1 Distance

Reading

[SB], Ch. 29.4, p. 811-816

A *metric space* is a set S with a given *distance* (or *metric*) function $d(x, y)$ which satisfies the conditions

(a) Positive definiteness $d(x, y) \geq 0, \quad d(x, y) = 0 \;\Leftrightarrow\; x = y$;

(b) Symmetry $d(x, y) = d(y, x)$;

(c) Triangle inequality $d(x, y) + d(y, z) \geq d(x, z)$.

For a given metric function $d(x, y)$:
A *closed ball* of radius r and center $x \in S$ is defined as

$$\bar{B}_r(x) = \{y \in R,\ d(x, y) \leq r\}.$$

An *open ball* of radius r and center $x \in S$ is defined as

$$\bar{B}_r(x) = \{y \in R,\ d(x, y) < r\}.$$

A *sphere* of radius r and center $x \in S$ is defined as

$$S_r(x) = \{y \in R,\ d(x, y) = r\}.$$

Example. Metrics on R^n:

1. *Euclidian metric* $d_E(x, y) = \sqrt{(x_1 - y_1)^2 + \ \dots\ + (x_n - y_n)^2}$.

2. *Manhattan metric* (or Taxi Cab metric) $d_M(x, y) = |x_1 - y_1| + \ \dots\ + |x_n - y_n|$.

3. *Maximum metric* $d_{max}(x, y) = max(|x_1 - y_1|,\ \dots\ , |x_n - y_n|)$.

Some exotic metrics:

4. *Discrete metric* $d_{disc}(x, y) = 0 \ \ if \ \ x = y \ \ and \ \ d_{disc}(x, y) = 1 \ \ if \ \ x \neq y$

5. *British Rail metric* $d_{BR}(x, y) = ||x|| + ||y||$ if $x \neq y$ and $d_{BR}(x, x) = 0$.

6*. *Hamming distance.* Let S be the set of all 8 vertices of a cube, in coordinates

$$S = \{(0,0,0), (0,0,1), (0,1,0), (1,0,0), (0,1,1), (1,0,1), (1,1,0), (1,1,1)\}.$$

Hamming distance between two vertices is defined as the number of positions for which the corresponding symbols are different.

2 Norm

Let V be a vector space, say R^n. A *norm* is defined as a real valued function $|| - || : V \to R,\ v \to ||v||$, which satisfies the following conditions:

(i) positive definiteness $||v|| \geq 0, \quad ||v|| = 0 \ \Leftrightarrow \ v = 0$;

(ii) positive homogeneity or positive scalability $||r \cdot v|| = |r| \cdot ||v||$;

(iii) triangle inequality or subadditivity $||v + w|| \leq ||v|| + ||w||$.

Note that from (ii) follows that $||O|| = 0$ (here $O = (0,\ ...\ , 0)$), indeed, $||O|| = ||0 \cdot x|| = |0| \cdot ||x|| = 0$.

There is the following general *weighted* Euclidian norm on R^n which depends on parameters $a_1, ..., a_n$:

$$||x||_{a_1,...,a_n} = \sqrt{a_1 \cdot x_1^2 + \ ... \ + a_n \cdot x_n^2}.$$

If each $a_i = 1$, then this norm coincides with ordinary *Euclidian norm*

$$||x|| = \sqrt{x_1^2 + \ ... \ + x_n^2}.$$

There is s series of norms which depend on parameter k:

$$||x||_k = \sqrt[k]{|x_1|^k + \ ... \ + |x_n|^k}.$$

Particularly

$||x||_2 = \sqrt{(x_1)^2 + \ ... \ + (x_n)^2} = |x||_E$,

$||x||_1 = |x_1| + \ ... \ + |x_n = ||x||_M$,

and

$||x||_\infty = lim_{k\to\infty}||x||_k = lim_{k\to\infty}\sqrt[k]{|x_1|^k + \ ... \ + |x_n|^k} = ||x||_{max}$.

