Math for Economists, Calculus 1

Tornike Kadeishvili

1 Homogenous and Homothetic Functions

1.1 Homogenous Functions

Definition 1 *A real valued function $f(x_1, ..., x_n)$ is homogenous of degree k if for all $t > 0$*

$$f(tx_1, ..., tx_n) = t^k f(x_1, ..., x_n). \tag{1}$$

Examples.

a) A monomial of degree 6 $f(x_1, x_2, x_3) = x_1^2 x_2^3 x_3$ is a homogenous function of degree 6:

$$f(tx_1, tx_2, tx_3) = \\ (tx_1)^2(tx_2)^3(tx_3) = t^2 x_1^2 t^3 x_2^3 t x_3 = t^6 x_1^2 x_2^3 x_3 = t^6 f(tx_1, tx_2, tx_3).$$

(b) The function $f(x_1, x_2) = \sqrt{x_1^3 + x_2^3}$ is a homogenous function of degree 3/2:

$$f(tx_1, tx_2) = \sqrt{(tx_1)^3 + (tx_2)^3} = \sqrt{t^3(x_1^3 + x_2^3)} = t^{3/2}\sqrt{x_1^3 + x_2^3}.$$

(c) A linear function $f(x_1, ..., x_n) = \sum_{k=1}^n a_k x_k$ is homogenous of degree 1 (**prove it**).

(d) A quadratic form $f(x_1, ..., x_n) = \sum_{k=1}^n a_{ij} x_i x_j$ is homogenous of degree 2 (**prove it**).

(e) If f is a homogenous function of degree k and g is a homogenous function of degree l then $f \cdot g$ is homogenous of degree $k + l$ and $\frac{f}{g}$ is homogenous of degree $k - l$ (**prove it**).

(f) If f and g are homogenous functions of same degree k then $f + g$ is homogenous of degree k too (**prove it**).

(g) Only homogenous function of degree k of one variable is $f(x) = ax^k$ (**prove it**).

(h) Only homogenous function of degree 0 of one variable is constant function (**prove it**).

(i) However there exist nonconstant homogenous functions of degree 0: the function $f(x, y) = \frac{x}{y}$ is homogenous of degree 0 (**prove it**).

1.1.1 Economical Examples

Constant return to scale - production function which is homogenous of degree $k = 1$.

Increasing return to scale - production function which is homogenous of degree $k > 1$.

Decreasing return to scale - production function which is homogenous of degree $k < 1$.

Cobb-Douglas function

$$q(x_1, ..., x_n) = Ax_1^{\alpha_1} \cdot \ ... \ \cdot x_n^{\alpha_n}$$

is homogenous of degree $k = \alpha_1 + ... + \alpha_n$.

Constant elasticity of substitution (CES) function $A(a_1 x_1^p + a_2 x_2^p)^{\frac{q}{p}}$ is homogenous of degree q.

Demand function that is derived from utility function is homogenous of degree 0: if the prices $(p_1, ..., p_n)$ and income I change say 10 times all together, then the demand will not change.

More precisely, let $U(x_1, ..., x_n)$ be the utility function, $p = (p_1, ..., p_n)$ be the price vector, $x = (x_1, ..., x_n)$ be a consumption bundle and let

$$p \cdot x = p_1 x_1 + \ ... \ + p_n x_n \leq I$$

be the budget constraint. The demand function $x = D(p_1, ..., p_n, I)$ associates to each price vector p and income level I the consumption bundle x which is a solution of the following constraint maximization problem

Maximize $U(x)$ **subject of** $p_1 x_1 + \ ... \ + p_n x_n \leq I$**.**

So we claim that demand function $x = D(p_1, ..., p_n, I)$ is homogenous of degree 0, that is

$$D(tp_1, ..., tp_n, tI) = t^0 D(p_1, ..., p_n, I) = D(p_1, ..., p_n, I).$$

Indeed, the bundle $D(tp_1, ..., tp_n, tI)$ is a solution of the following constraint maximization problem

Maximize $U(x)$ **subject of** $tp_1 x_1 + \ ... \ + tp_n x_n \leq tI$**.**

As we see this is the same problem, thus

$$D(tp_1, ..., tp_n, tI) = t^0 D(p_1, ..., p_n, I) = D(p_1, ..., p_n, I).$$

Profit and cost functions that are derived from production function are homogenous too.

