Cochain operations defining Steenrod $\smile_i$-products in the bar construction

Tornike Kadeishvili

Abstract

The set of cochain multioperations defining Steenrod $\smile_i$-products in the bar construction is constructed in terms of surjection operad. This structure extends a Homotopy G-algebra structure which defines $\smile$ on the bar construction.

The Adams's cobar construction $\Omega C^*(X)$ of the chain complex of a topological space X determines the homology $H_*(\Omega X)$ of the loop space just additively.

Lather on Baues [1] had constructed the *geometric* diagonal

$$\nabla_0 : \Omega(C_*(X)) \to \Omega(C_*(X)) \otimes \Omega(C_*(X))$$

which turns the cobar construction into a DG-Hopf algebra. This diagonal allows to produce the next cobar construction $\Omega\Omega(C_*(X))$ which models the double loop space.

Our aim here is to define on $\Omega(C_*(X))$ geometric cooperations (dual to Steenrod's $\smile_i$-products)

$$\{\nabla_i : \Omega(C_*(X)) \to \Omega(C_*(X)) \otimes \Omega(C_*(X)), \ i = 0, 1, ...\}$$

satisfying the conditions

$$deg\nabla_i = i, \ \nabla_i d + (d \otimes 1 + 1 \otimes d)\nabla_i = \nabla_{i-1} + T\nabla_{i-1}, \qquad (1)$$

(we work here on Z_2 so the signs are ignored). These cooperations are necessary (but of course not sufficient) for the further iteration of the cobar construction.

1

Namely we present particular elements $\{E^i_{p,q}, \ i = 0, 1, ...; \ p, q = 1, 2, ...\}$ in the surjection operad ([10]) χ such that the corresponding chain multico-operations

$$\{E^i_{p,q} : C_*(X) \to (C_*(X)^{\otimes p}) \otimes (C_*(X)^{\otimes q}), \ p, q = 1, 2, ...\}$$

define ∇_i-s on the cobar construction $\Omega(C_*(X))$ and the corresponding cochain multioperations

$$\{E^i_{p,q} : (C^*(X)^{\otimes p}) \otimes (C^*(X)^{\otimes q}) \to C^*(X), \ p, q = 1, 2, ...\}$$

define $\smile_i$-s on the bar construction $B(C^*(X))$.

It is known that the $\smile_i$-products in $C^*(X)$ are represented by the following elements of χ:

$$\smile = (1, 2); \ \smile_1 = (1, 2, 1); \ \smile_2 = (1, 2, 1, 2); ... \ .$$

Let us consider them as first the line cochain operations.

The second line let be presented by a *homotopy G-algebra* structure ([3]) on $C^*(X)$ which consists of the sequence of operations

$$\{E_{1,q} : (C^*(X)) \otimes (C^*(X)^{\otimes q}) \to C^*(X), \ q = 1, 2, ...\},$$

these operations in fact define a multiplication in the bar construction $BC^*(X)$. They are represented by the following elements of χ ([10]):

$$E_{1,k} = (1, 2, 1, 3, 1, ..., 1, k, 1, k + 1, 1). \tag{2}$$

Bellow we present the next line cochain operations. We introduce the notion of *extended homotopy G-algebra*, this is a DG-algebra with certain additional structure which defines $\smile_i$-s on the bar construction. A main example of such an object is again $C^*(X)$. This structure consists of multi-operations

$$\{E^i_{p,q} : (C^*(X)^{\otimes p}) \otimes (C^*(X)^{\otimes q}) \to C^*(X), \ i = 0, 1, ..., \ p, q = 1, 2, ...\}.$$

We present particular elements $\{E^i_{p,q} \in \chi\}$ representing these operations. Particularly, $E^0_{p,q}$ coincides with homotopy G-algebra structure (2),

$$\begin{aligned}
E^1_{p,q} = \ & (1; p + 1, 1, p + 2, 1, ..., p + q - 1, 1, p + q; \\
& 1, p + q, 2, p + q, 3, ..., p; p + q);
\end{aligned}$$

and

$$\begin{aligned}
E^2_{p,q} = \textstyle\sum_{k=0}^{q-1} \ & (1; p + 1, 1, p + 2, 1, ..., 1, p + k + 1; \\
& 1, p + k + 1, 2, p + k + 1, 3, ..., p + k + 1, p; \\
& p + k + 1, p, p + k + 2, p, ..., p + q; p).
\end{aligned}$$

1   Homotopy G-algebras

In this section we recall the notion of homotopy G-algebra from [3] In order to extend it in the next section.

