Reading [Simon], Chapter 18, p. 424-439.

1 Inequality Constraints

1.1 One Inequality constraint

Problem: maximize $f(x, y)$ subject to $g(x, y) \leq b$.

As we see here the constraint is written as inequality instead of equality.

An inequality constraint $g(x, y) \leq b$ is called *binding (or active) at a point* (x, y) if $g(x, y) = b$ and *not binding* (or inactive) if $g(x, y) < b$.

Again we consider the same Lagrangian function

$$L(x, y, \lambda) = f(x, y) - \lambda[g(x, y) - b].$$

Theorem 1 *Suppose (x^*, y^*) is a solution of the above problem: (x^*, y^*) maximizes f on the constraint set $g^*(x, y) \leq b$.*

*Suppose the following **qualification** is satisfied: If $g(x^*, y^*) = b$ (i.e. if (x^*, y^*) is binding) then $Dg(x^*, y^*) \neq (0, 0)$. Then there exists a multiplier λ^* such that*

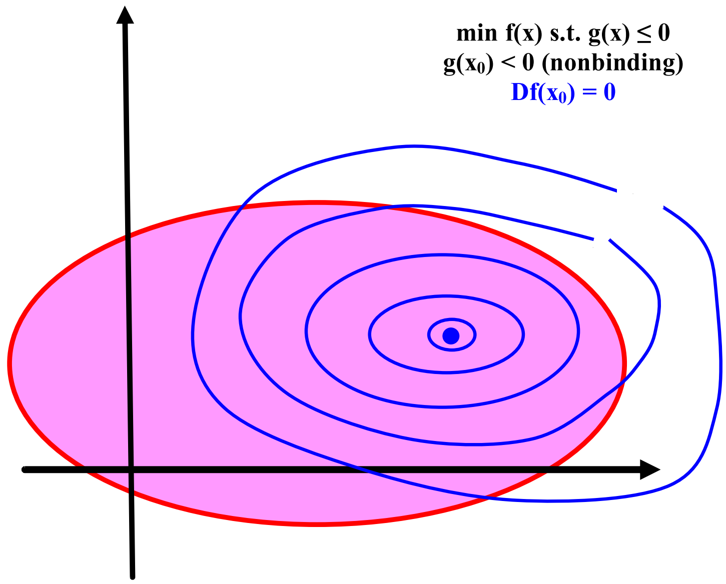
$$\begin{array}{ll} (a) & \frac{\partial L}{\partial x}(x^*, y^*, \lambda^*)) = 0, \\ (b) & \frac{\partial L}{\partial y}(x^*, y^*, \lambda^*)) = 0, \\ (c) & \lambda^*[g(x^*, y^*) - b] = 0, \\ (d) & \lambda^* \geq 0, \\ (e) & g(x^*, y^*) \leq b. \end{array}$$

Remark 1. These conditions, as well as the conditions from theorems bellow concerning with inequality conditions are called Karush-Kuhn-Tucker (KKT) conditions.

Remark 2. For the minimization problem the condition (d) must be replaced by
(d') $\lambda^* \leq 0$.

Almost a proof. Consider the following two cases: (x^*, y^*) is binding or not binding.

Case 1: (x^*, y^*) is not binding $g(x^*, y^*) < 0$.



This means that (x^*, y^*) is an *inner* (unconstraint) maximum, thus $f_x(x^*, y^*) = 0,\ f_y(x^*, y^*) = 0$. In this case we *can take* $\lambda = 0$ the conditions (a) - (e) are satisfied. Indeed

(a) $L_x(x^*, y^*, \lambda^*)) = f_x(x^*, y^*) - \lambda \cdot g_x(x^*, y^*) = f_x(x^*, y^*) - 0 \cdot g_x(x^*, y^*) = 0;$

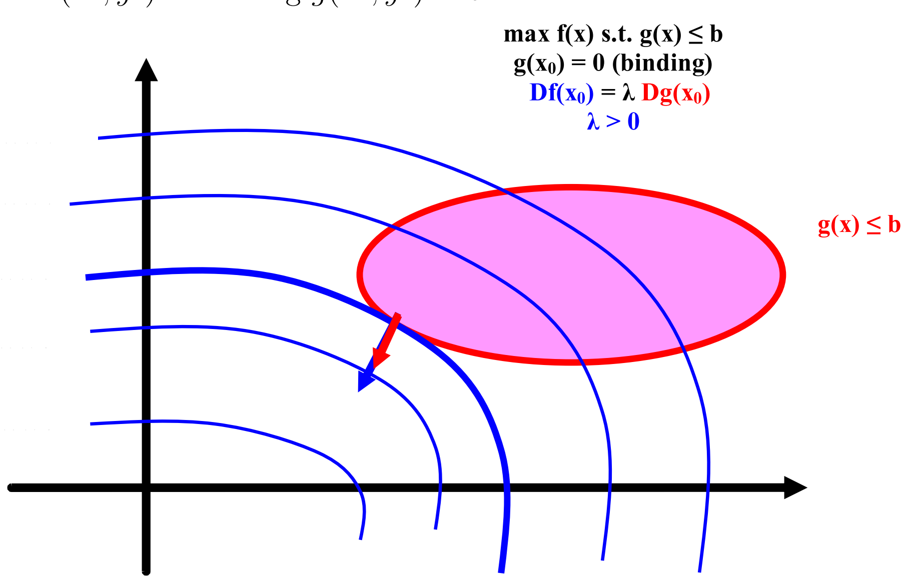
(b) $L_y(x^*, y^*, \lambda^*)) = f_y(x^*, y^*) - \lambda \cdot g_y(x^*, y^*) = f_y(x^*, y^*) - 0 \cdot g_x(x^*, y^*) = 0;$

