Reading [Simon], Chapter 24, p. 633-657.

1 Differential Equations

1.1 Definition and Examples

A *differential equation* is an equation involving an unknown function (say $y = y(t)$) and one or more of its derivatives

$$F(y, y', y'', ..., t) = 0.$$

This is general form of differential equation.

First-order differential equation is one which involves only y', y, t, that is $F(y', y, t) = 0$.

Ordinary differential equation is an equation of the form

$$y' = F(y, t).$$

Examples.

1. The equation $y' = 2t$ has a solution

$$y(t) = \int 2tdt = t^2 + C,$$

this is *general solution* which depends on the constant C. Assigning to C particular values we obtain *particular solutions*

$$y(t) = t^2, \quad y(t) = t^2 + 5, \quad y(t) = t^2 - 7, \quad ... \quad .$$

A particular solution which satisfies the *initial value condition* $y(0) = 1$ is $y(t) = t^2 + 1$.

2. The equation $y' = 2y$ has a solution

$$y(t) = Ce^{2t},$$

this is *general solution* which depends on the constant C. Assigning to C particular values we obtain *particular solutions*

$$y(t) = e^{2t}, \quad y(t) = 5e^{2t}, \quad y(t) = -7e^{2t}, \quad ... \quad .$$

A particular solution which satisfies the initial value condition $y(0) = 2$ is $y(t) = 2e^{2t}$.

This equation has also one important particular solution which corresponds to initial value condition $y(0) = 0$, the solution is *constant function* $y(t) = 0$. This solution is called: *steady state, stationary solution, stationary point, rest point, equilibrium.*

3. The equation $y' = 2ty$ has a solution

$$y(t) = Ce^{t^2},$$

this is *general solution* which depends on the constant C. Assigning to C particular values we obtain *particular solutions*

$$y(t) = 0, \quad y(t) = e^{t^2}, \quad y(t) = 5Ce^{t^2}, \quad y(t) = -7Ce^{t^2}, \quad \dots \ .$$

A particular solution which satisfies the initial value condition $y(1) = 3e$ is $y(t) = 3e^{t^2}$.

This equation also has stationary solution $y(t) = 0$.

4. The equation $y' = y^2$ has a solution

$$y(t) = \frac{1}{C - t},$$

this is *general solution* which depends on the constant C. Assigning to C particular values we obtain *particular solutions*

$$y(t) = -\frac{1}{t}, \quad y(t) = \frac{1}{5 - t}, \quad y(t) = -\frac{1}{7 + t}, \quad \dots \ .$$

A particular solution which satisfies the initial value condition $y(1) = 1$ is $y(t) = \frac{1}{2-t}$.

5. The Hooke's equation $y'' = -ky, \ k > 0$ has a solution

$$y(t) = C_1 cos\sqrt{k}t + C_2 sin\sqrt{k}t,$$

this is *general solution* which depends on the two constants C_1 and C_2. Assigning to C_1 and C_2 particular values we obtain *particular solutions*

$$y(t) = cos\sqrt{k}t, \quad y(t) = sin\sqrt{k}t, \quad y(t) = cos(\sqrt{k} + \frac{\pi}{4}), \quad \dots \ .$$

Remark. Most first order differential equations have exactly one particular solution that satisfies a given initial value condition. However there are examples where there are either no, or many solutions that satisfy a given initial value condition, see two examples bellow.

Example 6. The equation $y' = g(t)$ with

$$g(t) = \begin{cases} \frac{1}{t} & if \quad t \neq 0 \\ 0 & if \quad t = 0 \end{cases}$$

has no solution with initial value condition $y(0) = 0$.

Example 7. The equation $y' = \sqrt{t}$ with initial value condition $y(0) = 0$ has solutions: $y(t) = 0$ and $y(t) = \frac{t^2}{4}$.

1.1.1 Exponential Growth

Here are differential equations which describe various types of growth.

1. $y' = ky,\ k > 0$ describes *unlimited growth* or Malthus growth.

Here the rate of change of the quantity y with respect to time t is proportional to amount present.

General solution is $y(t) = Ce^{kt}$. There is the stationary solution $y(t) = 0$.

Malthus used this equation to describe the growth of population on the earth. The same equation describes the growth of money on the account in a bank that has a constant percent of rate k. The constant C in this case has the following meaning: $C = y(0)$, so C is the original deposit.

