ are cubical vertices of P_n. Thus the cubical vertices of P_3 are $1|2|3$, $2|1|3$, $3|1|2$ and $3|2|1$ since $1|2$ and $2|1$ are cubical vertices of P_2.

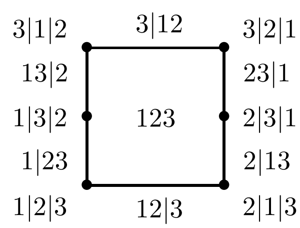


Figure 1: P_3 as a subdivision of $P_2 \times I$.

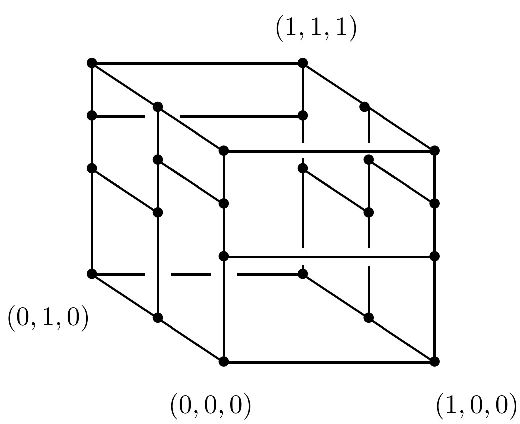


Figure 2a: P_4 as a subdivision of $P_3 \times I$.

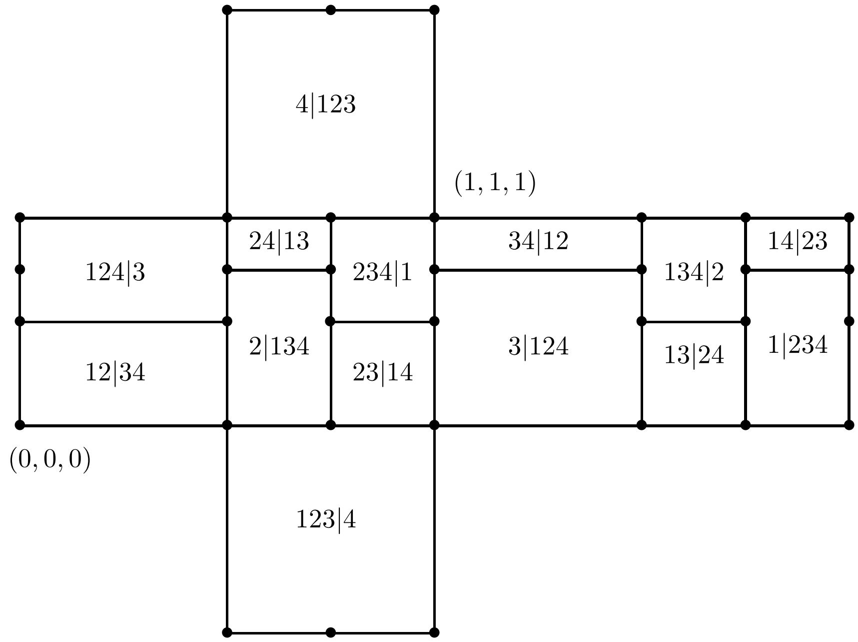


Figure 2b: The 2-faces of P_4.

2.2. **Singular Permutahedral Sets.** By way of motivation we begin with constructions of two singular permutahedral sets–our universal examples. Whereas the first emphasizes coface and codegeneracy operators, the second emphasizes cellular chains and is appropriate for homology theory. We begin by constructing the various maps we need to define singular coface and codegeneracy operators.

Fix a positive integer n. For $0 \leq p \leq n$, let

$$\underline{p} = \begin{cases} \varnothing, & p=0 \\ \{1,\ldots,p\}, & 1 \leq p \leq n \end{cases} \quad \text{and} \quad \overline{p} = \begin{cases} \varnothing, & p=0 \\ \{n-p+1,\ldots,n\}, & 1 \leq p \leq n; \end{cases}$$

then $\underline{p}$ and $\overline{p}$ contain the first and last p elements of $\underline{n}$, respectively; note that $\underline{p} \cap \overline{q} = \{p\}$ whenever $p+q = n+1$. Given integers $r,s \in \underline{n}$ such that $r+s = n+1$, there is a canonical projection $\Delta_{r,s} : P_n \to P_r \times P_s$ whose restriction to a vertex $v = a_1|\cdots|a_n \in P_n$ is given by

$$\Delta_{r,s}(v) = b_1|\cdots|b_r \times c_1|\cdots|c_s,$$

where $(b_1,\ldots,b_r; c_1,\ldots,c_{k-1},c_{k+1},\ldots,c_s)$ is the unshuffle of $(a_1,\ldots,a_n)$ with $b_i \in \underline{r}$, $c_j \in \overline{s}$, $c_k = r$. For example, $\Delta_{2,3}(2|4|1|3) = 2|1 \times 2|4|3$ and $\Delta_{3,2}(2|4|1|3) = 2|1|3 \times 4|3$. Since the image of the vertices of a cell of P_n uniquely determines a cell in $P_r \times P_s$ the map $\Delta_{r,s}$ is well-defined and cellular. Furthermore, the restriction of $\Delta_{r,s}$ to an $(n-k)$-cell $A_1|\cdots|A_k \subset P_n$ is given by

$$\Delta_{r,s}(A_1|\cdots|A_k) = \begin{cases} \underline{r} \times \left(A_1|\cdots|A_i \setminus \underline{r-1}\,|\cdots|A_k\right), & \text{if } \underline{r} \subseteq A_i, \text{ some } i, \\[2ex] \left(A_1|\cdots|A_j \setminus \overline{s-1}\,|\cdots|A_k\right) \times \overline{s}, & \text{if } \overline{s} \subseteq A_j, \text{ some } j, \\[2ex] \begin{array}{l}\left(A_1 \setminus \overline{s-1}\,|\cdots|A_k \setminus \overline{s-1}\right) \\ \qquad \times \left(A_1 \setminus \underline{r-1}\,|\cdots|A_k \setminus \underline{r-1}\right),\end{array} & \text{otherwise.} \end{cases}$$

Note that $\Delta_{r,s}$ acts homeomorphically in the first two cases and degeneratively in the third when $1 < k < n$. When $n = 3$ for example, $\Delta_{2,2}$ maps the edge $1|23$ onto the edge $1|2 \times 23$ and the edge $13|2$ onto the vertex $1|2 \times 3|2$ (see Figure 3).

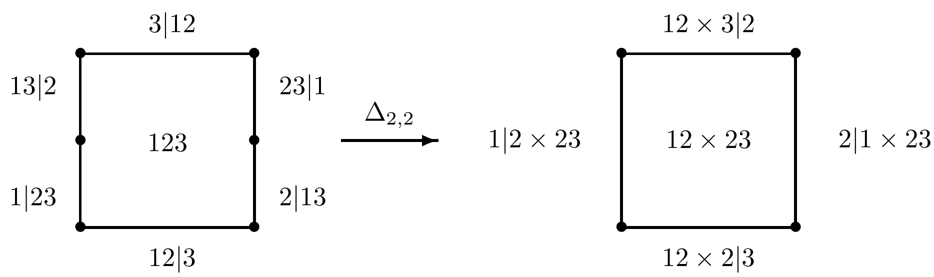


Figure 3: The projection $\Delta_{2,2} : P_3 \to I^2$.

Now identify the set $U = \{u_1 < \cdots < u_n\}$ with P_n and the ordered partitions of U with the faces of P_n in the obvious way. Then $(\Delta_{r,s} \times 1) \circ \Delta_{r+s-1,t} = (1 \times \Delta_{s,t}) \circ \Delta_{r,s+t-1}$ whenever $r + s + t = n + 2$ so that $\Delta_{*,*}$ acts coassociatively with respect to Cartesian product. It follows that each k-tuple $(n_1, \ldots, n_k) \in \mathbb{N}^k$ with $k \geq 2$ and $n_1 + \cdots + n_k = n + k - 1$ uniquely determines a cellular projection $\Delta_{n_1 \cdots n_k} : P_n \to P_{n_1} \times \cdots \times P_{n_k}$ given by the composition

$$\Delta_{n_1 \cdots n_k} = \left(\Delta_{n_1, n_2} \times 1^{\times k-2}\right) \circ \cdots \circ \left(\Delta_{n_{(k-2)}-k+3, n_{k-1}} \times 1\right) \circ \Delta_{n_{(k-1)}-k+2, n_k},$$

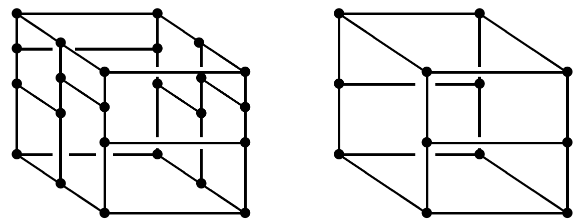
where $n_{(q)} = n_1 + \cdots + n_q$; and in particular,

(1) $\Delta_{n_1 \cdots n_k}(\underline{n}) = \underline{n_1} \times \underline{n_{(2)} - 1} \setminus \underline{n_1 - 1} \times \cdots \times \underline{n_{(k)} - (k-1)} \setminus \underline{n_{(k-1)} - (k-1)}.$

Note that formula 1 with $k = n - 1$ and $n_i = 2$ for all i defines a projection $\rho_n : P_n \to I^{n-1}$

$$\rho_n(\underline{n}) = \Delta_{2 \cdots 2}(\underline{n}) = 12 \times 23 \times \cdots \times \{n-1, n\}$$

(see Figure 4) acting on a vertex $u = u_1 | \cdots | u_n$ as follows: For each $i \in \underline{n-1}$, let $\{u_j, u_k \mid j < k\} = \{u_1, \ldots, u_n\} \cap \{i, i+1\}$ and set $v_i = u_j$, $v_{i+1} = u_k$; then $\rho_n(u) = v_1 | v_2 \times \cdots \times v_{n-1} | v_n$.



$$\begin{tikzcd} 1234 \arrow[r, "\Delta_{2,3}"] \arrow[d, "\Delta_{3,2}"'] & 12\times 234 \arrow[d, "1\times\Delta_{2,2}"] \\ 123\times 34 \arrow[r, "\Delta_{2,2}\times 1"'] & 12\times 23\times 34 \end{tikzcd}$$

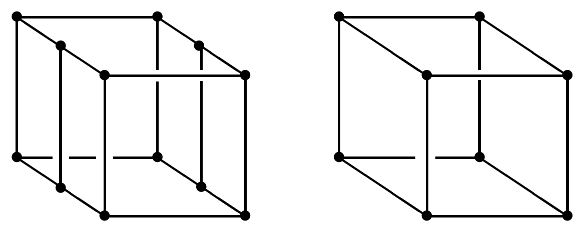


Figure 4: The projection $\rho_4 : P_4 \to I^3$.

Now choose a (non-cellular) homeomorphism $\gamma_n : I^{n-1} \to P_n$ whose restriction to a vertex $v = v_1|v_2 \times \cdots \times v_{n-1}|v_n$ can be expressed inductively as follows: Set $A_2 = v_1|v_2$; if A_{k-1} has been obtained from $v_1|v_2 \times \cdots \times v_{k-2}|v_{k-1}$, set

$$A_k = \begin{cases} A_{k-1}|k, & \text{if } v_k = k, \\ k|A_{k-1}, & \text{otherwise.} \end{cases}$$

For example, $\gamma_4\,(2|1 \times 3|2 \times 3|4) = 3|2|1|4$. Then γ_n sends the vertices of I^{n-1} to cubical vertices of P_n and the vertices of P_n fixed by $\gamma_n\rho_n$ are exactly its cubical vertices. Given a codimension 1 face $A|B \subset P_n$, index the elements of A and B as follows: If $n \in A$, write $A = \{a_1 < \cdots < a_m\}$ and $B = \{b_1 < \cdots < b_\ell\}$; if $n \in B$, write $A = \{a_1 < \cdots < a_\ell\}$ and $B = \{b_1 < \cdots < b_m\}$. Then $A|B$ uniquely embeds in P_n as the subcomplex

$$P_\ell \times P_m = \begin{cases} a_1|\cdots|a_m|B \times A|b_1|\cdots|b_\ell, & \text{if } n \in A \\ A|b_1|\cdots|b_m \times a_1|\cdots|a_\ell|B, & \text{if } n \in B. \end{cases}$$

For example, $14|23$ embeds in P_4 as $1|4|23 \times 14|2|3$. Let $\iota_{A|B} : A|B \hookrightarrow P_\ell \times P_m$ denote this embedding and let $h_{A|B} = \iota_{A|B}^{-1}$; then $h_{A|B} : P_\ell \times P_m \to A|B$ is an orientation preserving homeomorphism. Also define the cellular projection

$$\phi_{A|B} : P_n \to P_\ell \times P_m = \begin{cases} b_1 \cdots b_\ell \times a_1 \cdots a_m, & \text{if } n \in A \\ a_1 \cdots a_\ell \times b_1 \cdots b_m, & \text{if } n \in B \end{cases}$$

on a vertex $c = c_1|\cdots|c_n$ by $\phi_{A|B}\,(c) = u_1|\cdots|u_\ell \times v_1|\cdots|v_m$, where $(u_1, \ldots, u_\ell; v_1, \ldots, v_m)$ is the unshuffle of $(c_1, \ldots, c_n)$ with $u_i \in B$, $v_j \in A$ when $n \in A$ or with $u_i \in A$, $v_j \in B$ when $n \in B$. Note that unlike $\Delta_{r,s}$, the projection $\phi_{A|B}$ always degenerates on the top cell; furthermore, $\phi_{A|B} \circ h_{A|B} = \phi_{B|A} \circ h_{A|B} = 1$. We note that when A or B is a singleton set, the projection $\phi_{A|B}$ was defined by R.J. Milgram in [17].

The *singular codegeneracy operator associated with* $A|B$ is the map $\beta_{A|B} : P_n \to P_{n-1}$ given by the composition

$$P_n \xrightarrow{\phi_{A|B}} P_\ell \times P_m \xrightarrow{\rho_\ell \times \rho_m} I^{\ell-1} \times I^{m-1} = I^{n-2} \xrightarrow{\gamma_{n-1}} P_{n-1};$$

the *singular coface operator associated with* $A|B$ is the map $\delta_{A|B} : P_{n-1} \to P_n$ given by the composition

$$P_{n-1} \xrightarrow{\rho_{n-1}} I^{n-2} = I^{\ell-1} \times I^{m-1} \xrightarrow{\gamma_\ell \times \gamma_m} P_\ell \times P_m \xrightarrow{h_{A|B}} A|B \overset{i}{\hookrightarrow} P_n.$$

Unlike the simplicial or cubical case, $\delta_{A|B}$ need not be injective. We shall often abuse notation and write $h_{A|B} : P_\ell \times P_m \to P_n$ when we mean $i \circ h_{A|B}$.

We are ready to define our first universal example. For future reference and to emphasize the fact that our definition depends only on positive integers, let $(n_1, \ldots, n_k) \in \mathbb{N}^k$ such that $n_{(k)} = n$ and denote

$$\mathcal{P}_{n_1 \cdots n_k}(n) = \{\textit{Partitions } A_1|\cdots|A_k \textit{ of } \underline{n} \mid \#A_i = n_i\}.$$

Definition 2.1. *Let Y be a topological space. The* <u>*singular permutahedral set of Y*</u> *consists of the singular set*

$$\mathrm{Sing}_*^P Y = \bigcup_{n \geq 1} \left[\mathrm{Sing}_n^P Y = \{\textit{Continuous maps } P_n \to Y\}\right]$$

together with singular face and degeneracy operators

$$d_{A|B} : \mathrm{Sing}_n^P Y \to \mathrm{Sing}_{n-1}^P Y \quad \textit{and} \quad \varrho_{A|B} : \mathrm{Sing}_{n-1}^P Y \to \mathrm{Sing}_n^P Y$$

defined respectively for each $n \geq 2$ and $A|B \in \mathcal{P}_{**}(n)$ as the pullback along $\delta_{A|B}$ and $\beta_{A|B}$, i.e., for $f \in \mathrm{Sing}_n^P Y$ and $g \in \mathrm{Sing}_{n-1}^P Y$,

$$d_{A|B}(f) = f \circ \delta_{A|B} \quad \textit{and} \quad \varrho_{A|B}(g) = g \circ \beta_{A|B}.$$

$$\begin{tikzcd}
\delta_{A|B} : P_{n-1} \arrow[r] \arrow[drrrr, "d_{A|B}(f)"'] & I^{n-1} \arrow[r] & P_\ell \times P_m \arrow[r] & A|B \arrow[r, hook] & P_n \arrow[d, "f"] \\
 & & & & Y
\end{tikzcd}$$

Figure 5: The singular face operator associated with $A|B$.