(c)$\lambda^* \cdot [g(x^*, y^*) - b] = 0 \cdot [g(x^*, y^*) - b] = 0;$

(d) $0 = \lambda^* \geq 0;$

(e) $g(x^*, y^*) - b < 0.$

Case 2: (x^*, y^*) is binding $g(x^*, y^*) = 0$.



This means that (x^*, y^*) is a maximizer constrained by the equality condition, thus there exists λ^* such that

$$L_x(x^*, y^*) = 0,\ L_y(x^*, y^*) = 0,\ L_\lambda(x^*, y^*) = 0,$$

this again implies the needed conditions (a) - (e):

(a) $L_x(x^*, y^*, \lambda^*)) = 0$;
(b) $L_y(x^*, y^*, \lambda^*)) = 0$;
(c)$\lambda^* \cdot [g(x^*, y^*) - b] = \lambda^* \cdot 0 = 0$;
(d) since of maximality of (x^*, y^*) the gradients $\nabla f(x^*, y^*)$ and $\nabla g(x^*, y^*)$ must have the same directions, thus $\lambda \geq 0$;
(e) $g(x^*, y^*) - b = 0$.

Remark. What is the meaning of the zero $\lambda = 0$ multiplier in Case 1? The shadow price in this case is 0: the maximal value $f(x^*, y^*)$ does not change when we change b a little.

Example 1. Minimize $f(x, y) = x^2 + y^2$ subject of $g(x, y) = 2x + y \leq 2$.
Solution. There are no critical points of g at all, so the qualification is satisfied.

The lagrangian in this case is

$$L(x, y, \lambda) = x^2 + y^2 - \lambda(2x + y - 2),$$

and the KKT conditions from Theorem are

$$\begin{array}{ll} (a) & \frac{\partial L}{\partial x}(x, y, \lambda) = 2x - 2\lambda = 0, \\ (b) & \frac{\partial L}{\partial y}(x, y, \lambda) = 2y - \lambda y = 0, \\ (c) & \lambda[g(x, y) - b] = \lambda(2x + y - 2) = 0, \\ (d) & \lambda \leq 0, \\ (e) & g(x, y) = 2x + y \leq 2. \end{array}$$

We consider two cases:

Case 1. $\lambda = 0$, in this case our system looks as

$$\begin{array}{ll} (a) & x = 0, \\ (b) & y = 0, \\ (c) & 0 = 0, \\ (d) & 0 \leq 0, \\ (e) & 2x + y \leq 2, \end{array}$$

so the solution in this case is $(x, y, \lambda) = (0, 0, 0)$.
Case 2. $2x + y - 2 = 0$, in this case our system looks as

$$\begin{array}{ll} (a) & 2x = 2\lambda, \\ (b) & 2y = \lambda, \\ (c) & 2x + y - 2 = 0, \\ (d) & \lambda \leq 0, \\ (e) & 2x + y \leq 2, \end{array}$$

so $x = 2y,\ 2x + y = 2$, this gives the solution $x = 0.8,\ y = 0.4$ but $\lambda = 0.8 > 0$ so this solution *can not* be a minimizer.

So if this constrained minimization problem has a solution, it *can be* only $(0,0)$.

Example 2. Maximize $f(x,y) = xy$ subject of $g(x,y) = x^2 + y^2 \leq 2$.

Solution. The constraint function $g(x,y)$ has no critical points at all, so the qualification is satisfied.