2.0.1 From Norm to Metric

Theorem 1 *Any norm $||x||$ induces a metric by $d(x, y) = ||x - y||$.*

Proof. The condition (i) implies the condition (a):
$d(x, y) = ||x - y|| \geq 0$; besides, if $x = y$ then $x - y = O$, thus $d(x, y) = ||x - y|| = ||O|| = 0$; conversely, suppose $d(x, y) = 0$, thus $||x - y|| = 0$, then, according to (i) we obtain $x - y = O$, so $x = y$.

The condition (ii) implies (b):

$$d(y, x) = ||y - x|| = ||(-1) \cdot (x - y)|| = |(-1)| \cdot ||x - y|| = ||x - y|| = d(x, y).$$

The condition(iii) implies (c):

$$d(x, y) + d(y, z) = ||x - y|| + ||y - z|| \geq ||x - y + y - z|| = ||x - z|| = d(x, z).$$

2.0.2 From Metric to Norm

Conversely, *some* metrics on R^n, which fit with the vector space structure determine a norm.

Theorem 2 *Suppose a metric $d(u, v)$ is given on a vector space V, and assume that the following two additional conditions are satisfied*
(d) translation invariance $d(u, v) = d(u + w, v + w)$,
and
(e) homogeneity $d(ku, kv) = |k| \cdot d(u, v)$.
Then $||v|| := d(v, O)$ is a norm.

Proof. The condition (a) implies the condition (i): $||v|| = d(v, O) \geq 0$; besides, suppose $||v|| = 0$, then $d(v, O) = 0$, thus, according to (i) we obtain $v = O$.

The condition (e) implies the condition (ii):

$$||k \cdot v|| = d(k \cdot v, O) = d(k \cdot v, k \cdot O) = |k| \cdot d(v, O) = |k| \cdot ||v||.$$

The condition (d) implies the condition (iii):

$$||v|| + ||w|| = d(v, O) + d(w, O) = d(v + w, O + w) + d(w, O) = \\ d(v + w, w) + d(w, O) \geq d(v + w, O) = ||v + w||.$$

Examples.

The above metrics 1,2,3 satisfy the properties (d) and (e) (**prove this!**), thus they determine the following norms on R^n: for a vector $x = (x_1, ..., x_n)$

1'. Euclidian norm $||x||_E = \sqrt{(x_1)^2 + \ ... \ + (x_n)^2}$.
2'. Manhattan norm $||x||_M = |x_1| + \ ... \ + |x_n|$.
3'. Maximum norm $||x||_{max} = max(|x_1|, \ ... \ , |x_n|)$.

The discrete metric d_{disc} does not induce a norm Indeed, take $v \neq O$, then $||2 \cdot v|| = d(2 \cdot v, O) = 1 \neq 2 = 2 \cdot d(v, O) = 2 \cdot ||v||$.

2.1 Metric and Norm Induce Topology*

Any metric produces the notion of open ball. In its turn a notion of open ball produces the notion of open set, i.e. induces a *topology*.

Since any norm determines a metric, so it induces a topology too.

2.1.1 Equivalence of Norms*

Two norms $||x||$ and $||x||'$ are called equivalent if there exist two positive scalars a and b such that

$$a \cdot ||x|| \leq ||x||' \leq b \cdot ||x||.$$

This is an equivalence relation on the set of all possible norms on R^n.

If two norms are equivalent, then they induce the same notions of open sets (same topology). In particular, if a sequence $\{a_n\}$ converges to the limit a with respect to the norm $|| - ||$ then this sequence converges to the same limit with respect to the equivalent norm $|| - ||'$.

The three metrics $||v||_{max}$, $||v||_E$, $||v||_M$ are equivalent. This is a result of following geometrical inequalities

$$\begin{aligned}
&||v||_{max} \leq ||v||_E \leq ||v||_M;\\
&||v||_E \leq \sqrt{2}||v||_{max};\\
&||v||_M \leq 2||v||_{max};\\
&||v||_M \leq 2||v||_E.
\end{aligned}$$

So all these three metrics induce the same topology.

3 Ordering

Reading

[Debreu], Ch.1.4, p.7-9

The set of real numbers R is *ordered*: $x > y$ if the difference $x - y$ is positive.