Although coface operators $\delta_{A|B} : P_{n-1} \to P_n$ need not be inclusions, the top cell of P_{n-1} is always non-degenerate; however, the top cell of P_{n-2} may degenerate under quadratic compositions $\delta_{A|B}\delta_{C|D} : P_{n-2} \to P_n$. For example, $\delta_{12|34}\delta_{13|2} : P_2 \to P_4$ is a constant map, since $\delta_{12|34} : P_3 \to P_2 \times P_2 \hookrightarrow P_4$ sends the edge $13|2$ to the vertex $1|2 \times 3|2$.

Definition 2.2. *A quadratic composition of face operators $d_{C|D}d_{A|B}$ <u>acts on P_n</u> if the top cell of P_{n-2} is non-degenerate under the composition*

$$\delta_{A|B}\delta_{C|D} : P_{n-2} \to P_n.$$

For comparison, quadratic compositions of simplicial or cubical face operators always act on the simplex or cube. When $d_{C|D}d_{A|B}$ acts on P_n, we assign the label $d_{C|D}d_{A|B}$ to the codimension 2 face $\delta_{A|B}\delta_{C|D}\,(\underline{n})$. The various paths of descent from the top cell to a cell in codimension 2 gives rise to relations among compositions of face and degeneracy operators (see Figure 6).

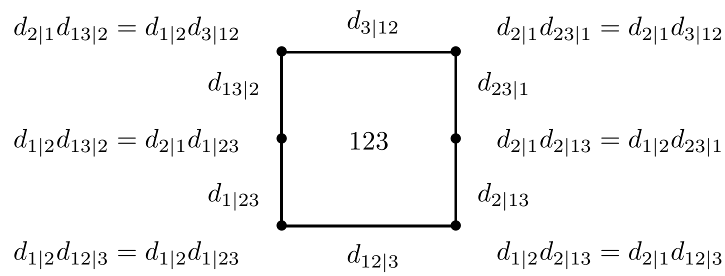


Figure 6: Quadratic relations on the vertices of P_3.

It is interesting to note that singular permutahedral sets have higher order structure relations, an example of which appears below in Figure 7 (see also (4)). This distinguishes permutahedral sets from simplicial or cubical sets in which relations are strictly quadratic. Our second universal example, called a "singular multipermutahedral set," specifies a singular permutahedral set by restricting to maps $f = \bar{f} \circ \Delta_{n_1 \cdots n_k}$ for some continuous $\bar{f} : P_{n_1} \times \cdots \times P_{n_k} \to Y$. Face and degeneracy operators satisfy those relations above in which $\Delta_{n_1 \cdots n_k}$ plays no essential role.

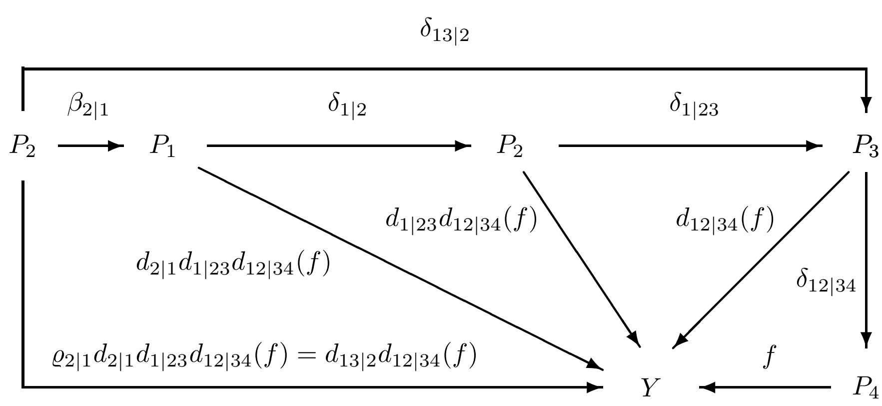


Figure 7: A quartic relation in $\mathrm{Sing}^P_* Y$.

Once again, fix a positive integer n, but this time consider $(n_1, \ldots, n_k) \in (\mathbb{N} \cup 0)^k$ with $n_{(k)} = n - 1$ and the projection $\Delta_{n_1+1 \cdots n_k+1} : P_n \to P_{n_1+1} \times \cdots \times P_{n_k+1}$ with

$\Delta_n : P_n \to P_n$ defined to be the identity. Given a topological space Y, let

$$\mathrm{Sing}^{n_1\cdots n_k} Y = \left\{ \bar f \circ \Delta_{n_1+1\cdots n_k+1} : P_n \to Y \mid \bar f \textit{ is continuous} \right\};$$

define $f, f' \in \mathrm{Sing}^{n_1\cdots n_k} Y$ to be equivalent if there exists $g : P_{n_1+1} \times \cdots \times P_{n_{i-1}+1} \times P_1 \times P_{n_{i+1}+1} \times \cdots \times P_{n_k+1} \to Y$ for some $i < k$ such that

$$f = g \circ (1^{\times i-1} \times \phi_{\underline{n_i+1}|n_i+1} \times 1^{\times k-i-1}) \circ \Delta_{n_1+1\cdots n_{i-1}+1, n_i+2, n_{i+2}+1\cdots n_k+1}$$

and

$$f' = g \circ (1^{\times i} \times \phi_{1|\underline{n_{i+2}+1}\backslash 1} \times 1^{\times k-i-2}) \circ \Delta_{n_1+1\cdots n_i+1, n_{i+2}+2, n_{i+3}+1\cdots n_k+1},$$

in which case we write $f \sim f'$. The geometry of the cube motivates this equivalence; the degeneracies in the product of cubical sets implies the identification (c.f. [15] or the definition of the cubical set functor $\mathbf{\Omega} X$ in [13]).

Define the singular set

$$\mathrm{Sing}_n^M Y = \bigcup_{\substack{(n_1,\ldots,n_k)\in(\mathbb{N}\cup 0)^k \\ n_{(k)}=n-1}} \mathrm{Sing}^{n_1\cdots n_k} Y / \sim .$$

Singular face and degeneracy operators

$$d_{A|B} : \mathrm{Sing}_n^M Y \to \mathrm{Sing}_{n-1}^M Y \quad \text{and} \quad \varrho_{A|B} : \mathrm{Sing}_{n-1}^M Y \to \mathrm{Sing}_n^M Y$$

are defined piece-wise for each $n \geq 2$ and $A|B \in \mathcal{P}_{*,*}(n)$, depending on the form of $A|B$. More precisely, for each pair of integers (p_i, q_i), $1 \leq i \leq k$, with

$$p_i = 1 + \sum_{j=1}^{i-1} n_j \quad \text{and} \quad q_i = 1 + \sum_{j=i+1}^{k} n_j, \quad \text{let}$$

$$\mathcal{Q}_{p_i,q_i}(n) = \left\{ U|V \in \mathcal{P}_{*,*}(n) \mid \left(\underline{p_i} \subseteq U \text{ or } \underline{p_i} \subseteq V \right) \text{ and } \left(\overline{q_i} \subseteq U \text{ or } \overline{q_i} \subseteq V \right) \right\};$$

in particular, when $r + s = n + 1$, set $k = 2$, $p_1 = q_2 = 1$, $p_2 = r$ and $q_1 = s$, then

$$\mathcal{Q}_{r,1}(n) = \{ U|V \in \mathcal{P}_{*,*}(n) \mid \underline{r} \subseteq U \text{ or } \underline{r} \subseteq V \} \text{ and}$$
$$\mathcal{Q}_{1,s}(n) = \{ U|V \in \mathcal{P}_{*,*}(n) \mid \overline{s} \subseteq U \text{ or } \overline{s} \subseteq V \}.$$

Since we identify $\underline{r}|\overline{s} \subset P_{n+1}$ with $P_r \times P_s = \Delta_{r,s}(P_n)$, it follows that $A|B \in \mathcal{Q}_{p_i,q_i}(n)$ for some i if and only if $\delta_{A|B}\delta_{\underline{r}|\overline{s}} : P_{n-1} \to P_{n+1}$ is non-degenerate; consequently we consider cases $A|B \in \mathcal{Q}_{p_i,q_i}(n)$ for some i and $A|B \notin \mathcal{Q}_{p_i,q_i}(n)$ for all i.

Since our definitions of $d_{A|B}$ and $\varrho_{A|B}$ are independent in the first case and interdependent in the second, we define both operators simultaneously. But first we need some notation: Given an increasingly ordered set $M = \{m_1 < \cdots < m_k\} \subset \mathbb{N}$, let $I_M : M \to \underline{\#M}$ denote the *indexing map* $m_i \mapsto i$ and let $M + z = \{m_i + z\}$ denote *translation by* $z \in \mathbb{Z}$. Of course, $M - z$ and $M + z$ are left and right translations when $z > 0$; we adopt the convention that translation takes preference over set operations.

Assume $A|B \in \mathcal{Q}_{p_i,q_i}(n)$ for some i, and let

$$C_i = \{p_i, p_i + 1, ..., p_i + n_i\};$$
$$A_i = (C_i \cap A) - n_{(i-1)}, \quad B_i = (C_i \cap B) - n_{(i-1)};$$

$$n_i' = \#(A \cap C_i) - 1, \quad n_i'' = \#(B \cap C_i) - 1. \tag{2}$$

For example, $n=6$, $n_1=3$ and $n_2=2$ determines the projection $\Delta_{4,3}:P_6\to 1234\times 456$ and pairs $(p_1,q_1)=(1,3)$ and $(p_2,q_2)=(4,1)$. Thus $A|B=1234|56\in \mathcal{Q}_{3,2}(6)$ and the composition $\delta_{\underline{4}|\overline{3}}\delta_{A|B}:P_5\to P_7$ is non-degenerate. Furthermore, $C_2=456$, $A_2=(456\cap 1234)-3=1$, $B_2=23$, $n_i'=0$, $n_i''=1$ and we may think of $d_{A|B}$ acting on 1234×456 as $1\times d_{1|23}$.

$$\begin{tikzcd}
P_{n-1} \arrow[d, "\Delta_{n_1'+1,n_1''+1,n_2+1}"'] \arrow[ddr, "d_{A|B}(f)"] & & P_n \arrow[d, "\Delta_{n_1+1,n_2+1}"] \arrow[ddl, "\varrho_{A|B}(g)"'] \\
P_{n_1'+1}\times P_{n_1''+1}\times P_{n_2+1} \arrow[dd, "h_{A_1|B_1}\times 1"'] \arrow[dr, "\tilde f"] & & P_{n_1+1}\times P_{n_2+1} \arrow[dd, "\phi_{A_1|B_1}\times 1"] \arrow[dl, "\tilde g"'] \\
& Y & \\
P_{n_1+1}\times P_{n_2+1} \arrow[ur, "\bar f"'] & & P_{n_1'+1}\times P_{n_1''+1}\times P_{n_2+1} \arrow[ul, "\bar g"] \\
P_n \arrow[u, "\Delta_{n_1+1,n_2+1}"] \arrow[uur, "f"'] & & P_{n-1} \arrow[u, "\Delta_{n_1'+1,n_1''+1,n_2+1}"'] \arrow[uul, "g"]
\end{tikzcd}$$

Figure 8: Face and degeneracy operators when $i=1$ and $k=2$.

For $f=\bar f\circ\Delta_{n_1+1\cdots n_k+1}\in \operatorname{Sing}^M_n Y$, let $\tilde f=\bar f\circ(1^{\times i-1}\times h_{A_i|B_i}\times 1^{\times k-i})$ and define

$$d_{A|B}(f)=\tilde f\circ\Delta_{n_1+1\cdots n_i'+1,n_i''+1\cdots n_k+1}.$$

Dually, note that $n_i'+n_i''=n_i-1$ implies the sum of coordinates $(n_1,\ldots,n_{i-1},n_i',n_i'',n_{i+1},\ldots,n_k)\in(\mathbb{N}\cup 0)^{k+1}$ is $n-2$. So for $g=\bar g\circ\Delta_{n_1+1\cdots n_i'+1,n_i''+1\cdots n_k+1}\in \operatorname{Sing}^M_{n-1}Y$, let $\tilde g=\bar g\circ(1^{\times i-1}\times\phi_{A_i|B_i}\times 1^{\times k-i})$ and define

$$\varrho_{A|B}(g)=\tilde g\circ\Delta_{n_1+1\cdots n_k+1}$$

(see Figure 8).

On the other hand, assume that $A|B\notin\mathcal{Q}_{p_i,q_i}(n)$ for all i and define $d_{A|B}$ inductively as follows: When $k=2$, set $r=n_1+1$, $s=n_2+1$ and let

$$K|L=\begin{cases}(\underline{r}\cap A)\cup\overline{s}\,|\,\underline{r}\cap B, & r\in A\\ \underline{r}\cap A\,|\,(\underline{r}\cap B)\cup\overline{s}, & r\in B\end{cases}$$

$$M|N=\begin{cases}(\overline{s}\cap A)-1\,|\,\underline{n-1}\setminus(\overline{s}\cap A)-1, & r\in B\\ \underline{n-1}\setminus(\overline{s}\cap B)-\#L\,|\,(\overline{s}\cap B)-\#L, & r\in A,\quad n\in A\\ I_{\underline{n}\setminus L}(A)\,|\,\underline{n-1}\setminus I_{\underline{n}\setminus L}(A), & r\in A,\quad n\in B\end{cases}$$

$$C|D=\begin{cases}I_{\underline{n}\setminus B}(\underline{r}\cap A)\,|\,\underline{n-1}\setminus I_{\underline{n}\setminus B}(\underline{r}\cap A), & r\in B,\quad n\in B\\ I_{\underline{n}\setminus A}(\overline{s}\cap B)\,|\,\underline{n-1}\setminus I_{\underline{n}\setminus A}(\overline{s}\cap B), & r\in A,\quad n\in B\\ \\ \underline{n-1}\setminus I_{\underline{n}\setminus B}(\overline{s}\cap A)\,|\,I_{\underline{n}\setminus B}(\overline{s}\cap A), & r\in B,\quad n\in A\\ \underline{n-1}\setminus I_{\underline{n}\setminus A}(\underline{r}\cap B)\,|\,I_{\underline{n}\setminus A}(\underline{r}\cap B), & r\in A,\quad n\in A.\end{cases}$$

Then define

$$d_{A|B} = \varrho_{C|D} d_{M|N} d_{K|L}. \tag{3}$$

Remark 2.1. *This definition makes sense since* $K|L \in \mathcal{Q}_{p_1,q_1}(n)$, $M|N \in \mathcal{Q}_{p_3,q_3}(n-1)$, $C|D \in \mathcal{Q}_{p_1,q_1}(n-1)$ *with either* $r, n \in B$ *or* $r, n \in A$ *and* $C|D \in \mathcal{Q}_{p_3,q_3}(n-1)$ *with either* $r \in B$, $n \in A$ *or* $r \in A$, $n \in B$*. Of course,* $\mathcal{Q}_{**}(n-1)$ *is considered with respect to the decomposition* $n-2 = m_1 + m_2 + m_3$ *fixed after the action of* $d_{K|L}(\underline{r} \times \underline{s})$.

If $k = 3$, consider the pair $(r, s) = (n_1 + 1, n - n_1)$, then $(r_1, s_1) = (n_2 + 1, n - n_1 - n_2 - 1)$ for $A_1|B_1 = I_{\underline{n}\setminus\underline{r}}(\overline{s} \cap A)|I_{\underline{n}\setminus\underline{r}}(\overline{s} \cap B) \in \mathcal{P}_{p_1,q_1}(n-r)$, and so on. Now dualize and use the same formulas above to define the degeneracy operator $\varrho_{A|B}$.

Definition 2.3. *Let* Y *be a topological space. The* <u>*singular multipermutahedral set of* Y</u> *consists of the singular set* $\operatorname{Sing}_*^M Y$ *together with the singular face and degeneracy operators*

$$d_{A|B} : \operatorname{Sing}_n^M Y \to \operatorname{Sing}_{n-1}^M Y \quad \textit{and} \quad \varrho_{A|B} : \operatorname{Sing}_{n-1}^M Y \to \operatorname{Sing}_n^M Y$$

defined respectively for each $n \geq 2$ *and* $A|B \in \mathcal{P}_{**}(n)$.

Remark 2.2. *The operator* $d_{A|B}$ *defined in (3) applied to* $d_{U|V}$ *for some* $U|V \in \mathcal{P}_{r,s}(n+1)$ *yields the higher order structural relation*

$$d_{A|B} d_{U|V} = \varrho_{C|D} d_{M|N} d_{K|L} d_{U|V} \tag{4}$$

discussed in our first universal example.