The Lagrangian in this case is

$$L(x,y,\lambda) = xy - \lambda(x^2 + y^2 - 2),$$

and the KKT conditions from Theorem are

$$\begin{array}{ll} (a) & \frac{\partial L}{\partial x}(x,y,\lambda^*)) = y - 2\lambda x = 0, \\ (b) & \frac{\partial L}{\partial y}(x,y,\lambda)) = x - 2\lambda = 0, \\ (c) & \lambda^*[g(x,y) - b] = \lambda(x^2 + y^2 - 2) = 0, \\ (d) & \lambda \geq 0, \\ (e) & g(x,y) = x^2 + y^2 \leq 2. \end{array}$$

We consider two cases:

Case 1: $\lambda = 0$. In this case our system looks as

$$\begin{array}{ll} (a) & y = 0, \\ (b) & x = 0, \\ (c) & 0 = 0, \\ (d) & 0 \geq 0, \\ (e) & x^2 + y^2 \leq 2 \end{array}$$

so the solution in this case is $(x^*, y^*, \lambda^*) = (0,0,0)$.

Case 2: $x^2 + y^2 - 2 = 0$. In this case our system looks as

$$\begin{array}{ll} (a) & y - 2\lambda x = 0, \\ (b) & x - 2\lambda y = 0, \\ (c) & (x^2 + y^2 - 2) = 0, \\ (d) & \lambda \geq 0, \\ (e) & x^2 + y^2 \leq 2. \end{array}$$

The first two equations yield

$$\lambda = \frac{y}{2x} = \frac{x}{2y}, \quad or \quad x^2 = y^2.$$

Together with the condition (c) it gives thew system

$$\begin{array}{l} x^2 = y^2 \\ x^2 + y^2 = 2. \end{array}$$

The solution gives $x^2 = 1,\ y^2 = 1$, or $x = \pm 1,\ y = \pm 1$. Combining with $\lambda = \frac{y}{2x}$ we obtain corresponding $\lambda = \pm\frac{1}{2}$.

Combining all these solutions we get the following candidates for maximizers

$$\begin{array}{lll} x^* = +1, & y^* = +1, & \lambda^* = +\frac{1}{2}; \\ x^* = -1, & y^* = -1, & \lambda^* = +\frac{1}{2}; \\ x^* = +1, & y^* = -1, & \lambda^* = -\frac{1}{2}; \\ x^* = -1, & y^* = +1 & \lambda^* = -\frac{1}{2}. \end{array}$$

The last two solution contradict to the condition (e) $\lambda \geq 0$, so, including $(0,0,0)$ there are three candidates which satisfy the first order conditions. Note that the constraint set is compact. Plugging these three into the objective function, we find that $f(1,1) = 1,\ f(-1,-1) = 1$ so both $(1,1)$ and $(-1,-1)$ are the needed maximizers.

Note that the two points with negative multipliers

$$\begin{array}{lll} x^* = +1, & y^* = -1, & \lambda^* = -\frac{1}{2}; \\ x^* = -1, & y^* = +1 & \lambda^* = -\frac{1}{2}. \end{array}$$

are the solutions of the problem of *minimizing* of $f(x,y) = xy$ subject to $g(x,y) = x^2 + y^2 \leq 1$.

Economical Application. Consider the standard problem of maximization of the utility function

$$U(x_1, x_2)$$

subject to the budget *inequality* constraint

$$p_1 x_1 + p_2 x_2 \leq I.$$

Suppose additionally that $p_1 > 0,\ p_2 > 0$ and the utility function is *monotonic* in both arguments, that is for each commodity bundle (x_1, x_2)

$$\frac{\partial U}{\partial x_1}(x_1,x_2) > 0, \quad \frac{\partial U}{\partial x_2}(x_1,x_2) > 0.$$

This means that our commodities are *goods*.

Then the KKT implies an important result: the optimal solution is *necessarily binding*

$$p_1 x_1 + p_2 x_2 = I,$$

that is at optimizer the *consumer spends all the available income.*

Indeed, for the Lagrangian

$$L(x_1,x_2) = f(x_1,x_2) - \lambda(p_1 x_1 + p_2 x_2 - I)$$

we have

$$\begin{array}{l} \frac{\partial L}{\partial x_1}(x_1,x_2) = \frac{\partial U}{\partial x_1}(x_1,x_2) - \lambda p_1 = 0, \\ \frac{\partial L}{\partial x_2}(x_1,x_2) = \frac{\partial U}{\partial x_2}(x_1,x_2) - \lambda p_1 = 0, \end{array}$$

now, since $\frac{\partial U}{\partial x_1}(x_1, x_2) > 0$ and (or) $\frac{\partial U}{\partial x_2}(x_1, x_2) > 0$, it follows that $\lambda > 0$. Then from the condition

$$\lambda(p_1 x_1 + p_2 x_2 - I) = 0$$

we get

$$p_1 x_1 + p_2 x_2 - I = 0.$$

Notice that it is enough to require that just one of commodities is a good.