But what about the ordering on the plane R^2? Well, we can say that the vector $(5,7) \in R^2$ is "bigger" than the vector $(1,2)$, but how can we compare the vectors $(1,2)$ and $(2,1)$?

Unfortunately (or fortunately) we do not have a *canonical* ordering on R^n for $n > 1$. It is possible to consider various notions of ordering suitable for each particular problem.

3.1 Preorderings and Orderings

A *partial preordering* on a set X is a relation $x \geq y$ which satisfies the following conditions

(i) *reflexivity*: $\forall x \in X,\ x \geq x$;

(ii) *transitivity*: $\forall x, y, z \in X,\ x \geq y,\ y \geq z\ \Rightarrow x \geq z$.

Note that each equivalence relation is automatically a preordering: it is reflexive and transitive (and additionally symmetric).

A preordering is called *total* if additionally it satisfies

(iii) *totality*: $\forall x, y \in X$ either $x \geq y$ or $y \geq x$.

A (total) preordering is called (total) *ordering* if it satisfies additionally the condition

(iv) *antisymmetricity*: $x \geq y,\quad y \geq x \quad \Rightarrow x = y$.

The above defined four notions: partial (total) (pre)ordering can be observed by the following diagram

$$
\begin{array}{ccccc}
 & & i, ii & & \\
 & & partial\ preordering & & \\
i, ii, iii & \nearrow & & \nwarrow & i, ii, iv \\
total\ preordering & & & & partial\ ordering \\
 & \nwarrow & & \nearrow & \\
 & & total\ ordering & & \\
 & & i, ii, iii, iv & &
\end{array}
$$

where an arrow indicates implication.

What is advantage of totality vs partiality? Each two elements are comparable: for $\forall x, y$ either $x \geq y$ or $y \geq x$.

What is advantage of ordering vs preordering? There is no indifference: $x \geq y$ and $y \geq x$ implies $x = y$.

Partially ordered sets are called *posets*.

One more notion: A directed set (directed preorder) is defined as a preordered set with the additional property that every pair of elements has an upper bound, in other words, for any a and b there must exist a c with $a \leq c$ and $b \leq c$. Any totaly preordered set is directed (why?).

3.1.1 Preorderings on R^2

1. Norm preordering:

$$v = (x, y) \geq v' = (x', y') \quad if \quad ||v|| = \sqrt{x^2 + y^2} \geq ||v'|| = \sqrt{x'^2 + y'^2}.$$

This is a total preordering. Why *"total"*? Why *"pre"*?

2. Product ordering: $(a, b) \leq (c, d)$ if $a \leq c$ and $b \leq d$. *This is a partial ordering.* Why *"partial"*? Why *"ordering"*?

3. Lexicographical ordering: $(a, b) \leq (c, d)$ if and only if $a < c$, but if $a = c$ then $b \leq d$. *This is a total ordering.* Why *"total"*? Why *"ordering"*?

3.1.2 Other Examples

1. The set of natural numbers N of course is ordered by the usual ordering "$m \geq n$ if $m - n$ is nonnegative".

2. There exists on N also the following *partial ordering* "$m \geq n$ if m is divisible by n" ("n divides m", notation $n|b$). For example $6 \geq 2,\ 6 \geq 3$, but 6 and 4 are not comparable (thus "partial").

3. Let S be a set, the set of all its subsets is denoted by 2^S. Let us introduce on 2^S the following relation: for arbitrary subsets $A \subseteq S,\ B \subseteq S$ we say $B \leq A$ if $B \subseteq A$. This is a partial ordering. Why *"partial"*?

4. Consider on R^3 the following relation:

$$(x, y, z) \geq (a, b, c)$$

if $x \geq a$ and $y \geq b$. This is partial preordering. Why *"partial"* and why *"pre"*?

3.1.3 Indifference Relation

Each preordering $\geq$ defines *indifference relation*:

$$x \sim y \ \ if \ \ x \geq y \ \ and \ \ y \geq x.$$

Theorem 3 *The relation $x \sim y$ is an* equivalence relation.