1.1.2 Properties of Homogenous Functions

Theorem 1 *If $f(x_1, ..., x_n)$ is homogenous of degree k then it's first order partial derivatives are homogenous of degree $k-1$.*

Proof. Differentiate $f(tx_1, ..., tx_n) = t^k f(x_1, ..., x_n)$ by x_i.

Theorem 2 *Level sets of a homogenous function are radial expansions of one another, that is for arbitrary $x = (x_1, ..., x_n),\ y = (y_1, ..., y_n),\ t > 0$ if x and y are on the same level set then their radial expansions tx and ty are on the same level set too.*

Proof. We must show that $f(x) = f(y)$ implies $f(tx) = f(ty)$, indeed

$$f(tx) = t^k f(x) = t^k f(y) = f(ty).$$

Theorem 3 *The tangent lines of the level curves of a homogenous function $f(x, y)$ have constant slopes along each ray from the origin.*

Proof. The slope of the tangent line of the level curve at (x_0, y_0) is

$$k = -\frac{\frac{\partial f}{\partial x}(x_0, y_0)}{\frac{\partial f}{\partial y}(x_0, y_0)}$$

and the slope of the tangent line of the level curve at (tx_0, ty_0) is

$$k_t = -\frac{\frac{\partial f}{\partial x}(tx_0, ty_0)}{\frac{\partial f}{\partial y}(tx_0, ty_0)} = -\frac{t^{k-1}\frac{\partial f}{\partial x}(x_0, y_0)}{t^{k-1}\frac{\partial f}{\partial y}(x_0, y_0)} = k$$

1.1.3 Calculus Criterion for Homogeneity

Observation. Let $f(x) = ax^k$ be a homogenous function of one variable of degree k. Then

$$x \cdot f'(x) = kf(x)$$

(check it!).

The following theorem generalizes this fact for functions of several variables.

Theorem 4 (Euler's theorem) *Let $f(x_1, ..., x_n)$ be a function that is homogenous of degree k. Then*

$$x_1 \frac{\partial\ f}{\partial\ x_1}(x) + \ \ ...\ \ + x_n \frac{\partial\ f}{\partial\ x_n}(x) = kf(x),$$

or, in gradient notation,

$$x \cdot \nabla f(x) = kf(x).$$

Proof. Differentiate $f(tx_1, ..., tx_n) = t^k f(x_1, ..., x_n)$ by t and then set $t = 1$.

Note that the converse result is also valid: $x \cdot \nabla f(x) = kf(x)$ implies the homogeneity of f.

1.1.4 Homogenizing of a Function

Theorem 5 *Suppose a function of $n+1$ variables $F(x_1,...,x_n,z)$ is homogenous of degree k, that is*

$$F(tx_1,...,tx_n,tz)=t^kF(x_1,...,x_n,z),$$

and let $f(x_1,..,x_n)$ be the function of n variables which is the restriction of F on arguments with $z=1$:

$$f(x_1,..,x_n)=F(x_1,..,x_n,1).$$

Then this restriction allows to reconstruct whole F:

$$F(x_1,...,x_n,z)=f(\frac{x_1}{z},...,\frac{x_1}{z}).$$

Proof.

$$F(x_1,...,x_n,z)=F(z\cdot\frac{x_1}{z},...,z\cdot\frac{x_n}{z},z\cdot 1)=z^kF(\frac{x_1}{z},...,\frac{x_n}{z},1)=f(\frac{x_1}{z},...,\frac{x_1}{z}).$$

Theorem 6 *For a function $f:R^n\to R$ and an integer k the $n+1$ variable function*

$$F(x_1,...,x_n,z)=z^kf(\frac{x_1}{z},...,\frac{x_n}{z})$$

is homogenous of degree k.

*This is unique extension of f to a homogenous function: if $G(x_1,...,x_n,z)$ is homogenous of degree k and $G(x_1,...,x_n,1)=f(x_1,...,x_n)$, then $G=F$.

Proof. Direct calculation (do it).