1.2 Two Inequality Constraints

Maybe it will be useful to consider separately the following problem:

$$max\ f(x_1, x_2, x_3) = 0 \quad s.t. \quad g_1(x_1, x_2, x_3) \leq b_1, \quad g_2(x_1, x_2, x_3) \leq b_2.$$

Lagrangian function in this case is

$$L(x_1, x_2, x_3) =$$
$$f(x_1, x_2, x_3) - \lambda_1(g_1(x_1, x_2, x_3) - a) - \lambda_2(g_2(x_1, x_2, x_3) - b_2).$$

The KKT conditions in this case look as

$$\begin{array}{ll}
(1) & \frac{\partial}{\partial x_1} f(x_1, x_2, x_3) - \lambda_1 \frac{\partial}{\partial x_1} g_1(x_1, x_2, x_3) - \lambda_2 \frac{\partial}{\partial x_1} g_2(x_1, x_2, x_3) = 0 \\
(2) & \frac{\partial}{\partial x_2} f(x_1, x_2, x_3) - \lambda_1 \frac{\partial}{\partial x_2} g_1(x_1, x_2, x_3) - \lambda_2 \frac{\partial}{\partial x_2} g_2(x_1, x_2, x_3) = 0 \\
(3) & \frac{\partial}{\partial x_3} f(x_1, x_2, x_3) - \lambda_1 \frac{\partial}{\partial x_3} g_1(x_1, x_2, x_3) - \lambda_2 \frac{\partial}{\partial x_3} g_2(x_1, x_2, x_3) = 0 \\
(4) & \lambda_1[g_1(x_1, x_2, x_3) - b_1] = 0 \\
(5) & \lambda_2[g_2(x_1, x_2, x_3) - b_2] = 0 \\
(6) & \lambda_1 \geq 0 \\
(7) & \lambda_2 \geq 0 \\
(8) & g_1(x_1, x_2, x_3) \leq b_1 \\
(9) & g_2(x_1, x_2, x_3) \leq b_2.
\end{array}$$

Consider, concerning complementary slackness conditions (4) and (5), the following 4 cases:
Case 1: $\lambda_1 = 0,\ \lambda_2 = 0.$
Case 2: $g_1(x_1, x_2, x_3) - b_1 = 0,\ \lambda_2 = 0.$
Case 3: $\lambda_1 = 0,\ g_2(x_1, x_2, x_3) - b_2 = 0.$
Case 4: $g_1(x_1, x_2, x_3) - b_1 = 0,\ g_2(x_1, x_2, x_3) - b_2 = 0.$

We rewrite the KKT conditions in these cases:

Case 1: $\lambda_1 = 0, \ \lambda_2 = 0.$

$$\begin{array}{ll}
(1) & \frac{\partial}{\partial x_1} f(x_1,x_2,x_3) = 0 \\
(2) & \frac{\partial}{\partial x_2} f(x_1,x_2,x_3) = 0 \\
(3) & \frac{\partial}{\partial x_3} f(x_1,x_2,x_3) = 0 \\
(4) & \\
(5) & \\
(6) & \\
(7) & \\
(8) & g_1(x_1,x_2,x_3) \leq b_1 \\
(9) & g_2(x_1,x_2,x_3) \leq b_2
\end{array}$$

so in this case we face ordinary nonconstrained optimization problem

$$max \ \ f(x_1,x_2,x_3),$$

but with additional conditions $g_1(x_1,x_2,x_3) \leq b_1$ and $g_2(x_1,x_2,x_3) \leq b_2$, that is ignore all candidates (critical points of f) which are out of feasible region.

Case 2: $g_1(x_1,x_2,x_3) - b_1 = 0, \ \lambda_2 = 0.$

$$\begin{array}{ll}
(1) & \frac{\partial}{\partial x_1} f(x_1,x_2,x_3) - \lambda_1 \frac{\partial}{\partial x_1} g_1(x_1,x_2,x_3) = 0 \\
(2) & \frac{\partial}{\partial x_2} f(x_1,x_2,x_3) - \lambda_1 \frac{\partial}{\partial x_2} g_1(x_1,x_2,x_3) = 0 \\
(3) & \frac{\partial}{\partial x_3} f(x_1,x_2,x_3) - \lambda_1 \frac{\partial}{\partial x_3} g_1(x_1,x_2,x_3) = 0 \\
(4) & g_1(x_1,x_2,x_3) - b_1 = 0 \\
(5) & \\
(6) & \lambda_1 \geq 0 \\
(7) & \\
(8) & \\
(9) & g_2(x_1,x_2,x_3) \leq b_2
\end{array}$$

so in this case we face the problem with one equality constraint

$$max \ \ f(x_1,x_2,x_3) \ \ s.t. \ \ g_1(x_1,x_2,x_3) = b_1$$

with additional conditions $g_2(x_1,x_2,x_3) \leq b_2, \ \ \lambda_1 \geq 0$, that is we ignore all candidates with $g_2(x_1,x_2,x_3) > b_2$ or $\lambda_1 < 0$.