Now $\operatorname{Sing}_*^M Y$ determines the singular (co)homology of a space Y in the following way: Let R be a commutative ring with identity. For $n \geq 1$, let $C_{n-1}(\operatorname{Sing}^M Y)$ denote the R-module generated by $\operatorname{Sing}_n^M Y$ and form the "chain complex"

$$(C_*(\operatorname{Sing}^M Y), d) = \bigoplus_{\substack{n_{(k)}=n-1\\ n\geq 1}} (C_{n-1}(\operatorname{Sing}^{n_1\cdots n_k} Y), d_{n_1\cdots n_k}),$$

where

$$d_{n_1\cdots n_k} = \sum_{A|B \in \bigcup_{i=1}^k \mathcal{Q}_{p_i,q_i}(n)} -(-1)^{n_{(i-1)}+n_i'} \ \textit{shuff}\,(C_i \cap A; C_i \cap B)\ d_{A|B}.$$

Refer to the example in Figure 7 and note that for $f \in C_4(\operatorname{Sing}^M Y)$ with $d_{13|2}d_{12|34}(f) \neq 0$, the component $d_{13|2}d_{12|34}(f)$ of $d^2(f) \in C_2(\operatorname{Sing}^M Y)$ is not cancelled and $d^2 \neq 0$. Hence d is not a differential. To remedy this, form the quotient

$$C_*^{\diamondsuit}(Y) = C_*\left(\operatorname{Sing}^M Y\right)/DGN,$$

where DGN is the submodule generated by the degeneracies, and obtain the *singular permutahedral chain complex* $\left(C_*^{\diamondsuit}(Y), d\right)$. Because the signs in d are determined by the index i, which is missing in our first universal example, we are unable to use our first example to define a chain complex with signs. However, we could use it to define a unoriented theory with $\mathbb{Z}_2$-coefficients.

The singular homology of Y is recovered from the composition

$$C_*(\operatorname{Sing} Y) \to C_*(\operatorname{Sing}^I Y) \to C_*(\operatorname{Sing}^M Y) \to C_*^{\diamondsuit}(Y)$$

arising from the canonical cellular projections

$$P_{n+1} \to I^n \to \Delta^n.$$

Since this composition is a chain map, there is a natural isomorphism

$$H_*(Y) \approx H_*^{\diamondsuit}(Y) = H_*(C_*^{\diamondsuit}(Y), d).$$

The fact that our diagonal on P and the A-W diagonal on simplices commute with projections allows us to recover the singular cohomology ring of Y as well. Finally, we remark that a cellular projection f between polytopes induces a chain map between corresponding singular chain complexes whenever chains on the target are normalized. Here $C_*(\operatorname{Sing} Y)$ and $C_*(\operatorname{Sing}^I Y)$ are non-normalized and the induced map f^* is not a chain map; but fortunately $d^2 = 0$ does not depend $df^* = f^*d$.

2.3. **Abstract Permutahedral Sets.** We begin by constructing a generating category $\mathbf{P}$ for permutahedral sets similar to that of finite ordered sets and monotonic maps for simplicial sets. The objects of $\mathbf{P}$ are the sets $n! = S_n$ of permutations of $\underline{n}$, $n \geq 1$. But before we can define the morphisms we need some preliminaries. First note that when P_n is identified with its vertices $n!$, the maps ρ_n and γ_n defined above become

$$\rho_n : n! \to 2!^{n-1} \quad \text{and} \quad \gamma_n : 2!^{n-1} \to n!.$$

Given a non-empty increasingly ordered set $M = \{m_1 < \cdots < m_k\} \subset \mathbb{N}$, let $M!$ denote the set of all permutations of M and let $J_M : M! \to k!$ be the map defined for $a = \left(m_{\sigma(1)}, ..., m_{\sigma(k)}\right) \in M!$ by $J_M(a) = \sigma$. For $n, m \in \mathbb{N}$ and partitions $A_1|\cdots|A_k \in \mathcal{P}_{n_1\cdots n_k}(n)$ and $B_1|\cdots|B_\ell \in \mathcal{P}_{m_1\cdots m_\ell}(m)$ with $n - k = m - \ell = \varkappa$, define the morphism

$$f_{A_1|\cdots|A_k}^{B_1|\cdots|B_\ell} : m! \to n!$$

by the composition

$$m! \xrightarrow{sh_B} \prod_{j=1}^{\ell} B_j \xrightarrow{\sigma_{\max}} \prod_{r=1}^{\ell} B_{j_r} \xrightarrow{J_B} \prod_{j=r}^{\ell} m_{j_r}! \xrightarrow{\rho_*} 2!^{\varkappa} \xrightarrow{\gamma_*}$$

$$\prod_{s=1}^{k} n_{i_s}! \xrightarrow{J_A^{-1}} \prod_{s=1}^{k} A_{i_s} \xrightarrow{\sigma_{\max}^{-1}} \prod_{i=1}^{k} A_i \xrightarrow{\iota_A} n!$$

where sh_B is a surjection defined for $b = \{b_1, ..., b_m\} \in m!$ by

$$sh_B(b) = (b_{1,1}, .., b_{m_1,1}; ...; b_{1,\ell}, .., b_{m_\ell,\ell}),$$

in which the right-hand side is the unshuffle of b with $b_{r,t} \in B_t$, $1 \leq r \leq m_t$, $1 \leq t \leq \ell$; $\sigma_{\max} \in S_\ell$ is a permutation defined by $j_r = \sigma_{\max}(r)$, $\max B_{j_r} = \max(B_1 \cup B_2 \cup \cdots \cup B_{j_r})$; $J_B = \prod_{r=1}^{\ell} J_{B_{j_r}}$; $\rho_* = \prod_{r=1}^{\ell} \rho_{j_r}$ and $\gamma_* = \prod_{s=1}^{k} \gamma_{i_s}$; finally, ι_A is the inclusion. It is easy to see that

$$f_{A_1|\cdots|A_k}^{B_1|\cdots|B_\ell} = f_{A_1|\cdots|A_k}^{\underline{\varkappa+1}} \circ f_{\underline{\varkappa+1}}^{B_1|\cdots|B_\ell} \quad \text{and} \quad f_{\underline{n}}^{\underline{n}} = \gamma_n \circ \rho_n.$$

In particular, the maps $f_{A|B}^{\underline{n-1}} : (n-1)! \to n!$ and $f_{\underline{n-1}}^{A|B} : n! \to (n-1)!$ are generator morphisms denoted by $\delta_{A|B}$ and $\beta_{A|B}$, respectively (see Theorem 2.1 below, the statement of which requires some new set operations).

Definition 2.4. *Given non-empty disjoint subsets $A, B, U \subset \underline{n+1}$ with $A \cup B \subseteq U$, define the* <u>*lower and upper disjoint unions*</u> *(with respect to U) by*

$$A\underline{\sqcup}B = \begin{cases} I_{U\diagdown A}(B) + \#A - 1, & \text{if } \min B > \min(U\diagdown A) \\ I_{U\diagdown A}(B) + \#A - 1 \cup \underline{\#A}, & \text{if } \min B = \min(U\diagdown A) \end{cases}$$

and

$$A\overline{\sqcup}B = \begin{cases} I_{U\diagdown B}(A), & \text{if } \max A < \max(U\diagdown B) \\ I_{U\diagdown B}(A) \cup \overline{\#B} - 1, & \text{if } \max A = \max(U\diagdown B). \end{cases}$$

If either A or B is empty, define $A\underline{\sqcup}B = A\overline{\sqcup}B = A \cup B$. Furthermore, given non-empty disjoint subsets $A, B_1, \ldots, B_k \subset \underline{n+1}$ with $k \geq 1$, set $U = A \cup B_1 \cup \cdots \cup B_k$ and define

$$A\square(B_1|\cdots|B_k) = (B_1|\cdots|B_k)\square A = \begin{cases} A\underline{\sqcup}B_1|\cdots|A\underline{\sqcup}B_k, & \text{if } \max A < \max U \\ B_1\overline{\sqcup}A|\cdots|B_k\overline{\sqcup}A, & \text{if } \max A = \max U. \end{cases}$$

Note that if $A|B$ is a partition of $\underline{n+1}$, then

$$A\underline{\sqcup}B = A\overline{\sqcup}B = \underline{n}.$$

Given a partition $A_1|\cdots|A_{k+1}$ of $\underline{n}$, define $A_1^1|\cdots|A_{k+1}^1 = A_1^1|\cdots|A_1^{k+1} = A_1|\cdots|A_{k+1}$; inductively, given $A_1^i|\cdots|A_{k-i+2}^i$ the partition of $\underline{n-i+1}$, $1 \leq i < k$, let

$$A_1^{i+1}|\cdots|A_{k-i+1}^{i+1} = A_1^i\square(A_2^i|\cdots|A_{k-i+2}^i)$$

be the partition of $\underline{n-i}$; and given $A_i^1|\cdots|A_i^{k-i+2}$ the partition of $\underline{n-i+1}$, $1 \leq i < k$, let

$$A_{i+1}^1|\cdots|A_{i+1}^{k-i+1} = (A_i^1|\cdots|A_i^{k-i+1})\square A_i^{k-i+2}$$

be the partition of $\underline{n-i}$.

Theorem 2.1. *For $A_1|\cdots|A_{k+1} \in \mathcal{P}_{n_1\cdots n_{k+1}}(n)$, $2 \leq k \leq n$, the map $f^{n-k}_{A_1|\cdots|A_{k+1}} : (n-k)! \to n!$ can be expressed as a composition of δ's two ways:*

$$f^{n-k}_{A_1|\cdots|A_{k+1}} = \delta_{A_1^1\,|\,A_2^1\cup\cdots\cup A_{k+1}^1}\cdots\delta_{A_1^k\,|\,A_2^k} = \delta_{A_1^1\cup\cdots\cup A_1^k\,|\,A_1^{k+1}}\cdots\delta_{A_k^1\,|\,A_k^2}.$$

Proof. The proof is straightforward and omitted. □

There is also the dual set of relations among the β's.

Example 2.1. *Theorem 2.1 defines structure relations among the δ's, the first of which is*

$$\delta_{A|B\cup C}\,\delta_{A\square(B|C)} = \delta_{A\cup B|C}\,\delta_{(A|B)\square C} \tag{5}$$

when $k = 2$. In particular, let $A|B|C = 12|345|678$. Since $A\underline{\sqcup}B = \{1234\}$, $A\underline{\sqcup}C = \{567\}$, $A\overline{\sqcup}C = \{12\}$ and $B\overline{\sqcup}C = \{34567\}$, we obtain the following quadratic relation on $12|345|678$:

$$\delta_{12|345678}\delta_{1234|567} = \delta_{12345|678}\delta_{12|34567};$$

similarly, on $345|12|678$ we have

$$\delta_{345|12678}\delta_{1234|567} = \delta_{12345|678}\delta_{34567|12}.$$

Definition 2.5. *Let $\mathcal{C}$ be the category of sets. A* <u>*permutahedral set*</u> *is a contravariant functor*

$$\mathcal{Z} : \mathbf{P} \to \mathcal{C}.$$

Thus a permutahedral set $\mathcal{Z}$ is a graded set $\mathcal{Z} = \{\mathcal{Z}_n\}_{n\geq 1}$ endowed with face and degeneracy operators

$$d_{A|B} = \mathcal{Z}(\delta_{A|B}) : \mathcal{Z}_n \to \mathcal{Z}_{n-1} \text{ and } \varrho_{M|N} = \mathcal{Z}(\beta_{M|N}) : \mathcal{Z}_n \to \mathcal{Z}_{n+1}$$

satisfying an appropriate set of relations, which includes quadratic relations such as

$$d_{A\square(B|C)}d_{A|B\cup C} = d_{(A|B)\square C}d_{A\cup B|C} \tag{6}$$

induced by (5) and higher order relations such as

$$d_{A|B}d_{U|V} = \varrho_{C|D}d_{M|N}d_{K|L}d_{U|V}$$

discussed in (4).

Let us define the abstract analog of a singular multipermutahedral set, which leads to a singular chain complex with arbitrary coefficients.

Definition 2.6. *For $n \geq 1$, let $X_n = \bigcup_{n_{(k)}=n-1,\ n_k\geq 0} X^{n_1\cdots n_k}$ and $X_{n-1} = \bigcup_{m_{(\ell)}=n-2,\ m_\ell\geq 0} X^{m_1\cdots m_\ell}$ be filtered sets; let $A|B \in \mathcal{Q}_{p_i,q_i}(n)$ for some i. A map $g : X_n \to X_{n-1}$ acts as an $\underline{A|B\text{-formal derivation}}$ if $g|_{X^{n_1\cdots n_k}} : X^{n_1\cdots n_k} \to X^{n_1\cdots n_i',n_i''\cdots n_k}$, where (n_i', n_i'') is given by (2).*

Let $\mathcal{C}_M$ denote the category whose objects are positively graded sets X_* filtered by subsets $X_n = \bigcup_{n_{(k)}=n-1,\ n_k\geq 0} X^{n_1\cdots n_k}$ and whose morphisms are filtration preserving set maps.

Definition 2.7. *A $\underline{\text{multipermutahedral set}}$ is a contravariant functor $\mathcal{Z} : \mathbf{P} \to \mathcal{C}_M$ such that*

$$\mathcal{Z}(\delta_{A|B}) : \mathcal{Z}(n!) \to \mathcal{Z}((n-1)!)$$

acts as an $A|B$-formal derivation for each $A|B \in \mathcal{Q}_{p_i,q_i}$, all $i \geq 1$.

Thus a multipermutahedral set $\mathcal{Z}$ is a graded set $\{\mathcal{Z}_n\}_{n\geq 1}$ with

$$\mathcal{Z}_n = \bigcup_{\substack{n_{(k)}=n-1 \\ n_k\geq 0}} \mathcal{Z}^{n_1\cdots n_k},$$

together with face and degeneracy operators

$$d_{A|B} = \mathcal{Z}(\delta_{A|B}) : \mathcal{Z}_n \to \mathcal{Z}_{n-1} \text{ and } \varrho_{M|N} = \mathcal{Z}(\beta_{M|N}) : \mathcal{Z}_n \to \mathcal{Z}_{n+1}$$

satisfying the relations of a permutahedral set and the additional requirement that $d_{A|B}$ respect underlying multigrading. This later condition allows us to form the chain complex of $\mathcal{Z}$ with signs mimicking the cellular chain complex of permutahedra (see below). Note that the chain complex of a permutahedral set is only defined with $\mathbb{Z}_2$-coefficients in general.

2.4. **The Cartesian product of permutahedral sets.** The objects and morphisms in the category $\mathbf{P}\times\mathbf{P}$ are the sets and maps

$$n!! = \bigcup_{r+s=n} r!\times s! \text{ and } \bigcup_{f,g\in\mathbf{P}} f\times g : m!! \to n!!$$

all $m,n \geq 1$. There is a functor $\Delta : \mathbf{P} \to \mathbf{P}\times\mathbf{P}$ defined as follows. If $A|B \in \mathcal{Q}_{r,1}(n) \cup \mathcal{Q}_{1,s}(n)$, define $\Delta_{r,s}(A|B) = A_1|B_1 \times A_2|B_2 \in r!\times s!$ and define $\delta_{A|B} : (n-1)! \to n!$ by

$$\Delta(\delta_{A|B}) = \delta_{A_1|B_1}\times\delta_{A_2|B_2},$$

where $\delta_{A_i|B_i}=1$ for either $i=1$ or $i=2$. Define $\Delta(\beta_{A|B})$ similarly. On the other hand, if $A|B\notin \mathcal{Q}_{r,1}(n)\cup\mathcal{Q}_{1,s}(n)$, define

$$\Delta(\delta_{A|B})=\Delta(\delta_{K|L})\Delta(\delta_{M|N})\Delta(\beta_{C|D}),$$

where $K|L$, $M|N$, $C|D$ are given by the formulas in (3). Dually, define $\Delta(\beta_{M|N})$. It is easy to check that Δ is well defined.