1.1   The notion of homotopy G-algebra

Definition 1 *A homotopy G algebra is a differential graded algebra (DG-algebra) $(A, d, \cdot)$ together with a given sequence of multioperations*

$$E_{1,k} : A \otimes (\otimes^k A) \to A, \quad k = 1, 2, 3, \dots \quad ,$$

subject of the following conditions

$deg E_{1,k} = -k, \; E_{1,0} = id;$

$$dE_{1,k}(a; b_1, ..., b_k) + E_{1,k}(da; b_1, ..., b_k) + \sum_i E_{1k}(a; b_1, ..., db_i, ..., b_k) =$$
$$b_1 E_{1k}(a; b_2, ..., b_k) + \sum_i E_{1k}(a; b_1, ..., b_i b_{i+1}, ..., b_k) + \qquad (3)$$
$$E_{1k}(a; b_1, ..., b_{k-1}) b_k;$$

$$a_1 E_{1,k}(a_2; b_1, ..., b_k) + E_{1,k}(a_1 \cdot a_2; b_1, ..., b_k) + E_{1,k}(a_1; b_1, ..., b_k) a_2 = \qquad (4)$$
$$\sum_{p=1,...,k-1} E_{p,1}(a_1; b_1, ..., b_p) \cdot E_{1,m-p}(a_2; b_{p+1}, ..., b_k);$$

$$E_{1,n}(E_{1,m}(a; b_1, ..., b_m); c_1, ..., c_n) =$$
$$\sum E_{1,n-\sum l_i + m}(a; c_1, ..., c_{k_1}, E_{1,l_1}(b_1; c_{k_1+1}, ..., c_{k_1+l_1}), c_{k_1+l_1+1}, ..., c_{k_m}, \qquad (5)$$
$$E_{1,l_m}(b_m; c_{k_m+1}, ..., c_{k_m+l_m}), c_{k_m+l_m+1}, ..., c_n).$$

Let us analyse these conditions in low dimensions.
For the operation $E_{1,1}$ the condition (3) gives

$$dE_{1,1}(a; b) + E_{1,1}(da; b) + E_{1,1}(a; db) = a \cdot b + b \cdot a, \qquad (6)$$

i.e. the operation $E_{1,1}$ is sort of $\cup_1$ product, which measures the noncommutativity of A. Bellow we use the notation $E_{1,1} = \smile_1$.
The condition (4) gives

$$(a \smile b) \smile_1 c + a \smile (b \smile_1 c) + (a \smile_1 c) \smile b = 0, \qquad (7)$$

3

that is our $E_{1,1} = \smile_1$ satisfies so called *Hirsch formula*, which states that the map $f_b : A \to A$ defined as $f_b(x) = x \smile_1 b$ is a derivation.

The condition (3) gives

$$a \smile_1 (b \smile c) + b \smile (a \smile_1 c) + (a \smile_1 b) \smile c =$$
$$dE_{1,2}(a; b, c) + E_{1,2}(da; b, c) + E_{1,2}(a; db, c) + E_{1,2}(a; b, dc), \qquad (8)$$

so the "left Hirsch formula" is satisfied just up to chain homotopy and a homotopy is operation $E_{1,2}$, so this operation measures the lack of "left Hirsch formula".

Besides, the condition (5) gives

$$(a \smile_1 b) \smile_1 c - a \smile_1 (b \smile_1 c) = E_{1,2}(a; b, c) + E_{1,2}(a; c, b), \qquad (9)$$

so this $\smile_1$ is not strictly associative, but the operation $E_{1,2}$ somehow measures the lack of associativity too.