Case 3: $\lambda_1 = 0, \ g_2(x_1,x_2,x_3) - b_2 = 0.$

$$\begin{array}{ll}
(1) & \frac{\partial}{\partial x_1} f(x_1,x_2,x_3) - \lambda_1 \frac{\partial}{\partial x_1} g_1(x_1,x_2,x_3) = 0 \\
(2) & \frac{\partial}{\partial x_2} f(x_1,x_2,x_3) - \lambda_1 \frac{\partial}{\partial x_2} g_1(x_1,x_2,x_3) = 0 \\
(3) & \frac{\partial}{\partial x_3} f(x_1,x_2,x_3) - \lambda_1 \frac{\partial}{\partial x_3} g_1(x_1,x_2,x_3) = 0 \\
(4) & \\
(5) & g_2(x_1,x_2,x_3) - b_2 = 0 \\
(6) & \\
(7) & \lambda_2 \geq 0 \\
(8) & g_1(x_1,x_2,x_3) \leq b_1 \\
(9) &
\end{array}$$

so in this case we face the problem with one equality constraint

$$max \quad f(x_1, x_2, x_3) \quad s.t. \quad g_2(x_1, x_2, x_3) = b_2$$

with additional conditions $g_1(x_1, x_2, x_3) \leq b_1, \quad \lambda_2 \geq 0$, that is we ignore all candidates with $g_1(x_1, x_2, x_3) > b_2$ or $\lambda_2 < 0$.

Case 4: $g_1(x_1, x_2, x_3) - b_1 = 0, \; g_2(x_1, x_2, x_3) - b_2 = 0$.

$$\begin{array}{ll}
(1) & \frac{\partial}{\partial x_1} f(x_1, x_2, x_3) - \lambda_1 \frac{\partial}{\partial x_1} g_1(x_1, x_2, x_3) - \lambda_2 \frac{\partial}{\partial x_1} g_2(x_1, x_2, x_3) = 0 \\
(2) & \frac{\partial}{\partial x_2} f(x_1, x_2, x_3) - \lambda_1 \frac{\partial}{\partial x_2} g_1(x_1, x_2, x_3) - \lambda_2 \frac{\partial}{\partial x_2} g_2(x_1, x_2, x_3) = 0 \\
(3) & \frac{\partial}{\partial x_3} f(x_1, x_2, x_3) - \lambda_1 \frac{\partial}{\partial x_3} g_1(x_1, x_2, x_3) - \lambda_2 \frac{\partial}{\partial x_3} g_2(x_1, x_2, x_3) = 0 \\
(4) & g_1(x_1, x_2, x_3) - b_1 = 0 \\
(5) & g_2(x_1, x_2, x_3) - b_2 = 0 \\
(6) & \lambda_1 \geq 0 \\
(7) & \lambda_2 \geq 0 \\
(8) & \\
(9) &
\end{array}$$

so in this case we face the problem with two equality constraint

$$max \quad f(x_1, x_2, x_3) \quad s.t. \quad g_1(x_1, x_2, x_3) = b_1, \quad g_2(x_1, x_2, x_3) = b_2$$

with additional conditions $\lambda_1 \geq 0, \quad \lambda_2 \geq 0$, that is we ignore all candidates with $\lambda_1 < 0$ or $\lambda_2 < 0$.

1.3 Several Inequality Constraints

Problem: maximize $f(x_1, ..., x_n)$ subject to k inequality constraints

$$g_1(x_1, ..., x_n) \leq b_1, \; ... \; , g_k(x_1, ..., x_n) \leq b_k.$$

Recall that constraint $g_i(x) \leq b$ is **binding** at a solution candidate $x^* = (x_1^*, ..., x_n^*)$ if $g_i(x^*) = b$, and it is called **not binding** or **slack** if $g(x^*) < b$.

Theorem 2 *Suppose $x^* = (x_1^*, ..., x_n^*)$ is a maximizer for our problem, and suppose first k_0 constraints are binding at x^* and the last $k - k_0$ are not binding.*

*Suppose that the following **qualification** is satisfied: the rank of the Jacobian of the binding constraints*

$$\begin{pmatrix} \frac{\partial g_1}{\partial x_1}(x^*) & ... & \frac{\partial g_1}{\partial x_n}(x^*) \\ ... & ... & ... \\ \frac{\partial g_{k_0}}{\partial x_1}(x^*) & ... & \frac{\partial g_{k_0}}{\partial x_n}(x^*) \end{pmatrix}$$

is k_0, as large as it can be. In other words the gradients of the active *inequality constraints are linearly independent at x^*.*