Given multipermutahedral sets $\mathcal{Z}',\mathcal{Z}'':\mathbf{P}\to\mathcal{C}_M$, first define a functor

$$\mathcal{Z}'\tilde{\times}\mathcal{Z}'':\mathbf{P}\times\mathbf{P}\to\mathcal{C}_M$$

on an object $n!!$ by

$$(\mathcal{Z}'\tilde{\times}\mathcal{Z}'')(n!!)=\bigcup_{r+s=n}\mathcal{Z}'(r!)\times\mathcal{Z}''(s!)/\sim,$$

where $(a,b)\sim(c,e)$ if and only if $a=\varrho'_{\underline{r}|r+1}(c)$ and $e=\varrho''_{1|\underline{s+1}\setminus\underline{1}}(b)$. On a map $h=\bigcup(f\times g):m!!\to n!!$,

$$(\mathcal{Z}'\tilde{\times}\mathcal{Z}'')(h):(\mathcal{Z}'\tilde{\times}\mathcal{Z}'')(n!!)\to(\mathcal{Z}'\tilde{\times}\mathcal{Z}'')(m!!)$$

is the map induced by $\bigcup(\mathcal{Z}'(f)\times\mathcal{Z}''(g))$. Now define the product $\mathcal{Z}'\times\mathcal{Z}''$ to be the composition of functors

$$\mathcal{Z}'\times\mathcal{Z}''=\mathcal{Z}'\tilde{\times}\mathcal{Z}''\circ\Delta:\mathbf{P}\to\mathcal{C}_M.$$

The face operator $d_{A|B}$ on $\mathcal{Z}'\times\mathcal{Z}''$ is given by

$$d_{A|B}(a\times b)=\begin{cases} d'_{\underline{r}\cap A|\underline{r}\cap B}(a)\times b, & \textit{if } A|B\in\mathcal{Q}_{1,s}(n),\\ a\times d''_{(\overline{s}\cap A)-r+1\,|\,(\overline{s}\cap B)-r+1}(b), & \textit{if } A|B\in\mathcal{Q}_{r,1}(n),\\ \varrho_{C|D}d_{M|N}d_{K|L}(a\times b), & \textit{otherwise},\end{cases}\tag{7}$$

with $M|N$, $K|L$, $C|D$ given by the formulas in (3).

Example 2.2. *The canonical map* $\iota:\operatorname{Sing}^P X\times\operatorname{Sing}^P Y\to\operatorname{Sing}^P(X\times Y)$ *defined for* $(f,g)\in\operatorname{Sing}^P_r X\times\operatorname{Sing}^P_s Y$ *by*

$$\iota(f,g)=(f\times g)\circ\Delta_{r,s}$$

is a map of permutahedral sets. Consequently, if X is a topological monoid, the singular permutahedral complex $Sing^P X$ inherits a canonical monoidal structure.

Definition 2.8. *A monoidal permutahedral set is a permutahedral set $\mathcal{Z}$ with a map $\mu:\mathcal{Z}\times\mathcal{Z}\to\mathcal{Z}$ of permutahedral sets which is associative and has the unit $e\in\mathcal{Z}_1$.*

Clearly, for a monoidal multipermutahedral set $\mathcal{Z}$, its chain complex $(C^{\diamondsuit}_*(\mathcal{Z};R), d)$ is a dg Hopf algebra.

Given a monoidal multipermutahedral set $\mathcal{Z}$, a $\mathcal{Z}$-*module* is a multipermutahedral set $\mathcal{L}$ together with associative action $\mathcal{Z}\times\mathcal{L}\to\mathcal{L}$ with the unit of $\mathcal{Z}$ acting as identity. In this case $C^*_{\diamondsuit}(\mathcal{L};R)$ is a dga comodule over the dg Hopf algebra $(C^*_{\diamondsuit}(\mathcal{Z};R),d)$.

2.5. **The permutahedral set functor $\mathbf{\Omega}Q$.** Let $Q = (Q_n, d_i^0, d_i^1, \eta_i)_{n\geq 0}$ be a cubical set. Recall that the diagonal

$$\Delta : C_*^{\square}(Q) \to C_*^{\square}(Q) \otimes C_*^{\square}(Q)$$

of Q is defined on $a \in Q_n$ by

$$\Delta(a) = \sum \mathit{shuff}(A;B)\, d_B^0(a) \otimes d_A^1(a),$$

where $d_B^0 = d_{j_1}^0 ... d_{j_q}^0$, $d_A^1 = d_{i_1}^1 ... d_{i_p}^1$, the summation is over all shuffles $\{A,B\} = \{i_1 < ... < i_q, j_1 < ... < j_p\}$ of the set $\underline{n}$. The primitive components of the diagonal are given by the extreme cases $A = \varnothing$ and $B = \varnothing$.

Assume Q is 1-reduced. Let $\bar{Q} = s^{-1}(Q_{>0})$ denote the desuspension of Q, let $\mathbf{\Omega}''Q$ be the free graded monoid generated by $\bar{Q}$ with the unit $e \in \bar{Q}_1 \subset \mathbf{\Omega}''Q$ and let Υ be the set of formal expressions

$$\Upsilon = \{\varrho_{M_k|N_k}((\cdots \varrho_{M_2|N_2}(\varrho_{M_1|N_1}(\bar{a}_1 \cdot \bar{a}_2) \cdot \bar{a}_3) \cdots) \cdot \bar{a}_{k+1}) |\, a_i \in Q_{r_i}\}_{r_i \geq 1; k\geq 2},$$

$M_i|N_i \in \mathcal{P}_{r_{(i)}, r_{i+1}}(r_{(i+1)})$ or $M_i|N_i \in \mathcal{P}_{r_{i+1}, r_{(i)}}(r_{(i+1)})$, $r_{(i)} = r_1 + \cdots + r_i$, $1 \leq i \leq k$. Note that one or more of the a_i's can be the unit e. Adjoin the elements of Υ to $\mathbf{\Omega}''Q$ and obtain the graded monoid $\mathbf{\Omega}'Q$ and let $\mathbf{\Omega}Q$ be the monoid

$$\mathbf{\Omega}Q = \mathbf{\Omega}'Q/\!\sim,$$

where $\varrho_{M|N}(\bar{a} \cdot \bar{b}) \sim \varrho_{N|M}(\bar{a} \cdot \bar{b})$, $\varrho_{j|\underline{n}\setminus j}(e \cdot \bar{a}) \sim \varrho_{\underline{n}\setminus j|j}(\bar{a} \cdot e) \sim \overline{\eta_j(a)}$, $a, b \in Q_{>0}$, and $\bar{a}_1 \cdots \varrho_{\underline{r_i}|r_i+1}(\bar{a}_i \cdot e) \cdot \bar{a}_{i+2} \cdots \bar{a}_{k+1} \sim \bar{a}_1 \cdots \bar{a}_i \cdot \varrho_{1|\underline{r_{i+2}+1}\setminus 1}(e \cdot \bar{a}_{i+2}) \cdots \bar{a}_{k+1}$ for $a_i \in Q_{r_i}, a_{i+1} = e$, $1 \leq i \leq k$. Then $\mathbf{\Omega}Q$ is canonically a multipermutahedral set in the following way: First, define the face operator $d_{A|B}$ on a monoidal generator $\bar{a} \in \bar{Q}_n$ by

$$d_{A|B}(\bar{a}) = \overline{d_B^0(a)} \cdot \overline{d_A^1(a)}, \quad A|B \in \mathcal{P}_{*,*}(n).$$

Next, use the formulas in the definition of a singular multipermutahedral set (3) to define $d_{A|B}$ and $\varrho_{M|N}$ on decomposables. In particular, the following identities hold for $1 \leq i \leq n$:

$$d_{i|\underline{n+1}\setminus i}\,(\overline{a}) = \overline{d_i^1(a)} \ \text{ and } \ d_{\underline{n+1}\setminus i|i}\,(\overline{a}) = \overline{d_i^0(a)}.$$

It is easy to see that $(\mathbf{\Omega}Q,\, d_{A|B},\, \varrho_{M|N})$ is a multipermutahedral set that depends functorially on Q.

Remark 2.3. *The fact that the definition of $\mathbf{\Omega}Q$ uses all cubical degeneracies is justified geometrically by the fact that a degenerate singular n-cube in the base of a path fibration lifts to a singular $(n-1)$-permutahedron in the fibre, which is degenerate with respect to Milgram's projections* [17] *(c.f., the definition of the cubical set $\mathbf{\Omega}X$ on a simplicial set X* [13]*).*

3. The permutocubes

The pertmutocube B_n is an n-dimensional polytope discovered by N. Berikashvili which can be thought of as a "twisted Cartesian product" of the cube and the permutahedron. Originally the permutocube B_n has been obtained from I^n by the following truncation procedure: First the n-cube is truncated at the minimal vertex $a_0 = (0, ..., 0)$, then it is truncated along those $(n-1)$-faces that contained a_0, and continuing so the last truncation is along those 1-faces (edges) of the n-cube that contained a_0. Hence, B_2 is a pentagon (Figure 9), for B_3 see Figure 10. In particular, this truncation procedure fixes the permutahedron P_n at the vertex a_0. So that we get the natural cellular embedding (see Figures 11 and 12)

$$\delta_{0]\underline{n}} : P_n \to B_n. \tag{8}$$

The notation for the above inclusion map is motivated by the following combinatorial description of B_n. We have that the faces of B_n are in one-to-one correspondence with partitions $C_0]C_1|...|C_p$ of all (non-empty) subsets of the set

$$\underline{n_0} = \{0, 1, ..., n\}$$

with $0 \in C_0$. More precisely, an $(n-k-p)$-face $u{=}C_0]C_1|...|C_p$ of B_n can be labelled by the composition of face operators, denoted by $d_u{=}d_{A_0]A_1|...|A_p;(i_k,...,i_1)}$ for $\{i_k < \cdots < i_1\} \subset \underline{n}$,

$$u = d_{A_0]A_1|...|A_p;(i_k,...,i_1)}(\underline{n_0}) = \\ d_{A_0]A_1} d_{A_0\cup A_1]A_2} \cdots d_{A_0\cup A_1\cup \cdots \cup A_{p-1}]A_p} d_{i_k} \cdots d_{i_1}(\underline{n_0}),$$

where d_i acts by deleting the $(i+1)^{th}$ integer, while $d_{A]M}$ forms the partition $A]M$ of the domain, and $A_i = I_{\underline{n_0}\setminus(i_k,...,i_1)}(C_i)$, $0 \le i \le p$. In particular, $d_{0]\underline{n}}(\underline{n_0})$ just corresponds to the single $(n-1)$-permutahedral face $\delta_{0]\underline{n}}(P_n) \subset B_n$. On the other hand, a face of the form $d_{i_k} \cdots d_{i_1}(\underline{n_0})$ may be identified with B_{n-k}, $0 \le k < n$.

For example, for $u = 038]14|6|79 \subset B_9$, we have

$$u = d_{026]13|4|57;\,(2,5)}(\underline{9_0}) = d_{024]13} d_{1235]4} d_{12346]57} d_2 d_5 (\underline{9_0}).$$

We have that B_n also admits the realization as a subdivision of the standard n-cube I^n compatible with the inclusion $P_n \subset B_n$ (see Figures 9,10). Indeed, let $B_0 = I^0$; for $1 \le i \le n$, let $e^{n-1}_{i,\epsilon}$ denote the $(n-1)$-face $(x_1, \ldots, x_{i-1}, \epsilon, x_{i+1}, \ldots, x_n) \subset I^n$ and label the endpoints of $B_1 = [0,1]$ via $e^0_{1,0} \leftrightarrow d_{0]1}$ and $e^0_{1,1} \leftrightarrow d_1$. Inductively, if B_{n-1} has been constructed, obtain B_n as a subdivision of $B_{n-1} \times I$ in the following way:

Face of B_n	Label	
$e^{n-1}_{n,0}$	$d_{\underline{(n-1)_0}\,]\,n}$	
$e^{n-1}_{i,1}$	$d_i,$	$i \in \underline{n}$
$d_{A]M} \times I_{0,\#M}$	$d_{A]M\cup n}$	
$d_{A]M} \times I_{\#M,\infty}$	$d_{A\cup n]M}.$	

Thus, proper cells of B_n are represented as the Cartesian product of the permutocube and permutahedra. In particular, on a proper cell $e = e_1 \times e_2 \subset B_n$ a permutahedral face operator $d_{M_1|M_2}$ acts as $d_{M_1|M_2}(e) = e_1 \times d_{M_1|M_2}(e_2)$.

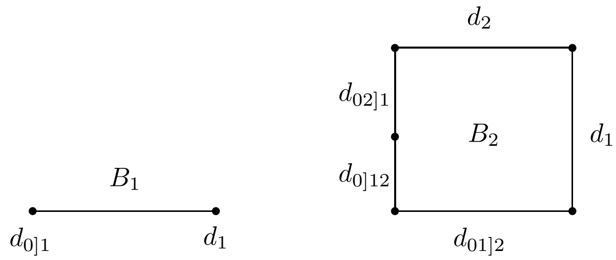


Figure 9: B_2 as a subdivision of $B_1 \times I$.

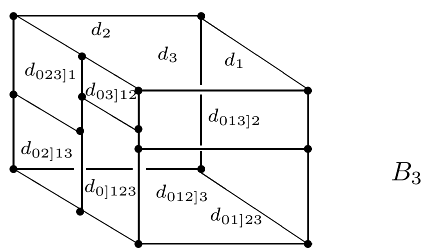


Figure 10: B_3 as a subdivision of $B_2 \times I$.

The above face operators are connected to each other by the following combinatorial relations: The relations between $d_{A]M}$ and $d_{M_1|M_2}$ reflect the associativity of the partition procedure, while the relations between d_i and either $d_{A]M}$ or $d_{M_1|M_2}$ reflect the commutativity of the deleting and the partition procedures. These relations together with those involving degeneracies are incorporated in the singular permutocubes (see Example 4.1) which motivate structural relations for a *permutocubical set* in the next section.

4. Permutocubical sets

Here we give the formal definition of a *permutocubical set*. The original motivation of that definition is the "twisted Cartesian product" of cubical and (multi) permutahedral sets (see Section 5). Note that for applications here structural relations may be assumed modulo degeneracies, so that we give the relations explicitly only for face operators; the other relations involving degeneracies can be written by examination of the universal example-the *singular permutocubical set* if necessary.

Let $\mathcal{P}^0_{**}(n) = \mathcal{P}_{**}(n) \cup \varnothing|\underline{n}$ with $\varnothing|\underline{n} \in \mathcal{P}^0_{0,n}(n)$.

Definition 4.1. *A permutocubical set is a bigraded set*

$$\mathcal{B} = \{\mathcal{B}_{p,q}\}_{p\geq 0, q\geq 1}$$

together with face and degeneracy operators

$$\begin{array}{rll} d_i & : \mathcal{B}_{p,q} \to \mathcal{B}_{p-1,q}, & \\ d_{A]M} & : \mathcal{B}_{p,q} \to \mathcal{B}_{p-r,q+r-1}, & A \setminus 0]M \in \mathcal{P}^0_{p-r,r}(p), \\ d_{M_1|M_2} & : \mathcal{B}_{p,q} \to \mathcal{B}_{p,q-1}, & \\ \eta_i & : \mathcal{B}_{p-1,q} \to \mathcal{B}_{p,q}, & i \in \underline{p}, \\ \varrho_{M_1|M_2} & : \mathcal{B}_{p,q-1} \to \mathcal{B}_{p,q}, & M_1|M_2 \in \mathcal{P}_{*,*}(q), \end{array}$$

such that for each $p \geq 0$, *the graded set*

$$\{\mathcal{B}_{p,q};\, d_{M_1|M_2},\, \varrho_{M_1|M_2}\}_{q\geq 1}$$

is a multipermutahedral set and the relations among face operators are:

$$\begin{array}{rcl} d_i d_j & = & d_{j-1} d_i, \qquad i<j, \\ d_i d_{A]M} & = & d_{A\setminus j]M} d_j, \quad i = I_{\underline{p}\setminus A}(j),\;\; i \in \underline{p-r}, \\ d_i d_{M_1|M_2} & = & d_{M_1|M_2} d_i, \\ d_{K|L} d_{A]M} & = & \begin{cases} d_{A]K\cap \underline{r}}\, d_{A\cup(K\cap\underline{r})]L\cap\underline{r}}, & K|L \in \mathcal{Q}_{1,q}(q+r), \\ d_{A]M}\, d_{K+1-r|L+1-r}, & K|L \in \mathcal{Q}_{r,1}(q+r). \end{cases} \end{array}$$

Example 4.1. *For a topological space* Y, *define the* *<u>singular permutocubical complex</u>* $\mathrm{Sing}^B Y$ *as follows: Let*

$$\mathrm{Sing}^B_{p,q} Y = \{\textit{continuous maps } B_p \times P_q \to Y\}_{p\geq 0;\, q\geq 1},$$

$B_p \times P_q$ *is a Cartesian product of the permutocube* B_p *and the permutohedron* P_q. *Let*

$$\begin{array}{lll} \delta_i \times 1 & : B_{p-1} \times P_q \to B_p \times P_q, & i \in \underline{p}, \\ \bar{\delta}_{A]M} & : B_{p-r} \times P_{q+r-1} \xrightarrow{1\times \Delta_{r,q}} B_{p-r} \times P_r \times P_q \xrightarrow{\delta_{A]M}\times 1} B_p \times P_q, & \end{array}$$

be the maps in which δ_i *and* $\delta_{A]M}$ *are the canonical inclusions. Consider also the map*

$$\varsigma_i \times 1 : B_p \times P_q \to B_{p-1} \times P_q, \quad i \in \underline{p},$$

where $\varsigma_i : B_p \to B_{p-1}$ *is the projection that identifies the faces* $d_{\underline{p+1}\setminus i]i}$ *and* d_i.