2. $y' = -ky,\ k > 0$ describes *unlimited decay.*

General solution is $y(t) = Ce^{-kt}$. There is the stationary solution $y(t) = 0$.

This equation describes depletion of natural resources, radioactive decay, price-demand curves.

3. $y' = k(M - y),\ k > 0$ describes *limited growth.*

Here the rate of change of the quantity y with respect to time t is proportional to the difference between a limiting value M and the amount present y.

General solution is $y(t) = M(1 - Ce^{-kt})$. There is also the stationary solution $y(t) = M$.

This equation describes sales, depreciations of equipment, company growth.

4. $y' = ky(M - y),\ k > 0$ describes *logistic growth*.

Here the rate of change of the quantity y with respect to time t is proportional to amount present and the difference between a limiting value M and the amount present y.

General solution is $y(t) = \frac{M}{1+Ce^{-kMt}}$. There is also the stationary solution $y(t) = M$.

This equation describes long-term population growth, epidemics, sales of new products, rumor spread.

1.2 First Order Equations

1.2.1 First Order Linear Differential Equations

Step 1. Bring the equation to the standard form $y' + f(t)y = g(t)$.
Step 2. Compute the integrating factor

$$I(t) = e^{\int f(t)dt},$$

then $I'(t) = I(t) \cdot f(t)$.
Step 3. Multiply both sides of the equation by $I(t)$:

$$I(t) \cdot y' + I(t) \cdot f(t) \cdot y = I(t) \cdot g(t),$$

$$I(t) \cdot y' + I'(t) \cdot y = I(t) \cdot g(t), \quad (I(t) \cdot y)' = I(t) \cdot g(t);$$

Step 4. Integrate both sides

$$I(t) \cdot y = \int I(t) \cdot g(t)dt;$$

Step 5. Solving for y we obtain the general solution

$$y = \frac{\int I(t) \cdot g(t)dt}{I(t)}.$$

Example. Solve $y' = 3(5 - y)$.

$$I(t) = e^{\int 3dt} = e^{3t},$$

$$y = \frac{\int e^{3t} \cdot 3 \cdot 5dt}{e^{3t}} = \frac{\int e^{3t} \cdot 5d3t}{e^{3t}} = \frac{5e^{3t} + C}{e^{3t}} = \frac{5e^{3t} + K}{e^{3t}} = 5 + Ce^{-3t}.$$

Example. Solve $y' + 2ty = 4t$.

$$I(t) = e^{\int 2tdt} = e^{t^2},$$

$$y = \frac{\int e^{t^2} \cdot 4tdt}{e^{t^2}} = \frac{2\int e^{t^2} dt^2}{e^{t^2}} = \frac{2e^{t^2} + C}{e^{t^2}} = 2 + \frac{C}{e^{t^2}}.$$

1.3 Separation of Variables

Step 1. Bring the equation to the form $f(y)y' = g(t)$

$$f(y)\frac{dy}{dt} = g(t), \quad f(y)dy = g(t)dt.$$

Step 2. Integrate both sides $\int f(y)dy = \int g(t)dt$.
Step 3. Solve the obtained equation for y.

Example. Solve $y' = 3(5 - y)$.
Step 1. $\frac{y'}{5-y} = 3$, $\frac{dy}{5-y} = 3dt$.
Step 2. $\int \frac{dy}{5-y} = \int 3dt, \quad -\int \frac{d(5-y)}{5-y} = \int 3dt, \quad -ln(5-y) = 3t + K.$
Step 3. $ln(5-y) = -3t - K$, $5 - y = e^{-3t-K}$, $y = 5 - e^{-3t}e^{-K} = 5 - Ce^{-3t}$.

Example. Solve $\frac{1}{y}y' = \frac{1}{t}$ if $y(2) = 6$.
Step 1. $\frac{1}{y}\frac{dy}{dt} = \frac{1}{t}$, $\frac{dy}{y} = \frac{dt}{t}$.
Step 2. $\int \frac{dy}{y} = \int \frac{dt}{t}$, $ln\ y + C_1 = ln\ t + C_2$, $ln\ y = ln\ t + C$.
Step 3. $ln\ y = ln\ e^C t$, $y = e^C t$, so the general solution is $y = kt$.
Let us find the particular solution: $6 = k \cdot 2$, $k = 3, y = 3t$.