Examples. (a) Let $f(x)=x^\alpha$, then its homogenization of degree 1 is

$$F(x,y)=yf(\frac{x}{y})=y\frac{x^\alpha}{y^\alpha}=x^\alpha y^{1-\alpha}.$$

(b) Consider nonhomogeneous function $f(x)=x-ax^2$. Its homogenization of degree one is

$$F(x,y)=yf(\frac{x}{y})=y[\frac{x}{y}-a\frac{x^2}{y^2}]=x-a\frac{x^2}{y}.$$

(c) Let $D_1(p_1,p_2)$ be the demand function for the good 1, where p_1 and p_2 are prices for the goods 1 and 2 respectively. As we know D_1 is homogenous of degree 0. Suppose we just know the demand function of the good 1 when the price of good 2 is fixed at p. Can we reconstruct D_1 for arbitrary price vector $D_1(p_1,p_2)$? Yes:

$$D_1(p_1,p_2)=D_1(p_1,\frac{p_2}{p}p)=D_1(\frac{p_2}{p}\frac{pp_1}{p_2},\frac{p_2}{p}p)=D_1(\frac{pp_1}{p_2},p).$$

1.2 Cardinal vs Ordinal

A property of a function is called **ordinal** if it depends only on the shape and location of level sets and does not depend on the actual values of the function.

A property is called **cardinal** if it also depends on actual values of the function.

In this context two functions are equivalent if they have the exact same level sets (they represent the same preferences), although they may assign different numbers to a given level sets.

For example the functions $u(x,y)$ and $u(x,y)+2$ differ as functions but they have exact same level sets.

Definition 2 *Let $u : R^n \to R$ be a function. Its* ***monotonic transformation*** *is the composition $g \circ u : R^n \to R \to R$ where $g : R \to R$ is a* ***strictly increasing*** *function.*

In this case u and $g \circ u$ are equivalent in the above sense: they have the exact same level sets.

Examples. Let $U(x,y) = xy$. Then the following functions

$$3xy + 2, \ (xy)^2, \ \ e^{xy}, \ \ lnx + lny$$

are monotonic transformations of u, the suitable strictly increasing g-s are respectively

$$3z + 2, \ \ z^2, \ \ e^z, \ \ lnz.$$

Monotonic transformations preserve ordinal properties of u but do not preserve cardinal properties.

Example. Let $u(x,y)$ be a utility function which is an increasing function of x. It means

$$\frac{\partial u}{\partial x}(x,y) > 0.$$

This property is ordinal: if we replace u by the function $v(x,y)$ which is the composition $v(x,y) = gu(x,y)$ with strictly increasing g then

$$\frac{\partial v}{\partial x}(x,y) = \frac{\partial (g \circ u)}{\partial x}(x,y) = g'(u(x,y)) \cdot \frac{\partial u}{\partial x}(x,y) > 0$$

since both factors here are positive.

Example. Marginal Utility (MU) is cardinal but marginal Rate of Substitution (MRS) is ordinal. Indeed, let $u(x,y)$ be a utility function and $v(x,y) = g(u(x,y))$ be its monotonic transformation. The functions u and v have different partial derivatives (different marginal utilities), but their MRS equal:

$$MRS(v(x,y)) = \frac{\frac{\partial v}{\partial x}(x,y)}{\frac{\partial v}{\partial y}(x,y)} = \frac{\frac{\partial}{\partial x} g(u(x,y))}{\frac{\partial}{\partial y} g(u((x,y))} =$$

$$\frac{g'(u(x,y)) \cdot \frac{\partial u}{\partial x}(x,y)}{g'(u(x,y) \cdot \frac{\partial u}{\partial y}(x,y)} = \frac{\frac{\partial u}{\partial x}(x,y)}{\frac{\partial u}{\partial y}(x,y)} = MRS(u(x,y)).$$

1.3 Homothetic Functions

Definition 3 *A function $\nu : R^n \to R$ is called homothetic if it is a monotonic transformation of a homogenous function, that is there exist a strictly increasing function $g : R \to R$ and a homogenous function $u : R^n \to R$ such that $\nu = g \circ u$.*

It is clear that homothetiticy is ordinal property: monotonic transformation of homothetic function is homothetic (prove it!).
Examples. Let $u(x, y) = xy$, a homogenous function of degree 2. Then the monotonic transformations

$$g_1(z) = z + 1,\ g_2(z) = z^2 + z,\ g_3(z) = ln\ z$$

generate the following homothetic (but not homogenous) functions

$$\nu_1(x, y) = xy + 1,\ \nu_2(x, y) = x^2y^2 + xy,\ \nu_3(x, y) = ln\ x + ln\ y.$$

1.3.1 Properties of Homothetic Functions

Theorem 7 *Level sets of a homothetic function are radial expansions of one another, that is $\nu(x) = \nu(y)$ implies $\nu(tx) = \nu(ty)$ for arbitrary $x = (x_1, ..., x_n),\ y = (y_1, ..., y_n),\ t > 0$.*

Proof. $\nu(tx) = g(u(tx)) = g(t^k u(x)) = g(t^k u(y)) = g(u(ty)) = \nu(ty)$.