Then for $f \in \mathrm{Sing}^B_{p,q} Y$, *define*

$$\begin{array}{rl} d_i & : \mathrm{Sing}^B_{p,q} Y \to \mathrm{Sing}^B_{p-1,q} Y, \\ d_{A]M} & : \mathrm{Sing}^B_{p,q} Y \to \mathrm{Sing}^B_{p-r,q+r-1} Y, \\ \eta_i & : \mathrm{Sing}^B_{p,q} Y \to \mathrm{Sing}^B_{p+1,q} Y \end{array}$$

as compositions

$$\begin{array}{rcl} d_i(f) & = & f \circ (\delta_i \times 1), \\ d_{A]M}(f) & = & f \circ \bar{\delta}_{A]M}, \\ \eta_i(f) & = & f \circ (\varsigma_i \times 1), \end{array}$$

while define

$$d_{M_1|M_2} : \mathrm{Sing}^B_{p,q} Y \to \mathrm{Sing}^B_{p,q-1} Y \quad \textit{and} \quad \varrho_{M_1|M_2} : \mathrm{Sing}^B_{p,q} Y \to \mathrm{Sing}^B_{p,q+1} Y$$

as $d_{M_1|M_2} = 1 \times d_{M_1|M_2}$ *and* $\varrho_{M_1|M_2} = 1 \times \varrho_{M_1|M_2}$ *to obtain the singular permutocubical set* $(\operatorname{Sing}^B Y,\ d_i,\ d_{A]M}, d_{M_1|M_2},\ \eta_i,\ \varrho_{M_1|M_2})$.

The singular permutocubical complex $\operatorname{Sing}^B Y$ *determines the singular (co)homology of* Y *in the following way: Form the "chain complex"*

$$(C_*(\operatorname{Sing}^B Y), d) = \bigoplus_{\substack{p+q=n-1\\ n\geq 1}} (C_{n-1}(\operatorname{Sing}^B_{p,q} Y), d_{p,q}),$$

where

$$d_{p,q} = \sum_{\substack{A\setminus 0\,|M\in \mathcal{P}^0_{**}(p)\\ 1\leq i\leq p}} \big((-1)^{i+1} d_i + (-1)^{\#A} \mathit{shuff}(A;M)\, d_{A]M}\big) + (-1)^p d_q,$$

$d_q : C_{q-1}(\operatorname{Sing}^B_{p,q} Y) \to C_{q-2}(\operatorname{Sing}^B_{p,q-1} Y)$, *the singular permutahedral differential. Let*

$$C^{\boxminus}_*(Y) = C_*(\operatorname{Sing}^B Y)/DGN,$$

where DGN *is the submodule of* $C_*(\operatorname{Sing}^B Y)$ *generated by the degenerate elements of* $\operatorname{Sing}^B Y$, *and obtain the singular permutocubical chain complex* $(C^{\boxminus}_*(Y), d)$ *of* Y.

Now let $\varphi : B_n \to I^n$ *be the cellular projection defined by the property that it maps homeomorphically the faces* $d_{\underline{n}\setminus i]i}(B_n)$ *and* $d_i(B_n)$ *onto the faces* $d^0_i(I^n)$ *and* $d^1_i(I^n)$ *respectively,* $1 \leq i \leq n$. *Then the composition of maps*

$$B_p \times P_q \xrightarrow{\varphi\times\rho} I^p \times I^{q-1} = I^{p+q-1} \xrightarrow{\psi} \Delta^{p+q-1},$$

clearly induces a composition of maps of graded sets

$$\operatorname{Sing} Y \xrightarrow{\psi_*} \operatorname{Sing}^I Y \xrightarrow{(\varphi\times\rho)_*} \operatorname{Sing}^B Y.$$

After the passage on the non-normalized chains (unless the last one) one gets a sequence of homomorphisms

$$C_*(\operatorname{Sing} Y) \to C_*(\operatorname{Sing}^I Y) \to C_*(\operatorname{Sing}^B Y) \to C^{\boxminus}_*(Y),$$

whose composition is a chain map inducing a natural isomorphism

$$H_*(Y) \approx H^{\boxminus}_*(Y) = H_*(C^{\boxminus}_*(Y), d).$$

Since the diagonal on the permutocube constructed in Section 6 is compatible with the A-W diagonal on the standard simplex under the above cellular projections, $H^{\boxminus}_*(Y)$ *determines the singular cohomology ring of* Y *as well.*

Basic examples of a permutocubical set are provided in the next section.

5. Truncating twisting functions and twisted Cartesian products

As in [13] in this section we introduce the notion of a twisting function between graded sets in which the domain and the target have face and degeneracy operators of different types; but this time we define a *truncating twisting function* $\vartheta : Q \to \mathcal{Z}$ from a cubical set Q to a monoidal permutahedral set $\mathcal{Z}$. For a permutahedral $\mathcal{Z}$-module with action $\mathcal{Z} \times \mathcal{L} \to \mathcal{L}$, such ϑ defines a *twisted Cartesian product* $Q \times_\vartheta \mathcal{L}$ as a *permutocubical set.*

These notions are motivated by the permutocubical set $\mathbf{P}Q$ (see below) which can be viewed as a twisted Cartesian product determined by the canonical inclusion $\vartheta : Q \to \mathbf{\Omega}Q$, $x \mapsto \bar{x}$ of degree -1, referred to as the *universal truncating twisting function.*

The geometrical interpretation of ϑ_U answers to the truncation procedure that converts I^n into B_n mentioned in Section 3. By this the permutocube is thought of as a "twisted Cartesian product" of the cube and the permutohedron (see Figures 11 and 12).

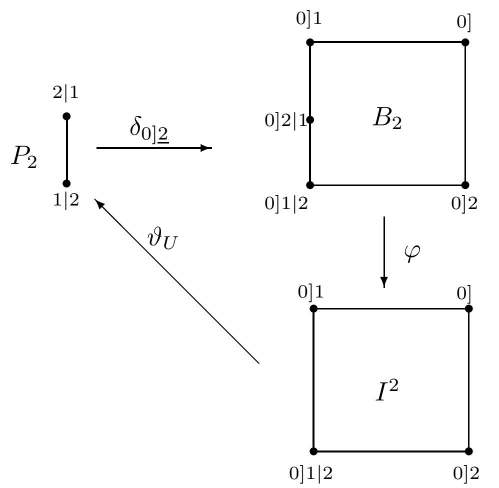


Figure 11: The universal truncating twisting function $\vartheta_U : I^2 \to P_2$.

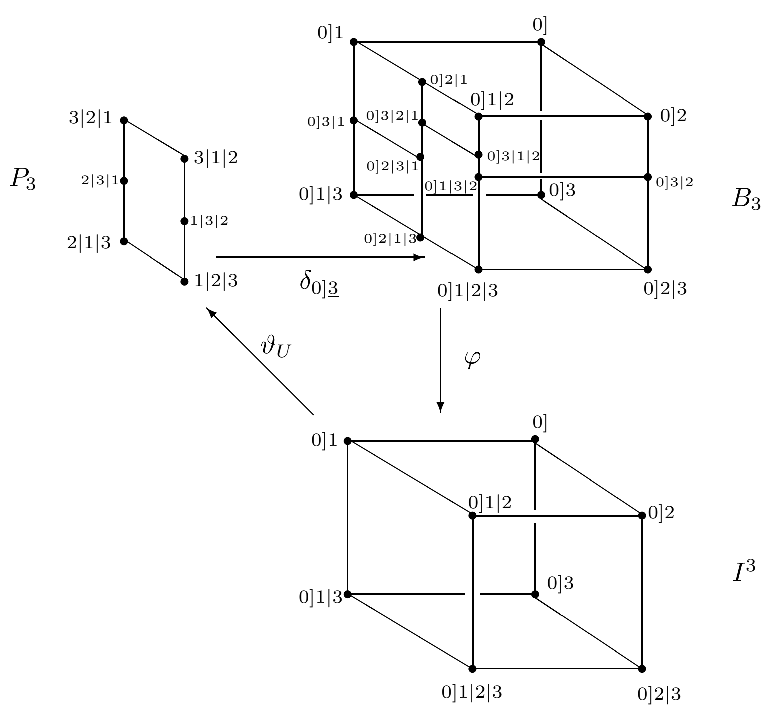


Figure 12: The universal truncating twisting function $\vartheta_U : I^3 \to P_3$.

Definition 5.1. *Let $Q = (Q_n, d_i^0, d_i^1, \eta_i)$ be a 1-reduced cubical set and $\mathcal{Z} = (\mathcal{Z}_n, d_{M_1|M_2}, \varrho_{M_1|M_2})$ be a monoidal permutahedral set. A sequence $\vartheta = \{\vartheta_n\}_{n\geq 1}$ of degree -1 functions $\vartheta_n : Q_n \to \mathcal{Z}_n$ is called a truncating twisting function if it satisfies:*

$$\begin{array}{rcll} \vartheta(a) & = & e, & a \in Q_1, \\ d_{M_1|M_2}\vartheta(a) & = & \vartheta d^0_{M_2}(a)\cdot \vartheta d^1_{M_1}(a), \quad M_1|M_2 \in \mathcal{P}_{*,*}(n), & a \in Q_n, \\ \varrho_{\underline{n}\setminus i|i}\vartheta(a) & = & \vartheta\eta_i(a), \quad i \in \underline{n}. & \end{array}$$

Note that since the first condition above we in particular get

$$d_{i|\underline{n}\setminus i}\vartheta(a) = \vartheta d_i^1(a) \text{ and } d_{\underline{n}\setminus i|i}\vartheta(a) = \vartheta d_i^0(a) \text{ for } i \in \underline{n} \text{ and } a \in Q_{n>0}.$$

Remark 5.1. *By definition a truncation twisting function commutes only with the permutahedral degeneracy operator $\varrho_{\underline{n}\setminus i|i}$, since it is in fact arisen by the cubical degeneracy operator η_i (c.f. Remark 2.3).*

We have the following

Proposition 5.1. *Let Q be a 1-reduced cubical set and $\mathcal{Z}$ be a monoidal permutahedral set. A sequence $\vartheta = \{\vartheta_n\}_{n\geq 1}$ of degree -1 functions $\vartheta_n : Q_n \to \mathcal{Z}_n$ is a truncating twisting function if and only if the monoidal map $f : \mathbf{\Omega}Q \to \mathcal{Z}$ defined by $f(\bar{a}_1\cdots\bar{a}_k) = \vartheta(a_1)\cdots\vartheta(a_k)$ is a map of permutahedral sets.*

Proof. Since f is completely determined by its restriction to monoidal generators, use the argument of verification of permutahedral identities for a given single generator $\bar{\sigma}$ in ΩQ being equivalent to that of identities of the universal truncating function $\vartheta_U : \sigma \to \bar{\sigma}$. □

Definition 5.2. *Let $Q = (Q_n, d_i^0, d_i^1, \eta_i)$ be a 1-reduced cubical set and $\mathcal{Z} = (\mathcal{Z}_n, d_{M_1|M_2}, \varrho_{M_1|M_2})$ be a monoidal permutahedral set and $\mathcal{L}$ be a permutahedral set with $\mathcal{Z}$-module structure. Let $\vartheta = \{\vartheta_n\}_{n\geq 1}$, $\vartheta_n : Q_n \to \mathcal{Z}_n$ be a truncating twisting function. The twisted Cartesian product $Q \times_\vartheta \mathcal{L}$ is the Cartesian product of sets*

$$Q\times\mathcal{L} = \{(Q\times\mathcal{L})_{p,q} = \bigcup_{p\geq 0, q\geq 1} Q_p\times\mathcal{L}_q\}$$

endowed with the face and degeneracy operators $d_i, d_{A]M}, d_{M_1|M_2}, \eta_i, \varrho_{M_1|M_2}$ defined for $(a,b)\in Q_p\times\mathcal{L}_q$ by :

$$\begin{array}{rcll} d_i(a,b) & = & (d_i^1(a),\ b), & i\in \underline{p}, \\ d_{A]M}(a,b) & = & (d^0_M(a),\ \vartheta d^1_A(a)\cdot b), & A\setminus 0\,|M \in \mathcal{P}^0_{*,*}(p), \\ d_{M_1|M_2}(a,b) & = & (a,\ d_{M_1|M_2}(b)), & M_1|M_2\in\mathcal{P}_{*,*}(q) \\ \eta_i(a,b) & = & (\eta_i(a),\ b), & i\in \underline{p+1}, \\ \varrho_{M_1|M_2}(a,b) & = & (a, \varrho_{M_1|M_2}(b)), & M_1|M_2\in\mathcal{P}_{*,*}(q+1). \end{array}$$

It is easy to check that $(Q\times_\vartheta\mathcal{L},\ d_i, d_{A]M}, d_{M_1|M_2}, \eta_i, \varrho_{M_1|M_2})$ is a permutocubical set.

Remark 5.2. *Note that for the twisted Cartesian product $Q\times_\vartheta\mathcal{L}$ we have the following sequence of graded sets*

$$\mathcal{L} \xrightarrow{\iota} Q\times_\vartheta\mathcal{L} \xrightarrow{\xi} Q$$

with $\iota(b) = (a_0, b)$ *and* $\xi(a,b) = a$, $a_0 \in Q_0$, $(a,b) \in Q \times \mathcal{L}$.

Example 5.1. *Let* $M = \{e_k\}_{k\geq 0}$ *be the free minoid on a single generator* $e_1 \in M_1$ *with trivial permutahedral set structure and let* $\vartheta : Q \to M$ *be the sequence of constant maps* $\vartheta_n : Q_n \to M_{n-1}$, $n \geq 1$. *Then the twisted Cartesian product* $Q \times_\vartheta M$ *can be thought of as a permutocubical resolution of the 1-reduced cubical set* Q.

5.1. **The permutocubical set functor $\mathbf{P}Q$.** The universal truncating twisting function ϑ_U defines a special (acyclic) twisted Cartesian product: Namely, we have

Definition 5.3. *A functor from the category of 1-reduced cubical sets to the category of permutocubical sets defined by* $Q \to Q \times_{\vartheta_U} \mathbf{\Omega} Q$ *is the universal permutocubical functor and denoted by* $\mathbf{P}$.

6. The diagonal of permutocubes

Here we construct the explicit diagonal $\Delta_B : C_*(B_n) \to C_*(B_n) \otimes C_*(B_n)$ for permutocubes which induces a diagonal for a permutocubical set too.

6.1. **The orthogonal stream.** Suppose that an n-dimensional polytope X is realized as a subdivision of the cube I^n so that each m-dimensional cell $e \subset X$, $0 \leq m \leq n$, is itself a subdivision of I^m (I^m need not be a face of I^n; c.f. B_n). In particular, we have an induced partial ordering on the set of all vertices of e defined by $x \leq y$ if there is an oriented broken line from x to y. For a cell $e' \subset e$, let $I^{m(e')} \subset I^m$ be the face of I^m of the minimal dimension $m(e')$ that contains e'. Then we introduce the following

Definition 6.1. *Let* $e \subset X$ *be an* m*-cell and* $x \in e$ *be a vertex. An orthogonal stream* $OS_x(e)$ *of* x *with the support* e *is a pair*

$$(U_x, V_x) = (\{u_1, ..., u_r\}, \{v_1, ..., v_s\})_{r,s\geq 1}$$

of collections of faces of e *satisfying the following conditions:*
1. $\max u_r = x = \min v_1$ *and* $\dim u_r + \dim v_1 = m$;
2. $I^{m(u_i)} = I^{m(u_r)}$, $\dim u_i = \dim u_r$ *and* $\max u_i \leq x$, $1 \leq i \leq r$;
3. $I^{m(v_j)} = I^{m(v_1)}$, $\dim v_j = \dim v_1$ *and* $\min v_j \geq x$, $1 \leq j \leq s$.

The union $\cup_{x\in e} SO_x(e)$ *is denoted by* $SO(e)$.

The pair $(u_r, v_1) \in OS_x(e)$ is referred to as *the strong complementary pair* (SCP) and denoted by (u_x, v_x); while a pair $(u,v) \in OS_x(e)$ is referred to as *a complementary pair* (CP) (compare, [19]).