1.2 Homotopy G-algebra structure and a multiplication in the bar construction

For a homotopy G-algebra $(A, d, \cdot, \{E_{1,k}\})$ the sequence $\{E_{1,k}\}$ defines in the bar construction BA of a DG-algebra $(A, d, \cdot)$ a multiplication, turning BA into a DG-Hopf algebra. In fact this means that a homotopy G-algebra is a $B(\infty)$-algebra in the sense of [4].

The sequence of operations $\{E_{1,k}\}$ defines a homomorphism

$$E : BA \otimes BA \to A$$

by $E(a_1, ..., a_m; b_1, ..., b_n) = 0$ if $m > 1$ and $E(a; b_1, ..., b_n) = E_{1,n}(a; b_1, ..., b_n)$.

Since the bar construction BA is a cofree coalgebra, a homomorphism E induces a graded coalgebra map $\mu_E : BA \otimes BA \to BA$.

Then the conditions (3) and (4) are equivalent to the condition

$$dE + E(d_{BA} \otimes id + id \otimes d_{BA}) + E \smile E = 0,$$

that is E is a twisting cochain, and this is equivalent to μ_E being a chain map. Besides, the condition (5) is equivalent to μ_E being associative. Finally we have

Proposition 1 *For a homotopy G-algebra $(A, d, \cdot, \{E_{1,k}\})$ the bar construction BA is a DG-Hopf algebra with respect to the standard coproduct $\nabla_B : BA \to BA \otimes BA$ and the multiplication $\mu_E : BA \otimes BA \to BA$.*

2 Extended homotopy G-algebras

In this section we introduce the notion of *extended homotopy G-algebra*. This is a DG-algebra with certain additional structure which defines $\smile_i$-s on the bar construction.

2.1 The notion of extended homotopy G-algebra

Definition 2 *An extended homotopy G-algebra we define as an object*

$$(A, d, \cdot, \{E_{p,q}^k, \ k = 0, 1, ...; \ p, q = 1, 2, ...\})$$

such that:

$E_{p>1,q}^0 = 0$ *and* $(A, d, \cdot, \{E_{1,q}^0\})$ *is a homotopy G-algebra;*

and

$$
\begin{aligned}
dE_{m,n}^k(a_1, ..., a_m; b_1, ..., b_n) &+ \sum_i E_{m,n}^k(a_1, ..., da_i, ..., a_m; b_1, .., b_n) + \\
&\sum_i E_{m,n}^k(a_1, ..., a_m; b_1, ..., db_i, .., b_n) + \\
&\sum_i E_{m-1,n}^k(a_1, ..., a_i \cdot a_{i+1}, ..., a_m; b_1, ..., b_n) + \\
&\sum_i E_{m,n-1}^k(a_1, ..., a_m; b_1, ..., b_i \cdot b_{i+1}, ..., b_n) + \\
a_1 E_{m-1,n}^k(a_2, ..., a_m; b_1, ..., b_n) &+ E_{m-1,n}^k(a_1, ..., a_{m-1}; b_1, ..., b_n) a_m \\
+ b_1 E_{m,n-1}^k(a_1, ..., a_m; b_2, ..., b_n) &+ E_{m,n-1}^k(a_1, ..., a_m; b_1, ..., b_{n-1}) b_n + \\
\sum_{i=0}^k \sum_{p,q} T^i E_{p,q}^{k-i}(a_1, ..., a_p; b_1, .., b_q) \cdot E_{m-p,n-q}^i&(a_{p+1}, ..., a_m; b_{q+1}, ..., b_n) = \\
E_{m,n}^{k-1}(a_1, ..., a_m; b_1, ..., b_n) &+ E_{n,m}^{k-1}(b_1, ..., b_n; a_1, ..., a_m),
\end{aligned}
$$

(10)

here $TE_{p,q}^i(x_1, ..., x_p; y_1, ..., y_q) = E_{q,p}^i(y_1, ..., y_q; x_1, ..., x_p).$

Let us analyse this condition in low dimensions.
For the operation $E_{1,1}^k$ the condition (10) gives

$$dE_{1,1}^k(a; b) + E_{1,1}^k(da; b) + E_{1,1}^k(a; db) = E_{1,1}^{k-1}(a; b) + E_{1,1}^{k-1}(b; a),$$

i.e. the operation $E_{1,1}^k$ is sort of $\smile_{k+1}$ product on A. Bellow we use the notation $E_{1,1}^k = \smile_{k+1}$.