1.4 Linear Second order Equations

1.4.1 Homogenous Linear Second order Equation with constant coefficients

We consider a differential equation

$$ay'' + by' + cy = 0. \tag{1}$$

To this differential equation corresponds the numerical quadratic equation

$$ar^2 + br + c = 0 \tag{2}$$

called *characteristic equation.*

Theorem 1 *If the characteristic equation (2) has two distinct roots* r_1 *and* r_2*, then the equation (1) has the general solution*

$$y(t) = k_1 e^{r_1 t} + k_2 e^{r_2 t}.$$

Example. Consider the problem

$$y'' - y' - 2y = 0, \quad y(0) = 3, \; y'(0) = 0.$$

Corresponding characteristic equation

$$r^2 - r - 2 = 0$$

has the roots $r_1 = 2, \; r_2 = -1$. Thus the general solution is

$$y(t) = k_1 e^{2t} + k_2 e^{-t}.$$

Plug in initial values

$$y(0) = k_1 + k_2 = 3$$
$$y'(0) = 2k_1 - k_2 = 0.$$

Solution gives $k_1 = 1, \; k_2 = 2$, so our particular solution is

$$y(t) = e^{2t} + 2e^{-t}.$$

Theorem 2 *If the characteristic equation (2) has two equal roots $r_1 = r_2$, then the equation (1) has the general solution*

$$y(t) = k_1 e^{r_1 t} + k_2 t e^{r_1 t}.$$

Example. Consider the problem

$$y'' - 2y' + y = 0, \quad y(0) = 6, \ y'(0) = 0.$$

Corresponding characteristic equation

$$r^2 - 2r + 1 = 0$$

has one root $r_1 = r_2 = 1$. Thus the general solution is

$$y(t) = k_1 e^t + k_2 t e^t.$$

Plug in initial values

$$\begin{aligned} y(0) &= k_1 = 6 \\ y'(0) &= k_1 + 2k_2 = 0. \end{aligned}$$

Solution gives $k_1 = 6, \ k_2 = -3$, so our particular solution is

$$y(t) = 6e^t - 3te^t.$$

Theorem 3 *If the characteristic equation (2) has complex roots $\alpha \pm i\beta$, then the equation (1) has the general solution*

$$y(t) = e^{\alpha t}(C_1 cos\beta t + C_2 sin\beta t).$$

Example. Consider the problem

$$y'' - 2y' + 2y = 0, \quad y(0) = 6, \ y'(0) = 0.$$

Corresponding characteristic equation

$$r^2 - 2r + 2 = 0$$

has the roots $r_1 = 1 + i,\ r_2 = 1 - i$. Thus the general solution is

$$y(t) = e^t(C_1 cos\ t + C_2 sin\ t).$$

Plug in initial values

$$\begin{array}{l} y(0) = C_1 = 6 \\ y'(0) = C_1 + C_2 = 0. \end{array}$$

Solution gives $k_1 = 6,\ k_2 = -6$, so our particular solution is

$$y(t) = 6e^t(cos\ t - sin\ t).$$

1.4.2 Nonhomogenous Linear Second order Equation with constant coefficients

Nonhomogenous equation looks as

$$ay'' + by' + cy = g(t).$$

Theorem 4 *Let $y_p(t)$ be any particular solution of the nonhomogenous equation*

$$ay'' + by' + cy = g(t),$$

and let $k_1y_1(t)+k_2y_2(t)$ be a general solution of the corresponding homogenous equation

$$ay'' + by' + cy = 0.$$

Then, a general solution of nonhomogenous equation is

$$y(t) = k_1y_1(t) + k_2y_2(t) + y_p(t).$$

We already know how to find general solution of homogenous equation, but how to find a particular solution of nonhomogenous one?

For this there exists so called *method of undetermined coefficients*. This method works only if the associated homogeneous equation has constant coefficients

This method is based on a guessing technique. That is, we will guess the form of $y_p(t)$ and then plug it in the equation to find it.

In this method one looks for a particular solution which has the same form as the right hand side function $g(t)$. Namely:

If $g(t) = g_0$ is constant, then $y_p(t) = g_0/c$.

If $g(t)$ is polynomial of order n then one looks for $y_p(t)$ which is a polynomial of same order.