Proof. We show that the relation $x \sim y$ satisfies the axioms of equivalence:
(1) Reflexivity $x \sim x$;
(2) Symmetricity $x \sim y \ \Rightarrow \ y \sim x$;
(3) Transitivity $x \sim y, \ y \sim z \ \Rightarrow \ x \sim z$.
Indeed,
(1) Since of (i) $x \geq x$, thus $x \sim x$.
(2) Suppose $x \sim y$, then $x \geq y$ and $y \geq x$, thus $y \sim x$.
(3) Suppose $x \sim y$, this implies $x \geq y$ and $y \geq x$, and suppose $y \sim z$, this implies $y \geq z$ and $z \geq y$. Then since of (ii) we have

$$x \geq y, \ y \geq z \ \Rightarrow x \geq z$$

and

$$z \geq y, \ y \geq x \ \Rightarrow z \geq x,$$

thus $x \sim z$.

The *indifference set* (or orbit, or equivalence class) of an element $x \in X$ is defined as

$$I(x) = \{y \in X, \ x \sim y\}.$$

Since indifference relation is an equivalence, the indifference sets form a *partition* of X.

Examples.

1. If the starting relation $\geq$ is an *ordering* then $x \sim y$ if and only if $x = y$. So the indifference sets are one point sets: $I(x) = \{x\}$.

2. For the norm preordering indifference sets are spheres centered at the origin: $I(x) = S_{|x|}(O)$.

3.1.4 Strict Preordering

Each preoredering $\geq$ induces the *strict preordering* $>$ defined by: $x > y$ if $x \geq y$ but not $y \geq x$. Equivalently $x > y$ if $x \geq y$ and not $x \sim y$.

If a starting preoredering $\geq$ is an ordering, then $x > y$ iff $x \geq y$ and $x \neq y$.

3.2 Maximal and Greatest

Let S be a partially preordered set.

An element $x \in S$ is called *maximal* if there exists no $y \in S$ such that $y > x$.

An element $x \in S$ is called *minimal* if there exists no $y \in S$ such that $y < x$.

An element $x \in S$ is called *greatest* if $x \geq y$ for all $y \in S$.

An element $x \in S$ is called *least* if $x \leq y$ for all $y \in S$.

Theorem 4 *If S is an ordered set, then a greatest (least) element is unique.*

Proof. Suppose x and x' are greatest elements. Then $x \geq x'$ since x is greatest, and $x' \geq x$ since x' is greatest. Thus, since S is an ordered set, we get $x = x'$.

Theorem 5 *A greatest element is maximal.*

Proof. Suppose $x \in S$ is greatest, that is $x \geq y$ for all $y \in S$, but not maximal, that is $\exists y \;\; s.t. \;\; y > x$. By definition of $>$ this means that $y \geq x$ but not $x \geq y$. The last contradicts to $x \geq y$.

Theorem 6 *If a preordering is total, then a maximal element is greatest.*

Proof. Suppose $x \in S$ is maximal, that is there exists no $y \in S$ such that $y > x$. Let us show that x is greatest, that is $x \geq z$ for each z. Suppose there exists z such that $x \geq z$ is wrong. This means that $x \neq z$ and, since of totality $z \geq x$, thus $z > x$ which contradicts to maximality.

So when the preordering is *total*, there is no difference between maximal and greatest. Similarly for minimal and least.

Examples.

1. The set $\{1, 2, 3, 4, 5, 6\}$ ordered by the partial ordering "divisible by" has three maximal elements $4, 5, 6$, no greatest element, one minimal element 1 and one least element 1:

$$\begin{array}{ccc} & 6 & \\ & \uparrow & \\ 4 & | & \\ \uparrow & | & \\ 2 & 3 & 5 \\ \nwarrow & \uparrow & \nearrow \\ & 1 & \end{array}$$

2. Let S be the set of all 8 vertices of a cube, in coordinates

$$S = \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (1, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}.$$

Hamming ordering on S is defined as follows: $v \geq w$ if v contains more 1-s than w.

The least (minimal) element here is $(0, 0, 0)$ and greatest (maximal) element is $(1, 1, 1)$.