Besides the condition (10) also gives

$$
\begin{aligned}
(a \smile b) \smile_k c + a \smile (b \smile_k c) &+ (a \smile_k c) \smile b + E_{2,1}^{k-2}(a, b; c) + E_{1,2}^{k-2}(c; a, b) = \\
dE_{2,1}^{k-1}(a, b; c) &+ E_{2,1}^{k-1}(da, b; c) + E_{2,1}^{k-1}(a, db; c) + E_{2,1}^{k-1}(a, b; dc)
\end{aligned}
$$

(11)

and

$$a \smile_k (b \smile c) + b \smile (a \smile_k c) + (a \smile_k b) \smile c + E_{1,2}^{k-2}(a; b, c) + E_{2,1}^{k-2}(b, c; a) =$$
$$dE_{1,2}^{k-1}(a; b, c) + E_{1,2}^{k-1}(a; db, c) + E_{1,2}^{k-1}(a; b, dc), \tag{12}$$

these are up to homotopy Hirsch type formulae connecting $\smile_k$ and $\smile$. We remark here that a homotopy G-algebra structure controls conection between $\smile$ and $\smile_1$, while the extended homotopy G-algebra structure controls the connections between $\smile$ and $\smile_k$-s (but not between $\smile_m$ and $\smile_n$ generally).

As we already know a homotopy G-algebra structure defines a multiplication in the bar construction. Bellow we are going to show that an extended homotopy G-algebra structure defines on the bar construction Steenrod $\smile_i$ products. But before we need some preliminary notions.

2.2 DG-Hopf algebras with Seenrod coproducts

Let a *DG-coalgebra with Steenrod's coproducts* be an object

$$(A; d; \nabla_0, \nabla_1, \nabla_2, \dots)$$

where $(A; d; \nabla_0)$ is a DG-coalgebra (with $\deg d = -1$), and $\nabla_i : A \to A \otimes A$, $i > 0$, are cooperations, dual to Steenrods $\smile_i$ products, i.e. they satisfy the conditions (1)

$$deg\nabla_i = i, \ \nabla_i d + (d \otimes 1 + 1 \otimes d)\nabla_i = \nabla_{i-1} + T\nabla_{i-1}.$$

Suppose now that A additionally is equipped with a multiplication $\cdot :$ $A \otimes A \to A$ which turns $(A, d, \cdot)$ into a *DG*-algebra. We are interested what kind of compatibility of ∇_i-s with the multiplication $\cdot$ should be required.

The following notion was introduced in [5], the dual notion was introduced by V. Smirnov in [11] and was called $\smile_\infty$-Hopf algebra.

Definition 3 *A DG-Hopf algebra with Steenrods coproducts we define as an object*

$$(A, d, \cdot, \nabla_0, \nabla_1, \nabla_2, \dots)$$

where $(A, d, \cdot)$ is a DG-algebra, ∇_i-s satisfy (1) and additionally we require the following connections between ∇_i-s and the product $\cdot$ (decomposition rule):

$$\nabla_n(a \cdot b) = \sum_{k=0}^n \nabla_k(a) \cdot T_{A \otimes A}^k \nabla_{n-k}(b), \tag{13}$$

6

where $T_{A \otimes A} : A \otimes A \to A \otimes A$ is the permutation map $T_{A \otimes A}(a \otimes b) = b \otimes a$ and T^k its iteration.

Particularly ∇_0 is a multiplicative map, that is $(A, d, \cdot, \nabla_0)$ is a DG-Hopf algebra; ∇_1 is a $(\nabla_0, T\nabla_0)$-derivation, etc.

This decomposition rule (13) has the following sense: if $(A, \cdot)$ is a free (i.e tensor) algebra, that is $A = T(V)$ (for example the cobar construction), then (13) allows to construct the cooperations ∇_i first on the generating vector space V and then to extend them on whole A by the suitable *extension rule*, which follows from the above decomposition rule (13).