If $g(t)$ is an exponential $e^{\beta t}$ then one looks for a particular solution of the following form:
(case 1) $Ae^{\beta t}$ if β is not a root of characteristic equation.
(case 2) $Ate^{\beta t}$ if β is a simple root (one of the two roots) of the characteristic equation.
(case 3) $At^2e^{\beta t}$ if β is a double (only) root of the characteristic equation.

Remark. Suppose the right hand side of the equation is the sum of two functions

$$ay'' + by' + cy = g_1(t) + g_2(t).$$

Then a particular solution of this equation is the sum of particular solutions of the equations

$$ay'' + by' + cy = g_1(t), \quad ay'' + by' + cy = g_2(t).$$

Example. Find the general solution of the equation

$$y'' - 2y' - 3y = 9t^2.$$

The general solution of homogenous equation $y'' - 2y' - 3y = 0$ is

$$y(t) = k_1e^{3t} + k_2e^{-t}.$$

Now look at a particular solution of $y'' - 2y' - 3y = 9t^2$. The right hand side is quadratic polynomial, so we look at

$$y_p(t) = At^2 + Bt + C.$$

Plugging this expression into the equation we obtain

$$(-3A)t^2 + (-4A - 3B)t + (2A - 2B - 3C) = 9t^2.$$

This gives the system

$$\begin{aligned}
&-3A = 9\\
&-4A - 3B = 0\\
&2A - 2B - 3C = 0
\end{aligned}$$

whose solution is $A = -3,\ B = 4,\ C = -14/3$. Therefore the particular solution is

$$y_p(t) = -3t^2 + 4t - 14/3,$$

and the general solution is

$$y(t) = k_1e^{3t} + k_2e^{-t} - 3t^2 + 4t - 14/3.$$

Example. Find the general solution of the equation

$$y'' - 2y' + y = e^{3t}.$$

The general solution of homogenous equation $y'' - 2y' + y = 0$, as we know, is

$$y(t) = k_1e^t + k_2te^t.$$

Now look at a particular solution of $y'' - 2y' + y = e^{3t}$. First let us mention that 3 is not a root of characteristic equation $r^2 - 2r + 1 = 0$, thus we look at such solution as $y(t) = Ae^{3t}$. Substitution gives

$$9Ae^{3t} - 6Ae^{3t} + Ae^{3t} = e^{3t},$$

and we calculate $A = \frac{1}{4}$.

Thus the general solution is

$$y(t) = k_1e^t + k_2te^t + \frac{1}{4}e^{3t}.$$

Example. Find the general solution of the equation

$$y'' - 2y' + y = e^t.$$

The general solution of homogenous equation $y'' - 2y' + y = 0$, as we know, is

$$y(t) = k_1e^t + k_2te^t.$$

Now look at a particular solution of $y'' - 2y' + y = e^t$. First let us mention that 1 is a double root of characteristic equation $r^2 - 2r + 1 = 0$, thus we look at such solution as $y(t) = t^2 A e^t$. Calculating $y'(t)$ and $y''(t)$ and substituting we obtain $A = \frac{1}{2}$. So the particular solution is $y_p(t) = \frac{1}{2} t^2 e^t$, and the general solution is

$$y(t) = k_1 e^t + k_2 t e^t + \frac{1}{2} t^2 e^t.$$

1.5 The Fundamental Existence and Uniqueness Theorem

Theorem 5 *Consider the initial value problem*

$$y' = f(t, y), \quad y(t_0) = y_0.$$

Suppose that f is continues at the point (t_0, y_0). Then there exists a function

$$y : I \to R$$

on an open interval $I = (t_0 - a, t_0 + a)$ about t_0 that is a solution of our problem on I.

Furthermore, if f is C^1 at (t_0, y_0) then this solution is unique.

Remark. This theorem explains the Examples 6 and 7: in first case the right hand side function is not continues, so there is no solution, and in second case this function is not C^1, so there are more then one solutions.

1.6 Economical Examples

1.6.1 Elasticity

Assume that the demand Q depends on price p as $Q = Q(p)$, and that elasticity of demand is constant

$$\epsilon(p) = \frac{p \cdot Q'(p)}{Q(p)} = -1$$

(constant elasticity demand). One of such functions is $Q(p) = \frac{C}{p}$, the inverse proportionality. Are there other functions with constant elasticity $\epsilon(p) = -1$?