Theorem 8 *For a homothetic function the slopes of level sets along rays from the origin are constant, that is*

$$-\frac{\frac{\partial \nu}{\partial x_i}(tx)}{\frac{\partial \nu}{\partial x_j}(tx)} = -\frac{\frac{\partial \nu}{\partial x_i}(x)}{\frac{\partial \nu}{\partial x_j}(x)}$$

for all i, j and $t > 0$.

In other words Marginal Rate of Substitution (MRS) for a homothetic function is a homogenous function of degree 0.
Proof.

$$\frac{\frac{\partial \nu}{\partial x_i}(tx)}{\frac{\partial \nu}{\partial x_j}(tx)} = \frac{\frac{\partial}{\partial x_i}g(u(tx))}{\frac{\partial}{\partial x_j}g(u(tx))} = \frac{g'(u(tx))\cdot\frac{\partial u}{\partial x_i}(tx)}{g'(u(tx))\cdot\frac{\partial u}{\partial x_j}(tx)} = \frac{\frac{\partial u}{\partial x_i}(tx)}{\frac{\partial u}{\partial x_j}(tx)} = \frac{t^{k-1}\frac{\partial u}{\partial x_i}(x)}{t^{k-1}\frac{\partial u}{\partial x_j}(x)} = \frac{\frac{\partial u}{\partial x_i}(x)}{\frac{\partial u}{\partial x_j}(x)}.$$

Substituting $t = 1$ we obtain

$$\frac{\frac{\partial \nu}{\partial x_i}(x)}{\frac{\partial \nu}{\partial x_j}(x)} = \frac{\frac{\partial u}{\partial x_i}(x)}{\frac{\partial u}{\partial x_j}(x)},$$

Thus

$$-\frac{\frac{\partial \nu}{\partial x_i}(tx)}{\frac{\partial \nu}{\partial x_j}(tx)} = -\frac{\frac{\partial u}{\partial x_i}(x)}{\frac{\partial u}{\partial x_j}(x)} = -\frac{\frac{\partial \nu}{\partial x_i}(x)}{\frac{\partial \nu}{\partial x_j}(x)},$$

this completes the proof.

Exercises

1. Which of the following functions is homogenous? What are degrees of homogenous ones?

$$
\begin{array}{llll}
(a) & 3x^5y + 2x^2y^4 - 3x^3y^3, & (b) & 3x^5y + 2x^2y^4 - 3x^3y^4, \\
(c) & x^{1/2}y^{-1/2} + 3xy^{-1} + 7, & (d) & x^{3/4}y^{1/4} + 6x, \\
(e) & x^{3/4}y^{1/4} + 6x + 4, & (f) & \frac{x^2-y^2}{x^2+y^2} + 3.
\end{array}
$$

2. If $f(x_1, x_2)$ is homogenous of degree r then

$$f''_{x_1x_1}x_1^2 + 2f''_{x_1x_2}x_1x_2 + f''_{x_2x_2}x_2^2 = r(r-1)f.$$

3. Write the degree one homogenization of each of the following functions (a) e^x, (b) $ln\ x$, (c) 5, (d) $x_1^2 + x_2^3$, (e) $x_1^2 + x_2^2$.

4. Is the zero function $f(x) = 0$ homogenous?

5. Show directly that each of the five utility functions

$$3xy + 2, \quad (xy)^2, \quad (xy)^3 + xy, \quad e^{xy}, \quad lnx + lny$$

are equivalent to xy. Show that they have the same marginal rates of substitution at the bundle $(2, 1)$. Show that they have different marginal utilities (of good one) at $(2, 1)$.

6. Use the monotonic transformation z^k to prove that every homogeneous function is equivalent to a homogeneous function of degree one.

7. Is having decreasing marginal utility, $\frac{\partial^2 U}{\partial x_i^2} < 0$ for all i, an ordinal property? Why?

8. Prove that any function $f : R^1 \to R^1$ with $f' > 0$ everywhere is equivalent to a homogeneous function of degree one.

9. Which of the following functions are homothetic? Give a reason for each answer.

$$
\begin{array}{llllll}
(a) & e^{x^2y}e^{xy^2}, & (b) & 2logx + 3logy, & (c) & x^3y^6 + 3x^2y^4 + 6xy^2 + 9, \\
(d) & x^2y + xy, & (e*) & \frac{x^2y^2}{xy+1}. & &
\end{array}
$$