Let (C, d, Δ) be a DG-coalgebra and ΩC be its cobar construction. By definition ΩC is the tensor algebra $T(s^{-1}\bar{C})$ generated by the desuspension $s^{-1}\bar{C}$ of the coaugmentation coideal $\bar{C}$. So coproducts $\nabla_i : \Omega C \to \Omega C \otimes \Omega C$ satisfying (13) are determined by their restrictions $E^i : C \to \Omega C \otimes \Omega C$, which are homomorphis of degree $i - 1$.

In order ∇_i to satisfy (1) E^i should satisfy the condition

$$d_{\Omega C \otimes \Omega C} E^i + E^i d + \sum_{k=0}^{i} E^k \smile T^k_{\Omega C \otimes \Omega C} E^{i-k} = E^{i-1} + T_{\Omega C \otimes \Omega C} E^{i-1}, \quad (14)$$

which is the restriction of (1) on C.

So if we want to construct on ΩC a sequence ∇_i forming a structure of DG-Hopf algebra with Steenrods coproducts we have to construct a sequence of *higher twisting cochains* - homomorphisms $\{E^i, \ i = 0, 1, ...; \ deg E^i = i - 1\}$ satisfying (14). Note that E^0 is an ordinary twisting cochain:

$$d_{\Omega C \otimes \Omega C} E^0 + E^0 d + E^0 \smile E^0 = 0.$$

2.3 DG-Hopf algebras with Seenrod products

Here we dualise the previous section.

Let a *DG-algebra with Steenrod's products* be an object

$$(A; d; \smile_0, \smile_1, \smile_2, ...)$$

where $(A; d; \smile_0)$ is a DG-algebra (with $\deg d = +1$), and $\smile_i \colon A \otimes A \to A, \ i > 0$, are Steenrods $\smile_i$ products, i.e. they satisfy the conditions

$$d(a \smile_i b) = da \smile_i b + a \smile_i db + a \smile_{i-1} b + b \smile_{i-1} b. \quad (15)$$

Suppose now that A additionally is equipped with a diagonal $\nabla : A \to A \otimes A$ which turns (A, d, ∇) into a *DG*-coalgebra. We are interested what kind of compatibility of $\smile_i$-s with the diagonal ∇ should be required.

Definition 4 *A DG-Hopf algebra with Steenrods products we define as an object* $(A, d, \nabla, \smile_0, \smile_1, \smile_2, ...)$ *where* (A, d, ∇) *is a DG-coalgebra, the products* $\smile_i \colon A \otimes A \to A$ *satisfy (15) and additionally we require the following connections between* $\smile_i$*-s and the diagonal* ∇:

$$\nabla \cdot \smile_n = \sum_{k=0}^{n} (\smile_k \otimes \smile_{n-k} \cdot T_{A \otimes A}^k) \nabla_{A \otimes A}. \tag{16}$$

Particularly $\smile_0$ is a coalgebra map, i.e. $(A, d, \nabla, \smile_0)$ is a DG-Hopf algebra.

Let $(C, d, \cdot)$ be a DG-algebra and BC be its bar construction. By definition BC is the tensor coalgebra $T^c(s^{-1}\bar{C})$ generated by the desuspension $s^{-1}\bar{C}$ of the augmentation ideal $\bar{C}$.

Since of cofreeness of T^c products $\smile_i \colon BC \otimes BC \to BC$ satisfying (16) are determined by their projections $E^i : BC \otimes BC \to BC \to C$, which are homomorphis of degree $1 - i$.

In order $\smile_i$ to satisfy (15) E^i should satisfy the condition

$$dE^i + E^i(d_{BC} \otimes id + id \otimes d_{BC}) + \sum_{k=0}^{i} E^k \smile E^{i-k} T_{BC \otimes BC}^k = E^{i-1} + E^{i-1} T_{BC \otimes BC}, \tag{17}$$

which is the projection of (15) on C.