Clearly, any vertex $x \in e \subset B_n$ uniquely defines the SCP (u_x, v_x) in $OS_x(e)$, and, consequently, the whole $OS_x(e)$ is uniquely determined by the vertex x. In particular, when x coincides with a vertex of I^m then $\dim u_x = m(u_x)$ and $\dim v_x = m(v_x)$, so that U_x and V_x actually lay on orthogonal faces of I^m at the vertex x.

For B_n, an orthogonal stream $OS_x(e)$ for each cell $e \subset B_n$ admits an explicit combinatorial description. For example, for the top cell of B_n we have the following: Think of a vertex $x{=}0\,]\,x_1|...|x_k \in B_n$ as an ordered sequence of integers for $1 \leq k \leq n$. Let $u_x{=}A_0]A_1|...|A_p$ and $v_x{=}C_0]C_1|...|C_q$ be partitions of $\underline{n_0}$ in which A_j for $1 \leq j \leq p$, and C_i for $0 \leq i \leq q$, are the $(j+1)^{th}$ decreasing and the $(i+1)^{th}$ increasing subsequence of maximal length of x respectively (compare, [19]); while for the vertex $x = 0] \in B_n$, let $(u_x, v_x) = \big(\underline{n_0}\,], 0]\big)$. For example, for $x{=}\ 0]1|...|n \in B_n$, one gets $(u_x, v_x) = \big(0]1|...|n\,, \underline{n_0}\,]\big)$; for $x{=}\ 0]2|1|3|6|5 \in B_6$, $(u_x, v_x){=}(04]12|3|56\,, 02]136|5)$.

Next for a partition $a = A_0]A_1|...|A_\ell$ of $\underline{n_0}$, we define the right-shift R and the left-shift L operators respectively as follows (compare, [19]): For proper subsets $M_i \subset A_i$ and $N_j \subset A_j$, $0 \leq i < \ell$, $0 < j \leq \ell$, let

$$\begin{array}{ll} R_{M_i}(a) = A_0]A_1|\cdots|A_i \setminus M_i|A_{i+1} \cup M_i|\cdots|A_\ell & \text{for} \quad \min M_i > \max A_{i+1}, \\ L_{N_j}(a) = A_0]A_1|\cdots|A_{j-1} \cup N_j|A_j \setminus N_j|\cdots|A_\ell & \text{for} \quad \min N_j > \max A_{j-1}, \end{array}$$

where $R_\varnothing = Id = L_\varnothing$. Then each CP $(u,v) \in (U_x, V_x)$ in the orthogonal stream $OS_x(B_n)$ can be obtained from the SCP (u_x, v_x) by successive application of the above operators as

$$(u,v) = (R_{M_{\ell-1}} \cdots R_{M_1} R_{M_0}(u_x)\,,\ L_{N_1} \cdots L_{N_\ell}(v_x))$$

for some $\{M_i\}_{0 \leq i < \ell}$ and $\{N_j\}_{0 < j \leq \ell}$.

For example, for the vertex $x = 0]2|1|3|6|5$, we obtain

$$\begin{array}{l} OS_x(B_6){=}(U_x, V_x){=} \\ (\{0]12|34|56,\ 0]124|3|56,\ 04]12|3|56\}\,, \{02]136|5],\, 023]16|5, 026]13|5, 0236]1|5\}). \end{array}$$

In particular, for the permutahedron P_n the above description of CP's agrees with that by configuration matrices in [19].

6.2. **The sign of $SO_x(B_n)$.** The sign for a pair $(u,v) \in OS_x(B_n)$ is deduced by motivation that the cellular projection $\varphi : B_n \to I^n$ preserves diagonals.

First for a partition $a = A_0]A_1|...|A_p$ of $\underline{n_0}$ fix the following signs:

$$\begin{gathered} sgn_1(a) = (-1)^{\epsilon_1}\, psgn(a) \ \text{ and } \ \epsilon_1 = \sum\nolimits_{i=1}^{p} i \cdot \#A_{p-i}, \\ sgn_2(a) = (-1)^{\epsilon_2}\, psgn(a) \ \text{ and } \ \epsilon_2 = \epsilon_1 + \tbinom{p}{2}, \\ rsgn(a) = (-1)^{\frac{1}{2}\left[(\#A_0)^2 + \cdots + (\#A_p)^2 - (n+1)\right]} \end{gathered}$$

and $psgn(a)$ is the permutation sign $\{0,1,...,n\} \to A_0]A_1|...|A_p$.

Now define

$$sgn(u,v) = (-1)^{\binom{q+1}{2}}\ rsgn(u) \cdot sgn_1(v) \cdot sgn_2(u) \cdot sgn_2(u_x),$$

where $q+1$ is the number of blocks in the partition v.

Note that the above sign agrees with that of a CP from $SO_x(P_n)$ established in [19].

6.3. **The diagonal of the permutocube.** By means of orthogonal streams we construct an explicit diagonal for permutocubes as follows.

Theorem 6.1. *The explicit diagonal of B_n*

$$\Delta_B : C_*(B_n) \to C_*(B_n) \otimes C_*(B_n)$$

is defined for a cell $e \subset B_n$ by

$$\Delta_B(e) = \sum_{(e_1,e_2) \in OS(e)} sgn(e_1,e_2)\, e_1 \otimes e_2.$$

Proof. The proof is straightforward and analogous to that of Theorem 1 in [19]. $\square$

In particular, in terms of orthogonal streams the diagonal Δ_P for permutahedra established in [19] can be formulated as follows.

Theorem 6.2. *The explicit diagonal of* P_n

$$\Delta_P : C_*(P_n) \to C_*(P_n) \otimes C_*(P_n)$$

is defined for a cell $e \subset P_n$ *by*

$$\Delta_P(e) = \sum_{(e_1,e_2)\in OS(e)} sgn(e_1,e_2)\, e_1 \otimes e_2.$$

Below all components of Δ_B for the top cell of B_n are written down for $n = 1, 2, 3$ in which rows correspond to the orthogonal streams.

Example 6.1.

$$\begin{array}{llcll} \Delta_B\,(1]) = & & & & \\ & 0]1 & \otimes & 1] & x = 0]1 \\ & +\ 1] & \otimes & 0] & x = 0]. \end{array}$$

Example 6.2.

$$\begin{array}{llcll} \Delta_B\,(12]) = & & & & \\ & 0]1|2 & \otimes & 12] & x = 0]1|2 \\ & -\ 0]12 & \otimes & 2]1 & x = 0]2|1 \\ & -\ (0]12 + 2]1) & \otimes & 1] & x = 0]1 \\ & +\ 1]2 & \otimes & 2] & x = 0]2 \\ & +\ 12] & \otimes & 0] & x = 0]. \end{array}$$

Example 6.3. *Up to sign, we have*

$$\begin{array}{llcll} \Delta_B\,(123]) = & & & & \\ & 0]1|2|3 & \otimes & 123] & x = 0]1|2|3 \\ & +\ 0]12|3 & \otimes & 2]13 & x = 0]2|1|3 \\ & +\ 0]1|23 & \otimes & 13]2 & x = 0]1|3|2 \\ & +\ (0]12|3 + 0]1|23) & \otimes & 3]12 & x = 0]3|1|2 \\ & +\ 0]12|3 & \otimes & (2]13 + 23]1) & x = 0]2|1|3 \\ & +\ 0]2|13 & \otimes & 23]1 & x = 0]2|3|1 \\ & +\ (0]12|3 + 2]1|3) & \otimes & 13] & x = 0]1|3 \\ & +\ 2]13 & \otimes & 3]1 & x = 0]3|1 \\ & +\ (0]1|23 + 0]13|2 + 3]1|2) & \otimes & 12] & x = 0]1|2 \\ & +\ (0]123 + 3]12) & \otimes & 2]1 & x = 0]2|1 \\ & +\ 1]2|3 & \otimes & 23] & x = 0]2|3 \\ & +\ 1]23 & \otimes & 3]2 & x = 0]3|2 \\ & +\ (0]123 + 3]12 + 2]13 + 23]1) & \otimes & 1] & x = 0]1 \\ & +\ (1]23 + 13]2) & \otimes & 2] & x = 0]2 \\ & +\ 12]3 & \otimes & 3] & x = 0]3 \\ & +\ 123] & \otimes & 0] & x = 0]. \end{array}$$

6.4. **The diagonal on a permutocubical set.** Now we use the above combinatorial description of an orthogonal stream to define an explicit diagonal for a permutocubical set $\mathcal{B} = \{\mathcal{B}_{p,q}\}_{p\geq 0; q\geq 1}$.

A coproduct
$$\Delta : C_*(\mathcal{B}) \to C_*(\mathcal{B}) \otimes C_*(\mathcal{B})$$
is defined for $a \in \mathcal{B}_{p,q}$ by

$$\Delta(a) = \sum_{\substack{(u_1,u_2)\in OS(B_p)\\ (v_1,v_2)\in OS(P_q)}} sgn(u_1,u_2)\cdot sgn(v_1,v_2)\cdot(-1)^{\epsilon}\, d_{u_1}d_{v_1}(a)\otimes d_{u_2}d_{v_2}(a), \tag{9}$$

$\epsilon = |d_{u_2}(a)||d_{v_1}(a)|$.

7. The permutocubical model for the path fibration

Let $\Omega Y \xrightarrow{i} PY \xrightarrow{\pi} Y$ be the Moore path fibration on a topological space Y. In [1] Adams constructed a dga map
$$\Omega C_*(Y) \to C_*^{\square}(\Omega Y)$$
being a weak equivalence for a simply connected Y, where C_* denotes the singular simplicial chain complex, while in [2] Adams and Hilton constructed a model for the path fibration using the singular cubical complex for each term of the fibration. Here we obtain a natural combinatorial model for the path fibration where for the base the singular cubical complex and for the fibre the singular multipermutahedral complex are taken; the total space in this case is modeled by the permutocubical set being a twisted Cartesian product described in Section 5. This model is naturally mapped into the singular permutocubical complex of the total space. The chain complex of the obtained model is a (comultiplicative) twisted tensor product, while the Adams-Hilton model is not. In particular, the acyclic cobar construction $\Omega\left(C_*^{\square}(Y); C_*^{\square}(Y)\right)$ coincides with the chain complex of the permutocubical set (compare, Theorem 5.1 in [13]).

For a space Y, let $\iota_0 : \operatorname{Sing}^M Y \to \operatorname{Sing}^B Y$ be an inclusion of sets induced by the identification $P_q = B_0 \times P_q$. Let denote $\iota_* = \iota_0 \circ i_* : \operatorname{Sing}^M \Omega Y \to \operatorname{Sing}^B PY$. Let $(\varphi\times\rho)_* : \operatorname{Sing}^I Y \to \operatorname{Sing}^B Y$ be a natural map of graded sets from Example 4.1. Then we have the following theorem (compare, [17], [8], [3]).

Theorem 7.1. *Let $\Omega Y \xrightarrow{i} PY \xrightarrow{\pi} Y$ be the Moore path fibration.*

(i) There are natural morphisms $\omega, p, (\varphi\times\rho)_*$ such that

$$\begin{array}{ccccc}
\operatorname{Sing}^P \Omega Y & \xrightarrow{\iota_*} & \operatorname{Sing}^B PY & \xrightarrow{\pi_*} & \operatorname{Sing}^B Y\\
\omega\uparrow & & p\uparrow & & (\varphi\times\rho)_*\uparrow\\
\mathbf{\Omega}\operatorname{Sing}^{1^I} Y & \xrightarrow{\iota} & \mathbf{P}\operatorname{Sing}^{1^I} Y & \xrightarrow{\xi} & \operatorname{Sing}^{1^I} Y,
\end{array} \tag{10}$$

$(\varphi\times\rho)_*$ is a map of graded sets induced by $\varphi\times\rho : B_p\times P_{n-p+1}\to I^n$, while p is a morphism of permutocubical sets, and ω is a morphism of monoidal permutahedral sets; p and ω are homotopy equivalences whenever Y is simply connected.

(ii) The chain complex $C_*^{\diamondsuit}(\mathbf{\Omega}\operatorname{Sing}^{1^I} Y)$ coincides with the cobar construction $\Omega C_*^{\square}(Y)$.

(iii) The chain complex $C_*^{\boxminus}(\mathbf{P}\mathrm{Sing}^{1^I}Y)$ coincides with the acyclic cobar construction $\Omega\left(C_*^{\square}(Y)\,;C_*^{\square}(Y)\right)$.

Proof. (i). Morphisms p and ω are constructed simultaneously by induction on the dimension of singular cubes in $\mathrm{Sing}^{1^I}Y$. For $i=0,1$ and $(\sigma,e)\in\mathbf{P}\mathrm{Sing}^{1^I}Y$, $\sigma\in\mathrm{Sing}^{1^I}_{i}Y$, define $p\,(\sigma,e)$ as the constant map $B_i\to PY$ to the base point y, where e denotes the unit of the monoid $\mathbf{\Omega}\mathrm{Sing}^{1^I}Y$ (and of the monoid $\mathrm{Sing}^P\,\Omega Y$ as well). Put $\omega(e)=e$.

Denote by $\mathbf{P}\mathrm{Sing}^{1^I}_{(i,j)}Y$ the subset in $\mathbf{P}\mathrm{Sing}^{1^I}Y$ consisting of the elements (σ,τ) with $|\sigma|\le i$ and $\tau\in\mathbf{\Omega}\mathrm{Sing}^{1^I}_{(j)}Y$, a submonoid in $\mathbf{\Omega}\mathrm{Sing}^{1^I}Y$ having (monoidal) generators $\bar\sigma$ with $|\bar\sigma|\le j$.

Suppose by induction that we have constructed p and ω on $\mathbf{P}\mathrm{Sing}^{1^I}_{(n-1,n-2)}Y$ and $\mathbf{\Omega}\mathrm{Sing}^{1^I}_{(n-2)}Y$ respectively such that

$$p\,(\sigma,\tau)=p\,(\sigma,e)\cdot\omega(\tau)\quad\text{and}\quad(\iota_*\circ\omega)(\bar\sigma)=p\,(d_{0]\underline{r}}(\sigma,e)),\ r=|\sigma|,\ 1\le r<n,$$

where the $\cdot$ product is determined by the action $PY\times\Omega Y\to PY$. Let $\bar B_n\subset B_n$ be the union of the all $(n-1)$-faces of B_n except the $d_{0]\underline{n}}(B_n)$, and then for a singular cube $\sigma:I^n\to Y$ define the map $\bar p:\bar B_n\to PY$ by

$$\bar p\,|_{d_i(B_n)}=p\,(d_i(\sigma,e)),\ 1\le i\le n,\ \text{ and }\ \bar p\,|_{d_{A]M}(B_n)}=p\,(d_{A]M}(\sigma,e)),\ A]M\neq 0]\underline{n}.$$

Then the following diagram commutes:

$$\begin{array}{ccccc}
\bar B_n & \xrightarrow{\bar p_\sigma} & P_\sigma Y & \xrightarrow{g_\sigma} & PY\\
{\scriptstyle\bar i}\downarrow & & {\scriptstyle\pi_\sigma}\downarrow & & {\scriptstyle\pi}\downarrow\\
B_n & \xrightarrow{\varphi} & I^n & \xrightarrow{\sigma} & Y.
\end{array}$$

Clearly, $\bar i$ is a strong deformation retraction and we define $p\,(\sigma,e):B_n\to PY$ as a lift of φ. Define $p\,(d_{0]\underline{n}}(\sigma,e))=p(\sigma,e)|_{d_{0]\underline{n}}(B_n)}$, and then $\omega(\bar\sigma)$ is determined by $(\iota_*\circ\omega)(\bar\sigma)=p(\sigma,e)\circ\delta_{0]\underline{n}}:P_n\to B_n\to PY$.

The proof of p and ω being homotopy equivalences (after the geometric realizations) immediately follows, for example, from the observation that ξ induces a long exact homotopy sequence. The last statement is a consequence of the following two facts: (1) $|\mathbf{P}\mathrm{Sing}^{1^I}Y|$ is contractible, (2) The projection ξ induces an isomorphism $\pi_*(|\mathbf{P}\mathrm{Sing}^{1^I}Y|,|\mathbf{\Omega}\mathrm{Sing}^{1^I}Y|)\xrightarrow{\xi_*}\pi_*(|\mathrm{Sing}^{1^I}Y|)$.

(ii)-(iii). This is straightforward. $\square$

Thus, by passing on chain complexes in diagram (10) we obtain the following comultiplicative model of π formed by dgc's (not necessarily coassociative ones).