3. In the partially ordered set 2^S the least (minimal) element his the empty set and greatest (maximal) element is S.

3.3 Utility Function

A real valued function $U : X \to R$ is said to *represent* a preordering $\geq$ if

$$\forall x, y \in X,\ x \geq y \Leftrightarrow\ U(x) \geq U(y).$$

In economics a preordering $\geq$ is called *preference preordering* and a representing function U is called *utility function*.

The norm preordering:

$$v = (x, y) \geq v' = (x', y') \quad if \quad ||v|| = \sqrt{x^2 + y^2} \geq ||v'|| = \sqrt{x'^2 + y'^2}.$$

is represented by the utility function

$$U(x, y) = \sqrt{x^2 + y^2},$$

or by the function $2U(x, y) = 2\sqrt{x^2 + y^2}$, or by $U^2(x, y) = x^2 + y^2$, etc. These functions differ but all of them have the same indifference sets.

3.3.1 Equivalent Utility Functions

A given preordering can be represented by various functions. Two utility functions are called *equivalent* if they have same indifferent sets.

A *monotonic transformation* of an utility function U is the composition $g \circ U(x) = g(U(x))$ where g is a strictly monotonic function.

It is clear that an utility function U and any its monotonic transformation $g \circ U$ represent the same or opposite preordering, so they are equivalent.

Example. The functions

$$3xy + 2, \quad (xy)^3, \quad (xy)^3 + xy, \quad e^{xy}, \quad ln\ x + ln\ y$$

all are monotonic transformations of the function xy: the corresponding monotonic transformations are respectively

$$3z + 2, \quad z^3, \quad z^3 + z, \quad e^z, \quad ln\ z.$$

Exercises

1. Draw the balls $\bar{B}_1(0,0)$, $\bar{B}_1(1,0)$, $\bar{B}_2(1,0)$ and $\bar{B}_3(1,0)$ for each of the following metrics

Euclidian metric $d_E(x,y) = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$.
Manhattan metric $d_M(x,y) = |x_1-y_1|+|x_2-y_2|$.
Maximum metric $d_{max}(x,y) = max(|x_1-y_1|,|x_2-y_2|)$.
British Rail metric $d_{BR}(x,y) = ||x||+||y||$.
Discrete metric $d_{disc}(x,y) = 0 \ \ if \ \ x=y \ \ and \ \ d(x,y)=1 \ \ if \ \ x\neq y$

Hint. $\bar{B}_3^{BR}(1,0) = \bar{B}_2^{E}(0,0)$, $\bar{B}_1^{disc}(1,0) = R^2$, $B_1^{disc}(1,0) = \{(0,0\}$, $S_1^{disc}(1,0) = R^2 - \{(0,0\}$.

2. Show that the discrete metric d_{disc} does not induce a norm.

Hint. Take $v \neq O$ and compare $d(2\cdot v, O)$ and $2\cdot d(v,O)$.

3. For a vector $v=(x,y)\in R^2$ let us define $||v||_{min} = min(|x|,|y|)$. Is this a norm?

Hint. No: take the nonzero vector $u=(0,1)$ and calculate $||u||_{min}$.

4. Does d_{BR} induce a *norm* $||x||_{BR} = d_{BR}(x,O)$?

5. Give examples of (a) partial preordering, (b) total preordering, (c) partial ordering, (d) total ordering. OK, the example (d) will automatically serve as an example of (a), (b), (c), but you undestand what I mean.

6. Is the relation defined on R^2 by

$$(x,y)\geq(x',y') \Leftrightarrow x\geq x',\ y\geq y'$$

a (a) partial preordering? (b) total preordering? (c) partial ordering? (d) total ordering?

7. What can you say about indifference sets of an ordering?

8. Draw indifference sets $I(0,0,0)$, $I(1,1,1)$, $I(2,2,2)$ in R^3 for the preordering

$$(x,y,z)\geq(x',y',z') \Leftrightarrow ||(x,y,z)||_E \geq ||(x',y',z')||_E$$

.

Hint. Particularly $I(1,1,1) = S_{\sqrt{3}}^{E}(0,0)$.