Consider the lagrangian

$$L(x_1, ..., x_n, \lambda_1, ..., \lambda_k) =$$
$$f(x_1, ..., x_n) - \lambda_1[g_1(x_1, ..., x_n) - b_1] - \ ... \ - \lambda_k[g_(x_1, ..., x_n) - b_1].$$

Then there exist multipliers $\lambda^* = (\lambda_1^*, ..., \lambda_k^*)$ *such that the following KKT conditions are satisfied*

$$\begin{array}{ll} (a) & \frac{\partial L}{\partial x_1}(x^*, \lambda^*) = 0, \ ... \ , \frac{\partial L}{\partial x_n}(x^*, \lambda^*) = 0, \\ (b) & \lambda_1^*[g_1(x^*) - b_1] = 0, \ ... \ , \lambda_k^*[g_k(x^*) - b_k] = 0, \\ (c) & \lambda_1^* \geq 0, \ ... \ , \lambda_k^* \geq 0, \\ (d) & g_1(x^*) \leq b_1, \ ... \ , g_k(x^*) \leq b_k. \end{array}$$

Remark. Actually, the Theorem can be reformulated as follows:

Suppose $x^* \in R^n$ is a maximizer of $f(x)$ s.t. $g_1(x) \leq b_1, \ ... \ , g_k(x) \leq b_k$, suppose also that firs k_0 constraints are binding at x^*, i.e.

$$g_1(x) = b_1, \ ... \ , g_{k_0}(x) = b_{k_0}$$

and others are nonbinding, and suppose the vectors

$$Dg_1(x^*), \ ... \ , Dg_{k_0}(x^*)$$

are linearly independent. Then

$$Df(x^*) \in span(Dg_1(x^*), \ ... \ , Dg_{k_0}(x^*)),$$

i.e.

$$Df(x^*) = \lambda_1 Dg_1(x^*) + \ ... \ + \lambda_{k_0} Dg_{k_0}(x^*),$$

and all the coefficients are nonnegative: $\lambda_1 \geq 0, \ ... \ , \lambda_{k_0} \geq 0$.

Remark. For the minimization problem the condition (c) must be replaced by
(c') $\lambda_1^* \leq 0, \ ... \ , \lambda_k^* \leq 0$.

Example 3. Solve the problem

maximize $f(x_1, x_2) = -(x_1 - 4)^2 - (x_2 - 4)^2$ subject to $g_1(x_1, x_2) = x_1 + x_2 \leq 4, \quad g_2(x_1, x_2) = x_1 + 3x_2 \leq 9$.

Solution. The Lagrangean looks as

$$L(x_1, x_2) = -(x_1 - 4)^2 - (x_2 - 4)^2 - \lambda_1(x_1 + x_2 - 4) - \lambda_2(x_1 + 3x_2 - 9).$$

The KKT conditions look as

$$\begin{aligned}
&-2(x_1-4)-\lambda_1-\lambda_2=0\\
&-2(x_2-4)-\lambda_1-3\lambda_2=0\\
&\lambda_1(x_1+x_2-4)=0\\
&\lambda_2(x_1+3x_2-9)=0\\
&\lambda_1\geq 0,\quad \lambda_2\geq 0\\
&x_1+x_2\leq 4,\quad x_1+3x_2\leq 9.
\end{aligned}$$

Analyzing the conditions $\lambda_1(x_1+x_2-4)=0,\quad \lambda_2(x_1+3x_2-9)=0$ we consider the following cases

Case 1: $x_1+x_2-4=0,\ x_1+3x_2-9=0$. In this case the solution gives $x_1=\frac{3}{2},\ x_2=\frac{5}{2},\ \lambda_1=6,\ \lambda_2=-1$ but the this solution violates the condition $\lambda_2\geq 0$, so NO SOLUTION of KKT in this case.

Case 2: $x_1+x_2-4=0,\ \lambda_2=0$. In this case the solution gives $x_1=2,\ x_2=2,\ \lambda_1=4,\ \lambda_2=0$. This solution is OK, it fulfills KKT.

Case 3: $\lambda_1=0,\ x_1+3x_2-9=0$. In this case the solution gives $x_1=3.3,\ x_2=1.8$ but this violates the condition $x_1+x_2\leq 4$, so NO SOLUTION of KKT in this case.

Case 4: $\lambda_1=0,\ \lambda_2=0$. In this case the solution gives $x_1=4,\ x_2=4,\ \lambda_1=0,\ \lambda_2=0$, this solution violates $x_1+x_2\leq 4$, so NO SOLUTION of KKT in this case.

Finally we have the single solution of KKT $x_1=2,\ x_2=2,\ \lambda_1=4,\ \lambda_2=0$. But KKT is just a necessary condition. So, is it a solution of our maximization problem?