Corollary 7.1. *For the path fibration $\Omega Y\xrightarrow{i}PY\xrightarrow{\pi}Y$ there is a comultiplicative model formed by dgc's which is natural in Y:*

$$\begin{array}{ccccc}
C_*^{\diamondsuit}(\Omega Y) & \xrightarrow{\iota_*} & C_*^{\boxminus}(PY) & \xrightarrow{\pi_*} & C_*^{\boxminus}(Y)\\
{\scriptstyle\omega_*}\uparrow & & {\scriptstyle p_*}\uparrow & & {\scriptstyle(\varphi\times\rho)_*}\uparrow\\
\Omega C_*^{\square}(Y) & \longrightarrow & \Omega\left(C_*^{\square}(Y);C_*^{\square}(Y)\right) & \xrightarrow{\xi_*} & C_*^{\square}(Y).
\end{array}$$

8. Permutocubical models for fibrations

Here we prove the main result in this paper. Let G be a topological group, F be a G-space $G\times F\to F$, $G\to P\xrightarrow{\pi} Y$ be a principal G-bundle and $F\to E\xrightarrow{\zeta} Y$ be the associated fibration with the fiber F. Let $Q=\mathrm{Sing}^{1^I}Y$, $\mathcal{Z}=\mathrm{Sing}^M G$ and $\mathcal{L}=\mathrm{Sing}^M F$. The group operation $G\times G\to G$ induces the structure of a monoidal multipermutahedral set on $\mathcal{Z}$, and the action $G\times F\to F$ induces $\mathcal{Z}$-module structure $\mathcal{Z}\times\mathcal{L}\to\mathcal{L}$ on $\mathcal{L}$ (c.f. Example 2.2).

Theorem 8.1. *The principal G-fibration $G\to P\xrightarrow{\pi} Y$ determines a truncating twisting function $\vartheta:\mathrm{Sing}^{1^I}Y\to\mathrm{Sing}^M G$ such that twisted Cartesian product $\mathrm{Sing}^{1^I}Y\times_\vartheta\mathrm{Sing}^M F$ models the total space E of the associated fibration $F\to E\xrightarrow{\zeta} Y$, that is, there exists a permutocubical map*

$$\mathrm{Sing}^{1^I}Y\times_\vartheta\mathrm{Sing}^M F\to\mathrm{Sing}^B E$$

inducing homology isomorphism.

Proof. Let $\omega:\mathbf{\Omega}Q\to\mathrm{Sing}^M\Omega Y$ be the map of monoidal multipermutahedral sets from Theorem 7.1. By Proposition 5.1 ω corresponds to a truncating twisting function $\vartheta':Q=\mathrm{Sing}^{1^I}Y\xrightarrow{\vartheta_U}\mathbf{\Omega}Q=\mathbf{\Omega}\mathrm{Sing}^{1^I}Y\xrightarrow{\omega}\mathrm{Sing}^M\Omega Y$. Composing ϑ' with the map of monoidal multipermutahedral sets $\mathrm{Sing}^M\Omega Y\to\mathrm{Sing}^M G=\mathcal{Z}$ induced by the canonical map $\Omega Y\to G$ of monoids we obtain a truncating twisting function $\vartheta:Q\to\mathcal{Z}$. The resulting twisted Cartesian product $\mathrm{Sing}^{1^I}Y\times_\vartheta\mathrm{Sing}^M F$ is a permutocubical model of E. Indeed, we have the canonical equality

$$Q\times_\vartheta\mathcal{L}=(Q\times_\vartheta\mathcal{Z})\times\mathcal{L}/\sim,$$

where $(xg,y)\sim(x,gy)$. Next the argument of the proof of Theorem 7.1 gives a permutocubical map $f':Q\times_{\vartheta_U}\mathbf{\Omega}Q\to\mathrm{Sing}^B P$ preserving the actions of $\mathbf{\Omega}Q$ and $\mathcal{Z}$. Hence, this map extents to a permutocubical map $f:Q\times_\vartheta\mathcal{Z}\to\mathrm{Sing}^B P$ by $f(x,g)=f'(x,e)g$. The map

$$(Q\times_\vartheta\mathcal{Z})\times\mathcal{L}\xrightarrow{f\times 1}\mathrm{Sing}^B P\times\mathcal{L}\xrightarrow{\lambda}\mathrm{Sing}^B(P\times F),$$
$$\lambda(h_1,h_2)=(h_1\times h_2)\circ(1_B\times\Delta_{r,s}),$$

induces the map of permutocubical sets

$$\mathrm{Sing}^{1^I}Y\times_\vartheta\mathrm{Sing}^M F\to\mathrm{Sing}^B E$$

as desired. □

For convenience, assume that $Q,\mathcal{Z}$ and $\mathcal{L}$ are as in Definition 5.2. On the chain level a truncating twisting function ϑ induces the twisting cochains $\vartheta_*:C_*^{\square}(Q)\to C_{*-1}^{\diamondsuit}(\mathcal{Z})$ and $\vartheta^*:C^*_{\diamondsuit}(\mathcal{Z})\to C^{*+1}_{\square}(Q)$ in the standard sense ([7], [5], [11]). It is straightforward to verify that the following equality holds:

$$C_*^{\boxminus}(Q\times_\vartheta\mathcal{L})=C_*^{\square}(Q)\otimes_{\vartheta_*}C_*^{\diamondsuit}(\mathcal{L}),\tag{11}$$

and, consequently, the obvious injection

$$C^*_{\boxminus}(Q\times_\vartheta\mathcal{L})\supset C^*_{\square}(Q)\otimes_{\vartheta^*}C^*_{\diamondsuit}(\mathcal{L})\tag{12}$$

of dg modules (which is an equality if the graded sets are of finite type).

The permutocubical structure of $Q\times_{\vartheta}\mathcal{L}$ induces a dgc sturcture on $C_*^{\boxminus}(Q\times_{\vartheta}\mathcal{L})$. Transporting this structure (diagonal (9)) on the right-hand side of (11) we obtain a *comultiplicative* model of $C_*^{\square}(Q)\otimes_{\vartheta}C_*^{\diamondsuit}(\mathcal{L})$ of our fibration. Dually, $C^*_{\boxminus}(Q\times_{\vartheta}\mathcal{L})$ is a dga, so a dga structure (a multiplication) arises on the right-hand side of (12) and we obtain a *multiplicative* model $C^*_{\square}(Q)\otimes_{\vartheta}C^*_{\diamondsuit}(\mathcal{L})$ of our fibration.

Below we describe these structures (the comultiplication on $C_*^{\square}(Q)\otimes_{\vartheta}C_*^{\diamondsuit}(\mathcal{L})$ and the multiplication on $C^*_{\square}(Q)\otimes_{\vartheta}C^*_{\diamondsuit}(\mathcal{L})$ in terms of certain (co)chain operations that form a *Hirsch (co)algebra* structure on the (co)chain complex of Q.

8.1. **The canonical Hirsch algebra structure on $C^*_{\square}(Q)$.** Consider the equality

$$C_*^{\diamondsuit}(\mathbf{\Omega}Q)=\Omega C_*^{\square}(Q)$$

from Theorem 7.1. As before, the permutahedral structure of $\mathbf{\Omega}Q$ induces a coproduct on $C_*^{\square}(\mathbf{\Omega}Q)$ ([19]); consequently, this structure also appears on the right-hand side of the above equality, so that the cobar construction $\Omega C_*(Q)$ becomes a dg Hopf algebra.

To describe the above coproduct in terms of generators (singular cubes) we need the following combinatorial analysis of the diagonal Δ_P on permutahedra (compare [4], [13]).

Given an ordered subset $B\subset\mathbb{N}\cup 0$ and $a,b\in B$ with $a<b$, let $[a\cdots b]=\{x\in B\,|\,a\leq x\leq b\}$ be a *block*; for $A=(a_1<\cdots<a_k)\subset B$, let

$$J_B(A)=[a_1\cdots a_2]\cdot[a_2\cdots a_3]\cdots[a_{k-1}\cdots a_k]$$

be a sequence of blocks; then $\bar J_B(A)$ can be thought of as a generator of the monoid $\mathbf{\Omega}Q$, i.e., $\bar J_B(A)\in\bar Q$ (recall Proposition 3.1 from [13] that we have the correspondence between sequences of such blocks and compositions of cubical face operators). On the other hand, Δ_P can be expressed on $\underline{n}$ as

$$\Delta_P(\underline{n})=\sum_{(u,v)\in OS(P_n)}\bar J_{\tilde u_1}(\tilde u_2)\cdots\bar J_{\tilde u_{p-1}}(\tilde u_p)\otimes\bar J_{\tilde v_1}(\tilde v_2)\cdots\bar J_{\tilde v_{q-1}}(\tilde v_q),$$

for $(\tilde u_i\,,\tilde v_j)=(u_i\cup 0\cup(n+1)\,,v_j\cup 0\cup(n+1))$, $(u_i,v_j)=(A_i\cup\cdots\cup A_p\,,C_j\cup\cdots\cup C_q)_{1\leq i\leq p,1\leq j\leq q}$, $(u,v)=(A_1|...|A_p\,,C_1|...|C_q)$. Consider the identification $J_{w_i}(w_{i+1})=d^0_{w_{i+1}}d^1_{\underline{n}\setminus w_i}([01\cdots n+1])$, $w=u,v$, to obtain the following formula for the coproduct $\Delta:\Omega C_*^{\square}(Q)\to\Omega C_*^{\square}(Q)\otimes\Omega C_*^{\square}(Q)$: For a generator $\sigma\in C_n^{\square}(Q)$, let

$$\Delta(\bar\sigma)=\sum_{(u,v)\in OS(P_n)}sgn(u,v)\left(\bigotimes_{i=1}^{p}\overline{d^0_{u_{i+1}}d^1_{\underline{n}\setminus u_i}(\sigma)}\right)\otimes\left(\bigotimes_{i=1}^{q}\overline{d^0_{v_{i+1}}d^1_{\underline{n}\setminus v_i}(\sigma)}\right).\tag{13}$$

Note that since Q is assumed to be 1-reduced, the image $\overline{d^0_{w_{i+1}}d^1_{\underline{n}\setminus w_i}(\sigma)}$ of a 1-dimensional face $d^0_{w_{i+1}}d^1_{\underline{n}\setminus w_i}(\sigma)$ for $w=u,v$, is the unit in $\Omega C_*^{\square}(Q)$ and hence can be omitted.

Actually the diagonal consists of *components*

$$E^{p,q}=pr\circ\Delta:C_*^{\square}(Q)\to\Omega C_*^{\square}(Q)\otimes\Omega C_*^{\square}(Q)\to C_*^{\square}(Q)^{\otimes p}\otimes C_*^{\square}(Q)^{\otimes q},\ p,q\geq 1,$$

where pr is the obvious projection.

The basic component $E^{1,1}$ is formed by those pairs $(u,v)\in OS(P_n)$ in which all but one pair satisfy $(\#A_i,\#C_j)=(1,1)$; this component is a chain operation dual to the cubical version of Steenrod's $\smile_1$-product.

Dualizing the operations $E^{p,q}$, we obtain the sequence of cochain operations

$$\{E_{p,q} : C^*_\square(Q)^{\otimes p} \otimes C^*_\square(Q)^{\otimes q} \to C^*_\square(Q)\}_{p+q\geq 0},$$

which define a multiplication on the bar construction $BC^*_\square(Q) \otimes BC^*_\square(Q) \to BC^*_\square(Q)$. These cochain operations form on $C^*_\square(Q)$ the structure of a *Hirsch algebra* (see the next section).

The operations $E_{p,q}$ are restrictions of more general cochain operations that arise on $\bar{C}^*_\square(Q)$ (the non-normalized chains) for a based space Y which is not necessarily 1-connected. In this case, for $Q = \operatorname{Sing}^I Y$, we have the operations

$$\{E_{p,q} : \bar{C}^*_\square(Q)^{\otimes p} \otimes \bar{C}^*_\square(Q)^{\otimes q} \to \bar{C}^*_\square(Q)\}_{p,q\geq 1}$$

given by the following explicit formulas: For $a_i \in \bar{C}^{\geq 2}(Q)$, $b_j \in \bar{C}^{\geq 2}(Q)$, $1 \leq i \leq p$, $1 \leq j \leq q$, let

$$E_{p,q}(a_1, ..., a_p;\, b_1, ..., b_q) = \sum_{s\geq p;\, t\geq q} \bar{E}_{s,t}(\epsilon^1, a_1, \epsilon^1, ..., \epsilon^1, a_p, \epsilon^1;\, \epsilon^1, b_1, \epsilon^1, ..., \epsilon^1, b_q, \epsilon^1),$$

$\epsilon^1 \in \bar{C}^1(Q)$ is the generator represented by the constant singular 1-cube at the base point $I \to y \in Y$ and the operations $\bar{E}_{s,t}$ are defined for $a_i \in \bar{C}^{k_i}_\square(Q)$, $b_j \in \bar{C}^{r_j}_\square(Q)$, $\sigma \in Q_n$, by

$$\bar{E}_{s,t}(a_1, ..., a_s\, ;\, b_1, ..., b_t) = c \in \bar{C}^n_\square(Q),$$

$$c(\sigma) = \sum_{\substack{u\in\mathcal{P}_{k_1,\ldots,k_s}(n)\\ v\in\mathcal{P}_{r_1,\ldots,r_t}(n)\\ (u,v)\in OS(P_n)}} sgn(u,v)\, a_1(\sigma_1)\cdots a_s(\sigma_s)\cdot b_1(\sigma'_1)\cdots b_t(\sigma'_t),$$

$$\sigma_i = d^0_{u_{i+1}} d^1_{\underline{n}\setminus u_i}(\sigma),\ 1 \leq i \leq s, \quad \sigma'_j = d^0_{v_{j+1}} d^1_{\underline{n}\setminus v_j}(\sigma),\ 1 \leq j \leq t,$$

where (u_i, v_j) is as in $\Delta_P(\underline{n})$ above, and where $\bar{E}_{s,t}(a_1, ..., a_s; b_1, ..., b_t) = 0$ otherwise.

Thus, the above formula for $p, q = 1$ defines $E_{1,1}$ as the cubical version of Steenrod's cochain $\smile_1$-operation without any restriction on Y.