So if we want to construct on BC a sequence $\smile_i$-s forming a structure of DG-Hopf algebra with Steenrod products we have to construct a sequence of *higher twisting cochains* - homomorphisms $\{E^i, \; i = 0, 1, ...; \; deg E^i = 1 - i\}$ satisfying (17). Note that E^0 is an ordinary twisting cochain:

$$E^0 d_{BC \otimes BC} E^0 + dE^0 + E^0 \smile E^0 = 0.$$

2.4 A structure of extended homotopy G-algebra and Steenrod products in the bar construction

As we already know the part of extended homotopy G-algebra - the sequence of operations $\{E_{p,q}^0\}$ (which in fact is a homotopy G-algebra structure) defines on the bar construction BA a multiplication, turning BA into a DG-Hopf algebra. Here we show that for an extended homotopy G-algebra $(A, d, \cdot, \{E_{p,q}^k\})$ the sequence $\{E_{p,q}^{k>0}\}$ defines in the bar construction BA of a DG-algebra

$(A, d, \cdot)$ the $\smile_i$-products turning BA into a DG-Hopf algebra with $\smile_i$-s.

The sequences of operations $\{E_{p,q}^k\}$ define homomorphisms

$$\{E^k : BA \otimes BA \to A, \ k = 0, 1, ...\}$$

by $E^k(a_1, ..., a_m; b_1, ..., b_n) = E_{m,n}^k(a_1, ..., a_m; b_1, ..., b_n)$.

The condition (10), which veirfy our $\{E_{p,q}^k\}$-s is equivalent to the condition (17) for the sequence $\{E^k\}$, so they define the correct $\smile_k$-s on BC.

Finally we have

Proposition 2 *For an extended homotopy G-algebra $(A, d, \cdot, \{E_{p,q}^k\})$ the bar construction BA is a DG-Hopf algebra with Steenrod $\smile_i$-products.*

3 Cochain complex $C^*(X)$ as an extended homotopy G-algebra

Main example of an extended homotopy G-algebra gives the following

Theorem 1 *The chain complex of a topological space $C_*(X)$ carries a structure of extended homotopy G-algebra.*

Bellow we construct particular elements in the surjection operad χ (which acts on $C_*(X)$), which represent the multicooperations $\{E_{p,q}^k\}$.

3.1 Operations $E_{p,q}^k$ in the surjection operad

Surjection operad χ [10] is defined as a sequence of chain complexes $\chi(n)$ where $\chi(n)_d$ is spanned by nondegenerate surjections $u : (1, 2, ..., n + d) \to (1, 2, ..., n)$, $u(i) \neq u(i+1)$. For the structure maps of this operad, action of χ on $C_*(X)$ (on $C^*(X)$) and the filtration $F_1\chi \subset ... \subset F_n\chi \subset ... \subset \chi$, with $F_n\chi$ equivalent to little n-cub operad, we refer to to [2].

A sirjection u is written as a string $(u(1), u(2), ..., u(n + d))$. The string $(1, 2) \in F_1\chi(2)_0$ corresponds to the cup-product $\smile$ in $C^*(X)$ (dually to the Alexander-Whitney diagonal Δ in $C_*(X)$);

the string $(1, 2, 1) \in F_2\chi(2)_1$ corresponds to $\smile_1$ in $C^*(X)$ (dually to Δ_1 in $C_*(X)$); the string $(1, 2, 1, 2) \in F_3\chi(2)_2$ corresponds to $\smile_2$ in $C^*(X)$ (dually to Δ_2 in $C_*(X)$) etc.

Here we present particular elements of χ representing operations $E_{p,q}^k$. They are obtained from *admissible tables* which we define now.

The first row of an admissible table consists of single number 1. The second looks as $p+1, 1, p+2, 1, ..., 1, p+k$. As we see it consists of the *stabile part* (1-s on even places) and of the *increasing part* $(p+1, p+2, ..., p+k$ on odd places).

Each next row starts with the stabile number of the previous row which gives rise to increasing part at odd places. As the stabile part at even places serves the maximal number from the previous row. For example if the previous row ends with $..., i-2, s, i-1, s, i$, then the next row looks as $s, i, s+1, i, s+2, i, ..., i, s+t$. Each row consists of odd number of terms. Important remark: the stabile part of a row may be *empty*, in this case under the *stabile part* of this row we mean the maximal number of the previous row.