1.4 Mixed constraints

Problem: maximize $f(x_1,...,x_n)$ subject to k inequality and m equality constraints

$$\begin{aligned}
&g_1(x_1,...,x_n)\leq b_1,\ \dots\ ,g_k(x_1,...,x_n)\leq b_k,\\
&h_1(x_1,...,x_n)=c_1,\ \dots\ ,h_m(x_1,...,x_n)=c_m.
\end{aligned}$$

Theorem 3 *Suppose $x^*=(x_1^*,...,x_n^*)$ is a maximizer for our problem, and suppose first k_0 inequality constraints are binding at x^* and the last $k-k_0$ are not binding.*

*Suppose that the following **qualification** is satisfied: the rank of the*

Jacobian of the binding constraints

$$\begin{pmatrix} \frac{\partial g_1}{\partial x_1}(x^*) & ... & \frac{\partial g_1}{\partial x_n}(x^*) \\ ... & ... & ... \\ \frac{\partial g_{k_0}}{\partial x_1}(x^*) & ... & \frac{\partial g_{k_0}}{\partial x_n}(x^*) \\ \frac{\partial h_1}{\partial x_1}(x^*) & ... & \frac{\partial h_1}{\partial x_n}(x^*) \\ ... & ... & ... \\ \frac{\partial h_m}{\partial x_1}(x^*) & ... & \frac{\partial h_m}{\partial x_n}(x^*) \end{pmatrix}$$

is $k_0 + m$, as large as it can be. In other words the gradients of the active *inequality constraints and the gradients of the equality constraints are linearly independent at x^* .*

Consider the lagrangian

$$\begin{array}{l} L(x_1, ..., x_n, \lambda_1, ..., \lambda_k, \mu_1, ...\mu_m)) = f(x_1, ..., x_n) + \\ -\lambda_1[g_1(x_1, ..., x_n) - b_1] - \ ... \ - \lambda_k[g_(x_1, ..., x_n) - b_1] + \\ -\mu_1[h_1(x_1, ..., x_n) - c_1] - \ ... \ - \mu_m[h_(x_1, ..., x_n) - c_1]. \end{array}$$

Then there exist multipliers $\lambda^* = (\lambda_1^*, ..., \lambda_k^*)$, $\mu^* = (\mu_1^*, ..., \mu_m^*)$ such that

$$\begin{array}{ll} (a) & \frac{\partial L}{\partial x_1}(x^*, \lambda^*, \mu^*) = 0, \ ... \ , \frac{\partial L}{\partial x_n}(x^*, \lambda^*, \mu^*) = 0, \\ (b) & \lambda_1^*[g_1(x^*) - b_1] = 0, \ ... \ , \lambda_k^*[g_k(x^*) - b_1] = 0, \\ (c) & h_1(x^*) = c_1, \ ... \ , h_m(x^*) = c_m, \\ (d) & \lambda_1^* \geq 0, \ ... \ , \lambda_k^* \geq 0, \\ (e) & g_1(x^*) \leq b_1, \ ... \ , g_1(x^*) \leq b_1. \end{array}$$

Remark. Actually, the Theorem can be reformulated as follows:

Suppose $x^* \in R^n$ is a maximizer of $f(x)$ s.t.

$$g_1(x) \leq b_1, \ ... \ , g_k(x) \leq b_k, \quad h_1(x) = c_1, \ ... \ , h_m(x) = c_m.$$

Suppose also that first k_0 inequality constraints are binding at x^*, i.e.

$$g_1(x) = b_1, \ ... \ , g_{k_0}(x) = b_{k_0}$$

and the others are nonbinding, and and suppose the vectors

$$Dg_1(x^*), \ ... \ , Dg_{k_0}(x^*), \ Dh_1(x^*), \ ... \ , Dh_m(x^*)$$

are linearly independent. Then

$$Df(x^*) \in span(Dg_1(x^*), \ ... \ , Dg_{k_0}(x^*), \ Dh_1(x^*), \ ... \ , Dh_m(x^*)$$

i.e.

$$Df(x^*) = \lambda_1 Dg_1(x^*) + \ ... \ + \lambda_{k_0} Dg_{k_0}(x^*) + \mu_1 Dh_1(x^*) + \ ... \ + \mu_m Dh_m(x^*)$$

and all the λ_i coefficients are nonnegative: $\lambda_1 \geq 0, \ ... \ , \lambda_{k_0} \geq 0$.

1.5 Economical Applications

The KKT conditions sometimes are used not for the finding of optimizers, rather for some important qualitative conclusions.

1.5.1 A Sales-Maximizing Firm with Advertizing

Let:

$y \in R_+$ production;

$C(y)$ - cost of manufacturing f y units (assume that $C' > 0$);

$a \in R_+$ - advertising cost;

$C(y) + a$ - total cost;

$R(y, a)$ - revenue (assume that $R_a > 0$);

$m \in R_+$ - minimal level of profit.