9. Draw indifference sets $I(0,0),\ I(1,1),\ I(2,2)$ in R^2 for the preordering defined by Manhattan norm

$$(x,y) \geq (x',y') \Leftrightarrow ||(x,y)||_M \geq ||(x',y')||_M$$

.

Hint. Particularly $I(1,1) = S_2^M(0,0)$.

10. Draw indifference sets $I(0,0),\ I(1,1),\ I(2,2)$ in R^2 for the preordering defined by maximum norm

$$(x,y) \geq (x',y') \Leftrightarrow ||(x,y)||_{max} \geq ||(x',y')||_{max}$$

.

Hint. Particularly $I(1,1) = S_1^{max}(0,0)$.

11. Suppose a set S has two greatest elements x and x'. Show that $x \sim x'$.

12. Find (draw) two sets

$$S = \{(x,y) \in R^2,\ (x,y) \leq (1,1)\}, \quad T = \{(x,y) \in R^2,\ (1,1) \leq (x,y)\}$$

where $\leq$ assumes the *product ordering* of R^2: $(x,y) \leq (x',y')$ if $x \leq x',\ y \leq y'$.

Hint. Notice that $T = \{(x,y) \in R^2,\ 1 \leq x,\ \ 1 \leq y\}$.

13. Find (draw) two sets

$$S = \{(x,y) \in R^2,\ (x,y) \leq (1,1)\}, \quad T = \{(x,y) \in R^2,\ (1,1) \leq (x,y)\}$$

where $\leq$ assumes the *lexicographical ordering* of R^2.

14. Find maximal, minimal, greatest, least elements of the set $S = \{2,3,4,5,6,12\}$ with respect of the ordering "$a \leq b$ if $a|b$" (a divides b).

15. Find maximal, minimal, greatest, least elements of the set $S = \{(x,y),\ 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$ with respect to the product ordering of R^2.

Hint. Use the pictures of "more than" set T and "less than" set S from the exercise 12.

16. Find maximal, minimal, greatest, least elements of the set $S = \{(x,y),\ x^2 + y^2 \leq 1\}$ with respect to the product ordering of R^2.

17. Find maximal, minimal, greatest, least elements of the set $S = \{(x,y),\ x^2 + y^2 \leq 1, x \geq 0,\ y \geq 0\}$ with respect to the product ordering of R^2.

18. For each of the functions

$$(a)\ \ 3xy+2, \quad (b)\ \ (xy)^2, \quad (c)\ \ (xy)^3+xy, \quad (d)\ \ e^{xy}, \quad (e)\ \ ln\ x+ln\ y$$

(which are equivalent to xy) identify the level sets which correspond to the level sets $xy = 1$ and $xy = 4$. For example to the level set $xy = 1$ corresponds the level set $3xy + 2 = 5$ for the function (a).

19. Which of the following functions are equivalent to xy? For those which are, what monotonic transformation provides this equivalence?
(a) $7x^2y^2 + 2$, (b) $ln\ x + ln\ y + 1$, (c) x^2y, (d) $x^{\frac{1}{3}}y^{\frac{1}{3}}$.

Homework
Exercises 3, 10, 13, 17, 19.

Short Summary
Metric and Norm

Axioms

	Metric		*Norm*
a	$d(x,y) \geq 0$	i	$\|\|v\|\| \geq 0$
	$d(x,y) = 0 \;\Leftrightarrow\; x = y;$		$\|\|v\|\| = 0 \;\Leftrightarrow\; v = 0;$
b	$d(x,y) = d(y,x);$	ii	$\|\|r \cdot v\|\| = \|r\| \cdot \|\|v\|\|;$
c	$d(x,y) + d(y,z) \geq d(x,z);$	iii	$\|\|v + w\|\| \leq \|\|v\|\| + \|\|w\|\|.$

From Norm to Metric: $d(x,y) = ||x - y||$.
From Metric to Norm: $||v|| := d(v,O)$ if $d(x,y)$ additionally satisfies $d(u,v) = d(u+w, v+w)$ and $d(ku,kv) = |k| \cdot d(u,v)$.
Examples of Metrics.