Remark 8.1. *The operations* $\{E^{p,q}\}$ *on* $C^\square_*(Q) = \Omega C_*(X)$, $Q = \mathbf{\Omega}\operatorname{Sing}^2 X$, *in fact have the form*

$$E^{p,q} = \sum \Delta_E^{p-1} \otimes \Delta_E^{q-1}$$

where $\Delta^k_E : \Omega C_*(X) \to \Omega C_*(X)^{\otimes k+1}$ *is the* k*-th iteration of the comultiplication* $\Delta_E : \Omega C_*(X) \to \Omega C_*(X) \otimes \Omega C_*(X)$ *being itself induced by the homotopy* G*-coalgebra structure* $\{E^{k,1}\}$ *on* $C_*(X)$ *(c.f.* [13]*).*

8.2. **Twisted multiplicative model for a fibration.** Next we further explore the twisted Cartesian product $Q\times_\vartheta \mathcal{L}$. To describe the corresponding coproduct and product on the right-hand sides of (11) and (12) respectively, it is very convenient to express the diagonal Δ_B in terms of combinatorics of the cubes and permutahedra. Namely, for the top cell $\underline{n_0}$ of B_n, let

$$\Delta_B(\underline{n_0}) = [01\cdots n+1] \otimes [0, n+1] + \sum_{\substack{\underline{n}\setminus N | N\in\mathcal{P}^0_{n-s,s}(n)\\ (u,v)\in OS(P_s)}} J_{u_0}(u'_1)\cdot \bar{J}_{u'_1}(u'_2)\cdots \bar{J}_{u'_{p-1}}(u'_p) \otimes J_{v'_1}(v'_2)\cdot \bar{J}_{v'_2}(v'_3)\cdots \bar{J}_{v'_{q-1}}(v'_q), \tag{14}$$

for $u_0 = \{01\cdots n+1\}$, $(u'_i, v'_j) = (I^{-1}_{N'}(\tilde{u}_i), I^{-1}_{N'}(\tilde{v}_j))$, $N' = N \cup 0 \cup (n+1)$, where $(\tilde{u}_i, \tilde{v}_j)_{1\leq i\leq p, 1\leq j\leq q}$ is as in $\Delta_P(\underline{s})$ above. In particular, the summand $[01\cdots n{+}1]\otimes[0, n+1]$ is a primitive component of the diagonal, while the second one is obtained by $N = \underline{n}$, $(u,v) = (1|2|...|n\,,\underline{n})$, and is equal to

$$[01][12]...[n,n+1]\cdot\overline{[012][23]...[n,n+1]}\cdot\overline{[023][34]...[n,n+1]}\cdots\overline{[0,n,n+1]}\otimes \otimes[01...n+1].$$

Remark 8.2. *Note that we abuse the notation when we mean under* $[01\cdots n+1]$ *an* n*-permutocube, since this notation was used for combinatorial description of the* n*-cube in* [13]. *Accordingly here* $\overline{[01\cdots n+1]}$ *corresponds to the* $(n-1)$*-permutahedron.*

Furthermore, the action $\mathcal{Z}\times\mathcal{L}\to\mathcal{L}$ induces a comodule structure $\Delta_{\mathcal{L}} : C^*_{\diamondsuit}(\mathcal{L}) \to C^*_{\diamondsuit}(\mathcal{Z})\otimes C^*_{\diamondsuit}(\mathcal{L})$ and it is not hard to see that the permutocubical multiplication of (12) can be expressed by this comodule structure, the diagonal (14), the twisting cochain ϑ^*, and the operations $\{E_{p,q}\}_{p,q\geq 1}$ by the following formula: Let $a_1\otimes m_1, a_2\otimes m_2 \in C^*_{\square}(Q)\otimes_{\vartheta^*} C^*_{\diamondsuit}(\mathcal{L})$ and $\Delta^k_L : C^*_{\diamondsuit}(\mathcal{L}) \to C^*_{\diamondsuit}(\mathcal{Z})^{\otimes k}\otimes C^*_{\diamondsuit}(\mathcal{L})$ be the iterated $\Delta_{\mathcal{L}}$ with $\Delta^0_{\mathcal{L}} = \mathrm{Id} : C^*_{\diamondsuit}(\mathcal{L}) \to C^*_{\diamondsuit}(\mathcal{L})$; let $\Delta^p_{\mathcal{L}}(m_1) = \sum c^1_1\otimes\ldots\otimes c^p_1\otimes m^{p+1}_1$ and $\Delta^{q-1}_{\mathcal{L}}(m_2) = \sum c^1_2\otimes\ldots\otimes c^{q-1}_2\otimes m^q_2$; then

$$\mu((a_1\otimes m_1)\otimes(a_2\otimes m_2)) = \sum_{p\geq 0;\ q\geq 1} (-1)^{\epsilon} a_1 E_{p,q}(\vartheta(c^1_1),...,\vartheta(c^p_1); a_2, \vartheta(c^1_2),...,\vartheta(c^{q-1}_2))\otimes m^{p+1}_1 m^q_2, \tag{15}$$

$\epsilon = |m^{p+1}_1|(|a_2| + |c^1_2| + \cdots + |c^{q-1}_2|)$.

Corollary 8.1. *Under the circumstances of Theorem 8.1, the twisted differential* d_ϑ *and multiplication* μ *turn the tensor product* $C^*_{\square}(Z)\otimes C^*_{\diamondsuit}(F)$ *into a dga* $(C^*_{\square}(Z)\otimes C^*_{\diamondsuit}(F), d_\vartheta, \mu_\vartheta)$ *weakly equivalent to the dga* $C^*_{\diamondsuit}(E)$.

Corollary 8.2. *There exists on the acyclic bar construction* $B(C^*_{\square}(Z); C^*_{\square}(Z))$ *the following multiplication: For* $a = a_0\otimes[\bar{a}_1|\cdots|\bar{a}_n]$, $\ b = b_0\otimes[\bar{b}_1|\cdots|\bar{b}_m]$, $\ a_i, b_j \in C^*_{\square}(Z)$, $\ 0\leq i\leq n$, $\ 0\leq j\leq m$, *let*

$$ab = \sum_{p\geq 0;\, q\geq 1} (-1)^{\epsilon} a_0 E_{p,q}(a_1,...,a_p\,; b_0, b_1,...,b_{q-1})\otimes[\bar{a}_{p+1}|\cdots|\bar{a}_n]\circ[\bar{b}_q|\cdots|\bar{b}_m], \tag{16}$$

$\epsilon = (|\bar{a}_{p+1}| + \cdots + |\bar{a}_n|)\left(|b_0| + |\bar{b}_1| + \cdots + |\bar{b}_{q-1}|\right).$

Proof. Take $\mathcal{Z} = \mathcal{L} = \mathbf{\Omega}Q$. Then the multiplication (15) looks as (16). □

Using the fact that $BC^*(Y)$ has an associative multiplication [13] we canonically introduce on the acyclic bar construction $B(BC^*(Y); BC^*(Y))$ the multiplication by (16) that agrees with the one on the double bar construction $BBC^*(Y)$ [19].

9. Twisted tensor products for Hirsch algebras

The notion of Hirsch (co)algebra naturally generalizes that of a homotopy G-(co)algebra. We generalize the theory of *multiplicative* twisted tensor products for homotopy G-algebras, and, consequently, for commutative dga's [13]. Namely we define a twisted tensor product with both twisted differential and *twisted multiplication* inspired by formulas (15) and (16) established in the previous section.

Let A be a dga and consider the dg module $(Hom(BA\otimes BA,A),\nabla)$ with differential ∇. The $\smile$-product induces a dga structure (the tensor product $BA\otimes BA$ is a dgc with the standard coalgebra structure).

Definition 9.1. *A Hirsch algebra is a 1-reduced associative dga A eqwipped with multilinear maps*

$$E_{p,q}:A^{\otimes p}\otimes A^{\otimes q}\to A,\ p,q\geq 0,\ p+q>0,$$

satisfying the following conditions:

(i) $E_{p,q}$ *is of degree* $1-p-q$*;*
(ii) $E_{1,0}=Id=E_{0,1}$ *and* $E_{p>0,0}=0=E_{0,q>0}$;
(iii) *The homomorphism $E:BA\otimes BA\to A$ defined by*

$$E([\bar a_1|\cdots|\bar a_p]\otimes[\bar b_1|\cdots|\bar b_q])=E_{p,q}(a_1,...,a_p;b_1,...,b_q)$$

is a twisting element in the dga $(Hom(BA\otimes BA,A),\nabla)$, i.e., it satisfies $\nabla E=-E\smile E$.

Condition (iii) implies that μ_E is a chain map; thus BA becomes a dg Hopf algebra with not necessarily associative multiplication μ_E (c.f. [10], [21]). Condition (iii) can be rewritten in terms of components $E_{p,q}$. In particular, the operation $E_{1,1}$ satisfies conditions similar to Steenrod's $\smile_1$ product:

$$dE_{1,1}(a;b)-E_{1,1}(da;b)+(-1)^{|a|}E_{1,1}(a;db)=(-1)^{|a|}ab-(-1)^{|a|(|b|+1)}ba,$$

so it measures the non-commutativity of the product of A (thus, a Hirsch algebra with $E_{p,q}=0$ for $p,q\geq 1$ is just a commutative dga).

The dual notion is that of a Hirsch coalgebra. For a Hirsch coalgebra $(C,d,\Delta,\{E^{p,q}:C\to C^{\otimes p}\otimes C^{\otimes q}\})$, the cobar construction ΩC is a dg Hopf algebra with a comultiplication induced by $\{E^{p,q}\}$.

Main examples of Hirsch (co)algebras are: $C^*_{\square}(Q)$ (see the previous section), in particular, Adams' cobar construction $\Omega C_*(X)$ ([19]), and the singular simplicial cochain complex $C^*(X)$: In [16] a twisting cochain $E:BC^*(X)\otimes BC^*(X)\to C^*(X)$ satisfying (i)-(iii) is constructed and these conditions determined E uniquely up to the standard equivalence of twisting cochains.

9.1. **Multiplicative twisted tensor products.** Let A be a Hirsch algebra, C be a dg Hopf algebra, and M be a dga being a dg comodule over C.

Definition 9.2. *A twisting cochain $\vartheta:C\to A$ in $Hom(C,A)$ is multiplicative if the comultiplicative extension $C\to BA$ is an algebra map.*

It is clear that if $\vartheta:C\to A$ is a multiplicative twisting element and if $g:B\to C$ is a map of dg Hopf algebras then the composition $\vartheta g:B\to A$ is again a multiplicative twisting cochain. The canonical projection $BA\to A$ provides an example of the universal multiplicative cochain. The argument for the proof of formula (15) immediately yields the following:

Theorem 9.1. *Let $\vartheta:C\to A$ be a multiplicative twisting cochain. Then the tensor product $A\otimes M$ with the twisting differential $d_\vartheta=d\otimes Id+Id\otimes d+\vartheta\cap_-$ becomes a dga $(A\otimes M,d_\vartheta,\mu_\vartheta)$ with the twisted multiplication μ_ϑ determined by formula (15).*

The above theorem includes the twisted tensor product theory both for homotopy G-algebras ([13]) and for commutative algebras ([18]).

Corollary 9.1. *For a Hirsch algebra A, the acyclic bar construction $B(A;A)$ endowed with the twisted multiplication determined by formula (16) asquires a dga structure.*

9.2. **Examples.** For simplicity we assume that the ground ring R is a field, and all spaces are path connected. In examples below we further explore the fact that for a space being a suspension the corresponding homotopy G-algebra structure is extremely simple: it consists just of $E_{1,1} = \smile_1$ and all other operations $E_{k>1,1}$ are trivial [13], and so does the corresponding Hirsch algebra structure for the loop space on a double suspension.

1. *Multiplicative models for* Ω^2S^2X. Given a polyhedron X, consider the space $Y = \Omega S^2X$. As in [13] we regard a suspension SX as the geometric realization of quotient simplicial set $C_+X \cup C_-X/C_-X$. It is immediate to check by (13) that $E^{p,q} = 0$ for $(p,q) \neq (1,1)$ on $C_*^{\square}(Q)$, where $Q = \mathbf{\Omega}S^2X$, $\mathbf{\Omega}$ is the cubical set functor constructed in [13] (if we had $Q = \mathbf{\Omega}SX$, then the Hircsh coalgebra structure would be reduced to that of homotopy G-coalgebra on $C_*^{\square}(Q)$; verification of this fact is left to the interested reader). Furthermore, $E^{1,1} : C_*^{\square}(Q) \to C_*^{\square}(Q) \otimes C_*^{\square}(Q)$ becomes a coassociative chain map of degree 1. Since the comultiplication on $C_*^{\square}(Q)$ is cocomutative (more precisely, $C_*^{\square}(Q)$ is a primitively generated Hopf algebra), $E^{1,1}$ also induces a binary cooperation of degree 1 on the homology denoted by $Sq^{1,1} : H_*(Q) \to H_*(Q) \otimes H_*(Q)$.

Notice that both $(C_*^{\square}(Q), d, \Delta, E^{1,1})$ and $(H_*(Q), d = 0, \Delta_*, Sq^{1,1})$ are Hirsch coalgebras, thus $\Omega C_*^{\square}(Q)$ and $\Omega H_*(Q)$ both are dg Hopf algebras.

Similarly to [13] the cycle choosing homomorphism $\iota : H_*(Q) \to C_*^{\square}(Q)$ is a dg coalgebra map and induces an isomorphism of dg Hopf algebras

$$H(\Omega T\tilde{H}_*(SX)) \xrightarrow{\approx} H(\Omega H_*(Q)) \xrightarrow{(\Omega\iota)_*} H_*(\Omega C_*^{\square}(Q)) = H_*(\Omega Y).$$

2. Let $\Omega Y \to PY \xrightarrow{\pi} Y$ be the Moore path fibration with the base $Y = \Omega S^2X$. Let $f : Y \to Z$ be a map, $\Omega Y \times \Omega Z \to \Omega Z$ be the induced action via the composition

$$\Omega Y \times \Omega Z \xrightarrow{\Omega f \times \mathrm{Id}} \Omega Z \times \Omega Z \to \Omega Z,$$

and $\Omega Z \to E_f \xrightarrow{\zeta} Y$ be the associated fibration; for simplicity assume that Z is the suspension and simply connected CW-complex of finite type, as well. We present two multiplicative models for the fibration ζ using the permutocubical model $Q \times_\vartheta \mathbf{\Omega}Z$ with the universal truncating twisting function $\vartheta = \vartheta_U : Q \to \mathbf{\Omega}Q$.

Notice that the twisted differential of the cochain complex $(C_{\boxminus}^*(Q \times_\vartheta \mathbf{\Omega}Z), d) = (C_{\square}^*(Q) \otimes C_{\diamondsuit}^*(\mathbf{\Omega}Z), d_{\vartheta\#}) = (C_{\square}^*(Y) \otimes BC_{\square}^*(Z), d_{\vartheta\#})$ with universal $\vartheta^\# : BC_{\square}^*(Q) \to C_{\square}^*(Q)$ becomes the form

$$d_{\vartheta\#}(a \otimes [\bar{m}^1|...|\bar{m}^n]) = da \otimes [\bar{m}^1|...|\bar{m}^n] + \sum_{k=1}^{n} a \otimes [\bar{m}^1|...|d\bar{m}^k|...|\bar{m}^n] + \\ a \cdot m_1 \otimes [\bar{m}^2|...|\bar{m}^n].$$

Since the simplified structure of the Hirsch algebra $(C_{\square}^*(Q), d, \mu, E_{1,1})$ formula (15) becomes the following form:

(17)

$$\mu_{\vartheta\#}((a_1 \otimes m_1)(a_2 \otimes m_2)) = a_1a_2 \otimes m_1m_2 + a_1E_{1,1}(f^\#(m_1^1), a_2) \otimes [\bar{m}_1^2|...|\bar{m}_1^n] \cdot m_2,$$

where $f^{\#}: C^*_{\square}(Z) \to C^*_{\square}(Q)$, $a_1, a_2 \in C^*_{\square}(Q)$, $m_1 = [\bar{m}_1^1|...|\bar{m}_1^n]$, $m_2 \in BC^*_{\square}(Z)$, $n \geq 0$.

So that we get that $H(C^*_{\square}(Y) \otimes BC^*_{\square}(Z), d_{\vartheta\#}, \mu_{\vartheta\#})$ and $H^*(E_f)$ are isomorphic as algebras.

On the other hand, let us consider the following multiplicative twisted tensor product $(H^*(Y) \otimes H^*(\mathbf{\Omega} Z), d_{\vartheta^*}) = (H^*(Y) \otimes BH^*(Z), d_{\vartheta^*})$ with universal $\vartheta^*: BH^*(Y) \to H^*(Y)$. The differential here is of the form:

$$d_{\vartheta^*}(a \otimes [\bar{m}^1|...|\bar{m}^n]) = a \cdot m_1 \otimes [\bar{m}^2|...|\bar{m}^n].$$

Again since the simplified structure of the Hirsch algebra $(H^*(Y), d = 0, \mu^*, Sq_{1,1})$ the formula (15) becomes the following form:

(18)

$$\mu_{\vartheta^*}((a_1 \otimes m_1)(a_2 \otimes m_2)) = a_1 a_2 \otimes m_1 m_2 + a_1 Sq_{1,1}(f^*(m_1^1), a_2) \otimes [\bar{m}_1^2|...|\bar{m}_1^n] \cdot m_2,$$

where $f^*: H^*(Z) \to H^*(Y)$, $a_1, a_2 \in H^*(Y)$, $m_1 = [\bar{m}_1^1|...|\bar{m}_1^n]$, $m_2 \in BH^*(Z)$, $n \geq 0$. Remark that for an element $a \in H^*(Y)$, one gets $Sq_{1,1}(a, a) = Sq_1(a)$, the Steenrod square.

We claim that $(H^*(Y) \otimes BH^*(Z), d_{\vartheta^*})$ is a "small" multiplicative model of the fibration ζ, i.e., $H(H^*(Y) \otimes BH^*(Z), d_{\vartheta^*})$ and $H^*(E_f)$ are isomorphic as algebras. Indeed, since the explicit formulas (17) and (18) it is straightforward to check that a "cocyle choosing" homomorphism $(H^*(Y) \otimes BH^*(Z), d_{\vartheta^*}) \to (C^*(Y) \otimes BC^*(Z), d_{\vartheta\#})$ induces an algebra isomorphism

$$H\left(H^*(Y) \otimes BH^*(Z), d_{\vartheta^*}\right) \xrightarrow{\approx} H\left(C^*(Y) \otimes BC^*(Z), d_{\vartheta\#}\right) \approx H^*(E_f)$$

as required.

As a byproduct we obtain that the multiplicative structure of the total space E_f does not depend on a map f in a sense that if $f^* = g^*$ then $H^*(E_f) = H^*(E_g)$ as algebras. Note also that this multiplicative structure is purely defined by the $\smile$, $\smile_1$ and $\smile_2$ operations on the simplicial cochain complex $C^*(S^2X)$.

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