$\Pi = R(y, a) - C(y) - a$ - profit.

Problem:

$$max \quad R(y, a) \quad s.t. \quad \Pi \geq m, \quad a \geq 0.$$

Equivalently

$$max \quad R(y, a) \quad s.t. \qquad -a \leq 0, \quad m - R(y, a) + C(y) + a \leq 0.$$

Lagrangian:

$$L(y, a, \lambda_1, \lambda_2) = R(y, a) + \lambda_1 a - \lambda_2(m - R(y, a) + C(y) + a).$$

The KKT conditions for a maximizer y^*:

(1) $L_y = (1 + \lambda_2)R_y(y^*, a) - \lambda_2 C'(y^*) = 0$

(2) $L_a = (1 + \lambda_2)R_a(y^*, a) + \lambda_1 - \lambda_2 = 0$

(3) $\lambda_1 a = 0$

(4) $\lambda_2(m - R(y^*, a) + C(y^*) + a) = 0$

(5) $\lambda_1 \geq 0$

(6) $\lambda_2 \geq 0$

(7) $-a \leq 0$

(8) $m - R(y^*, a) + C(y^*) + a \leq 0.$

Observation 1. In (2) we have $(1 + \lambda_2)R_a(y^*, a) > 0, \ \lambda_1 \geq 0 \ \Rightarrow \lambda_2 > 0$. This, (4) gives

$$m - R(y^*, a) + C(y^*) + a = 0$$

i.e. at maximizer y^* we have $R(y^*) - C(y^*) - a = m$, *that is the revenue is maximal when the profit is at minimal allowed level!*

Observation 2. Let us estimate the marginal profit at the revenue maximizer y^* using (1):

$$(1 + \lambda_2)\Pi_y(y^*, a) =$$
$$(1 + \lambda_2)(R_y(y^*) - C'(y^*)) = (1 + \lambda_2)R_y(y^*) - (1 + \lambda_2)C'(y^*) =$$
$$(1 + \lambda_2)R_y(y^*) - \lambda_2 C'(y^*) - C'(y^*) = L_y(y^*, a) - C'(y^*) = 0 - C'(y^*) =$$
$$-C(y^*) < 0,$$

thus the revenue maximizer y^* *is greater than the profit maximizer.*

Exercises

1. Compare the solutions of following problems
(a) Find the maximizer of $f(x,y) = 10 - x^2 - y^2$.

(b) Find the maximizer of $f(x,y) = 10 - x^2 - y^2$, subject to the constraint $h(x,y) = 2x^2 + y^2 = 2$.

(c) Find the maximizer of $f(x,y) = 10 - x^2 - y^2$, subject to the constraint $g(x,y) = 2x^2 + y^2 \leq 2$.

(d) Find the maximizer of $f(x,y) = 10 - x^2 - y^2$, subject to the constraint $g(x,y) = 2x^2 + y^2 \geq 2$.

2. Compare the solutions of following problems
(a) Find the minimizer of $f(x,y) = 10 + x^2 + y^2$.

(b) Find the minimizer of $f(x,y) = 10 + x^2 + y^2$, subject to the constraint $h(x,y) = 2x^2 + y^2 = 2$.

(c) Find the minimizer of $f(x,y) = 10 + x^2 + y^2$, subject to the constraint $g(x,y) = 2x^2 + y^2 \leq 2$.

(d) Find the minimizer of $f(x,y) = 10 + x^2 + y^2$, subject to the constraint $g(x,y) = 2x^2 + y^2 \geq 2$.

3. Find the dimensions of the box with largest volume if the total surface area is 24 cm^2.

4. Find the maximum and minimum of $f(x,y) = 5x - 3y$ subject to the constraint $x^2 + y^2 = 136$.

5. Find the maximum and minimum of $f(x,y) = 4x^2 + 10y^2$ subject to the constraint $x^2 + y^2 \leq 4$.

6. Write down the KKT conditions for the problem:
Minimize $f(x_1, x_2, x_3) = -x_1^3 + x_2^2 - 2x_1x_3^2$ subject to the constraints: $2x_1 + x_2^2 + x_3 - 5 = 0$, $5x_1^2 - x_2^2 - x_3 \geq 2$, $x_1 \geq 0$, $x_2 \geq 2$, $x_3 \geq 0$. Verify the KKT conditions for $(1, 0, 3)$.

7. Write down the KKT conditions for the problem:
Minimize $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$ subject to the constraints: $-x_1 + x_2 - x_3 \geq -10$, $x_1 + x_2 + 4x_3 \geq 20$. Find all the solutions.

Homework

Exercises 18.10, 18.11, 18.12, 18.15, 18.17 from [Simon].