1. *Euclidian metric* $d_E(x,y) = \sqrt{(x_1 - y_1)^2 + \ ... \ + (x_n - y_n)^2}$.
2. *Manhattan metric* (or Taxi Cab metric) $d_M(x,y) = |x_1 - y_1| + \ ... \ + |x_n - y_n|$.
3. *Maximum metric* $d_{max}(x,y) = max(|x_1 - y_1|, \ ... \ , |x_n - y_n|)$.
4. *Discrete metric* $d_{disc}(x,y) = 0 \;\; if \;\; x = y \;\; and \;\; d_{disc}(x,y) = 1 \;\; if \;\; x \neq y$
5. *British Rail metric* $d_{BR}(x,y) = ||x|| + ||y||$ if $x \neq y$ and $d_{BR}(x,x) = 0$.

Examples of Norms

1. $||x||_{a_1,\dots,a_n} = \sqrt{a_1 \cdot x_1^2 + \ ... \ + a_n \cdot x_n^2}$.
   If each $a_i = 1$ this norm coincides with *Euclidian norm*

$$||x||_E = \sqrt{x_1^2 + \ ... \ + x_n^2}.$$

2. Manhattan norm $||x||_M = |x_1| + \ ... \ + |x_n|$.
3. Maximum norm $||x||_{max} = max(|x_1|, \ ... \ , |x_n|)$.
4. The k-norm $||x||_k = \sqrt[k]{|x_1|^k + \ ... \ + |x_n|^k}$. Particularly $||x||_E = ||x||_2$, $||x||_M = ||x||_1$ and in some sense $||x||_{max} = ||x||_\infty$.

Short Summary
Orderings

Axioms

(i) *reflexivity*: $\forall x \in X,\ x \geq x$;
(ii) *transitivity*: $\forall x, y, z \in X,\ x \geq y,\ y \geq z\ \Rightarrow x \geq z$.
(iii) *totality*: $\forall x, y \in X$ either $x \geq y$ or $y \geq x$.
(iv) *antisymmetricity*: $x \geq y,\ \ y \geq x\ \ \Rightarrow x = y$.

$$
\begin{array}{ccccc}
 & & i, ii & & \\
 & & partial\ preordering & & \\
i, ii, iii & \nearrow & & \nwarrow & i, ii, iv \\
total\ preordering & & & & partial\ ordering \\
 & \nwarrow & & \nearrow & \\
 & & total\ ordering & & \\
 & & i, ii, iii, iv & &
\end{array}
$$

Examples

1. Norm total preordering on R^2:

$$v = (x, y) \geq v' = (x', y') \quad if \quad ||v|| = \sqrt{x^2 + y^2} \geq ||v'|| = \sqrt{x'^2 + y'^2}.$$

2. Product partial ordering on R^2: $(a, b) \leq (c, d)$ if $a \leq c$ and $b \leq d$.
3. Lexicographical total ordering on R^2: $(a, b) \leq (c, d)$ if and only if $a < c$, but if $a = c$ then $b \leq d$.
4. Standard total ordering on N: "$m \geq n$ if $m - n$ is nonnegative".
5. Divisibility partial ordering on N: $m \geq n$ if $n|b$.
6. Standard partial ordering on 2^S: $B \leq A$ if $B \subseteq A$.
7. Partial preordering on R^3: $(x, y, z) \geq (a, b, c)$ if $x \geq a$ and $y \geq b$.

Indifference Relation: $x \sim y\ \ if\ \ x \geq y\ \ and\ \ y \geq x$. The indifference set (orbit) of x: $I(x) = \{y \in X,\ x \sim y\}$. For an ordering $x \sim y$ iff $x = y$ and $I(x) = \{x\}$.

Strict Preordering: $x > y$ if $x \geq y$ but not $y \geq x$.

Greatest and Maximal.
$x \in S$ is **maximal** if there exists no $y \in S$ s.t. $y > x$.
$x \in S$ is **greatest** if $x \geq y$ for all $y \in S$.
Greatest is always maximal.
If a preordering is total, then maximal is greatest.
If S is an ordered set, then a greatest element is unique.

A **utility** function $f : S \to R$ determines a total (pre) ordering $x \leq y$ if $f(x) \leq f(y)$.