Let us solve the above differential equation:

$$\frac{dQ}{Q} = -\frac{dp}{p}, \quad \int \frac{dQ}{Q} = -\int \frac{dp}{p}, \quad ln\ Q = -ln\ p + c,$$
$$ln\ Q + ln\ p = C, \quad \ln Q \cdot p = C, \quad Q \cdot p = e^C, \quad Q = \frac{C_1}{p},$$

thus ANY solution is of the form $Q(p) = \frac{C_1}{p}$.

1.6.2 Equilibrium Price

Let $Q_d(p)$ be the demand function and $Q_s(p)$ be the supply function. The equilibrium prise is the solution of the equation $Q_d(p) = Q_s(p)$.

Assume that the demand and supply functions are linear: $Q_d = c + bp$ and $Q_s = g + hp$. Then the equilibrium prise is $\overline{p} = \frac{c-g}{h-b}$.

Suppose now that we start with some initial prise $p_0 = p(0)$ which may be differs from equilibrium prise $\overline{p}$. How this prise will change in time?

It is natural to assume that the rate of change of price $\frac{dp}{dt}$ is proportional to the *excess demand* $Q_d - Q_s$ that is satisfies the equation

$$\frac{dp}{dt} = m(Q_d - Q_s)$$

where m is a positive constant. Note that

$$Q_d - Q_s = (c + bp) - (g + hp) = (c - g) + (b - h)p,$$

so the equation looks as

$$\frac{dp}{dt} = m(c - g) + m(b - h)p.$$

This is first order linear differential equation.

The general solution of this equation is

$$p(t) = \frac{C \cdot e^{m(b-h)t}}{m(b-h)} + \frac{c-g}{h-b} = C_1 \cdot e^{m(b-h)t} + \frac{c-g}{h-b}.$$

Note that usually $b < 0$ (the demand function $Q_d = c + bp$ decreases in p) and $h > 0$ (the supply function is $Q_s = g + hp$ increases in p). Thus $\lim_{t\to\infty} e^{m(b-h)t} = 0$. Then the limit of prize $p(t)$ as $t \to \infty$ is

$$\lim_{t\to\infty} p(t) = \lim_{t\to\infty}(C_1 \cdot e^{m(b-h)t} + \frac{c-g}{h-b}) =$$
$$\lim_{t\to\infty} C_1 \cdot e^{m(b-h)t} + \lim_{to\infty} \frac{c-g}{h-b} = 0 + \frac{c-g}{h-b} = \frac{c-g}{h-b}.$$

Thus the prise $p(t)$, starting at any initial value $p(0) = p_0$, converges to equilibrium prise $\overline{p} = \frac{c-g}{h-b}$

Exercises

1. Find the general solution for each differential equation using separation of variables. Then find the particular solution satisfying the initial condition.

(a) $y' = ye^x,\ y(0) = 3e$; (b) $y' = xy + y,\ y(0) = 2$;
(c) $xyy' = ln\ x,\ y(1) = 1$; (d) $xy' = x\sqrt{y} + 2\sqrt{y},\ y(1) = 4$;
(e) $yy' = xe^{-y^2},\ y(0) = 1$; (f) $yy' = x(1 + y^2),\ y(0) = 1$.

2. Find the general solution for each differential equation using integrating factor. Then find the particular solution satisfying the initial condition.

(a) $y' + 3x^2y = 9x^2,\ y(0) = 7$; (b) $y' - 2y = e^{3x},\ y(0) = 3$;
(c) $y' - 3y = 6\sqrt{x}e^{3x},\ y(0) = -2$.

3. Find the general solution for each differential equation using integrating factor.

(a) $y' + y = x^2$; (b) $xy' + 2y = xe^{3x}$; (c) $xy' + y = xln\ x$.

4. Solve the following initial value problems

(a) $y'' - y = 0,\ \ y'(0) = y(0) = 1$; (b) $y'' + 6y + 9 = 0,\ \ y(0) = 0,\ y'(0) = 1$;
(c) $y'' + 2y' + 10y = 0,\ \ y(0) = 2,\ y'(0) = 1$.

5. Find general solution for the following nonhomogenous equations

(a) $y'' - 2y' - y = 7$; (b) $y'' + y' - 2y = 6t$; (c) $y'' - y' - 2y = 4e^{-1}$.

6. Solve the *logistic growth* equation $y' = ky(M - y),\ k > 0$ using separation of variables and the following hint

$$\frac{1}{y(M-y)} = \frac{1}{My} + \frac{1}{M(M-y)}.$$