As we see a table consists of two increasing sequences of integres $1, 2, ...$ and $p+1, p+2, ...$ (of course with repetitions and permutations). The main restriction is that the first sequence should necessarily end by p. A table always ends with one term row.

After this we put all rows of the admissible table in one string and obtain an *admissible string* in χ. We say that this string belongs to $E_{p,q}^k$ if: its table consists of $k+3$ rows; the first element of the second row is $p+1$ and the maximal number which occurs in the string is $p+q$.

Here are some examples (by ; we indicate the ends of rows in admissible tables). The admissible string

$$(1; 5, 1, 6, 1, 7; 1, 7, 2, 7, 3, 7, 4; 7, 4, 8, 4, 9; 4)$$

belongs to $E_{4,5}^2$ and

$$(1; 4, 1, 5, 1, 6; 1, 6, 2; 6; 2, 6, 3; 6; 3)$$

belongs to $E_{3,3}^4$.

An element $E_{p,q}^k \in \chi$ we define as the sum of all admissible strings belonging to it.

Particularly $E_{1,1}^{2k} = (1, 2, 1, ..., 1, 2)$ and $E_{1,1}^{2k-1} = (1, 2, 1, ..., 1, 2, 1)$. They correspond to $\smile_{2k}$ and $\smile_{2k+1}$ respectively.

Besides $E_{1,q}^0 = (1, 2, 1, 3, ...1, q+1, 1)$. These elements generate $F_2\chi$ ([10]) and they determine on $C^*(X)$ a structure of homotopy G/-algebra..

Here are the examples of more higher operations:

$$\begin{aligned} E_{p,q}^1 = \quad &(1; p+1, 1, p+2, 1, ..., p+q-1, 1, p+q; \\ &1, p+q, 2, p+q, 3, ..., p; p+q); \end{aligned}$$

$$E^2_{p,q} = \sum_{k=0}^{q-1} \quad (1; p+1, 1, p+2, 1, ..., 1, p+k+1;$$
$$1, p+k+1, 2, p+k+1, 3, ..., p+k+1, p;$$
$$p+k+1, p, p+k+2, p, ..., p+q; p).$$

Generally $E^k_{p,q}$ belong to filtration $F_{k+2}\chi$.

The way how these elements $E^k_{p,q}$ are obtained is following. In [9] it is shown that the bar construction $BC^*(X)$ actually is the cochain complex of certain cubical set, and in [5] the Steenrods $\smile_i$ products are constructed in the cochains of a cubical set.

Remark 1 *The elements $E^0_{1,q}$-s satisfy the defining conditions of a Homotopy G-algebra already in χ. So do $E^{k>0}_{p,q}$-s: they satisfy (10) already in χ. For example the condition (7) is a result of*

$$(1, 2, 1) \circ_1 (1, 2) + (1, 2) \circ_2 (1, 2, 1) + (id \times T)(1, 2) \circ_1 (1, 2, 1) = 0, \quad (18)$$

(what is not a case in, say, Barrat-Eccles operad: the suitable combination there is just homological to zero. Note also that Barrat-Eccles operad acts on $C(X)$ via χ, see [2]). The condition (8) is a result of

$$(1, 2, 1) \circ_2 (1, 2) + (T \times id)(1, 2) \circ_2 (1, 2, 1) + (1, 2) \circ_1 (1, 2, 1) = d(1, 2, 1, 3, 1).$$

Remark 2 *The extended homotopy G-coalgebra structure, that is the operations $E^k_{p,q}$, establishes connections just between Δ_k and Δ (equivalently between $\smile_k$ and $\smile$ or between $E^k_{1,1}$ and $1, 2$ in χ), but not a connections between Δ_m and Δ_n generally. Here are two operations establishing connections between Δ_2 and Δ_1: $G_{1,2} = (1, 2, 1, 3, 1, 3, 2)$, $G_{2,1} = (1, 2, 3, 2, 3, 1, 3) \in F_3\chi(3)_4$ satisfy the conditions*

$$dG_{2,1}(a, b; c) + G_{2,1}(da, b; c) + G_{2,1}(a, db; c) + G_{2,1}(a, b; dc) =$$
$$(a \smile_1 b) \smile_2 c + a \smile_1 (b \smile_2 c) + (a \smile_2 c) \smile_2 b + E^1_{2,1}(a, b; c) + E^1_{2,1}(b, a; c),$$

and

$$dG_{1,2}(a; b, c) + G_{1,2}(da; b, c) + G_{1,2}(a; db, c) + G_{1,2}(a; b, dc) =$$
$$a \smile_2 (b \smile_1 c) + b \smile_1 (a \smile_2 c) + (a \smile_2 c) \smile_1 b + E^1_{1,2}(a; b, c) + E^1_{1,2}(a; c, b),$$

already in the operad χ. Note that the element $(1, 2) \in F_1\chi$ generates the operad $F_1\chi$. furthermore, $(1, 2)$ and $E^1_{1,k} = (1, 2, 1, 3, 1, ..., 1, k, 1, k+1, 1) \in F_2\chi$ generate the operad $F_2\chi$ [10] (but not freely: for example (18) is a relation). The elements $G_{1,2} = (1, 2, 1, 3, 1, 3, 2)$, $G_{2,1} = (1, 2, 3, 2, 3, 1, 3) \in F_3\chi(3)_4$ should be a part of some rich structure, which, together with $(1, 2)$ and $(1, 2, 1, 3, ..., 1, k+1, 1)$ generates $F_3\chi$.

Remark 3 *The elements $E_{1,q}^0 \in F_2\chi$ determine on the Hochschild cochain complex of an associative algebra $C^*(A, A)$ operations from [6], [4] and on the cochain complex $C^*(X)$ operations from [1] forming in both cases a Homotopy G-algebra structures [3]. There is one more example where these operations act, this is the cobar construction of a Hopf algebra [7]. We remark also that the Hochschild cochain complex is not an extented homotopy G-algebra, that is $E_{p,q}^{k>0} \in F_3\chi$ do not act on $C^*(A, A)$ since of nontriviality of Gerstehaber bracket in the Hochschild cohomology (no $\smile_2$ in $C^*(A, A)$). Here acts only the suboperad $F_2\chi$ (Deligne conjecture), since it is generated by $(1, 2) \in F_1\chi$ and $E_{1,q}^0 \in F_2\chi$-s [10], whereas on $C^*(X)$ acts whole χ.*

References

[1] H.-J. Baues, The double bar and cobar constructions, Compositio Math., 43 (1981), 331-341.

[2] C. Berger, B. Fresse, Combinatorial operad actions on cochains, preprint, arXi:math.AT/0109158v1, 2001.

[3] M. Gerstenhaber, A. Voronov, Higher Operations in the Hochschild Complex, Functional Analysis and its Applications, 29 (1995), 1-5.

[4] E. Getzler , J. Jones, Operads, homotopy algebras and iterated integrals for double loop spaces, preprint, arXiv.org/abs/hep-th/9403055, 1994.

[5] T. Kadeishvili, DG Hopf Algebras with Steenrods i-th coproducts. Bull. Georgian Acad. Sci. 158,2 (1998), 203-206.

[6] T. Kadeishvili, The $A(\infty)$-algebra Structure and Cohomology of Hochschild and Harrison, Proc. of Tbil. Math. Inst., 91 (1988), 19-27.

[7] T. Kadeishvili, Measuring the noncommutativity of DG-algebras, preprint.

[8] T. Kadeishvili, S. Saneblidze, On a multiplicative model of a fibration. Bull. Georg. Acad. Sci. 153,3, 1996, 345-346.

[9] T. Kadeishvili and S. Saneblidze, The cobar construction as a cubical complex. Bull. Georgian Acad. Sci. 158,3, (1998).

[10] J. McClure, J. H. Smith, A solution of Delignes conjecture, preprint, arXiv:math,QA/9910126, 1999.

[11] V. Smirnov, $A(\infty)$-structures and differentials of the Adams spectral sequence, preprint, Tbilisi conference, 1998.

A. Razmadze Mathematical Institute of the Georgian Academy of Sciences,
M. Alxidze str. 1, Tbilisi, 380093, Georgia
kade@rmi.acnet.ge