ISET MATH IV Midterm

Problem 1.

(a) Suppose the sum of n real numbers is k. Calculate the minimal value of the sum of squares of these numbers.

(b) Suppose the sum of squares of n real numbers is k. Calculate the maximal value of the sum of these numbers.

(c) Suppose the sum of squares of n real numbers is k. Calculate the minimal value of the sum of these numbers.

Solution.

(a)
$f(x_1, ..., x_n) = x_1^2 + ... + x_n^2, \quad g(x_1, ..., x_n) = x_1 + ... + x_n = k$
$L(x_1, ..., x_n, \lambda) = x_1^2 + ... + x_n^2 - \lambda(x_1 + ... + x_n - k)$

$$\left\{ \begin{array}{l} L_{x_1} = 2x_1 - \lambda = 0 \\ ... \\ L_{x_1} = 2x_n - \lambda = 0 \\ x_1 + ... + x_n = k \end{array} \right|$$

$x_i = \frac{\lambda}{2}$, $n \cdot \frac{\lambda}{2} = k$, $\lambda = \frac{2k}{n}$, $x_i = \frac{k}{n}$, $x_1^2 + ... + x_n^2 = n \cdot \frac{k^2}{n^2} = \frac{k^2}{n}$.

(b)
$f(x_1, ..., x_n) = x_1 + ... + x_n, \quad g(x_1, ..., x_n) = x_1^2 + ... + x_n^2 = k$
$L(x_1, ..., x_n, \lambda) = x_1 + ... + x_n - \lambda(x_1^2 + ... + x_n^2 - k)$

$$\left\{ \begin{array}{l} L_{x_1} = 1 - 2\lambda x_1 = 0 \\ ... \\ L_{x_1} = 1 - 2\lambda x_n = 0 \\ x_1^2 + ... + x_n^2 = k \end{array} \right|$$

$x_i = \frac{1}{2\lambda}$, $n \cdot \frac{1}{4\lambda^2} = k$, $\lambda = \pm\sqrt{\frac{n}{4k}}$, $x_i = \sqrt{\frac{k}{n}}$, $x_1 + ... + x_n = \sqrt{nk}$.

(c)
$x_i = -\sqrt{\frac{k}{n}}$, $x_1 + ... + x_n = -\sqrt{nk}$.

Problem 2.
(a) Use Lagrange multipliers to find min and max of the function $f(x, y) = x^2 + y^2$ subject of $g(x, y) = x^2 + 2y^2 = 4$.

(b) Suppose that the constraint in (a) is changed to 4.01. Use the Lagrange multiplier to estimate new minimal and maximal values of f.

(c) Use Lagrange multipliers to find min and max of the function $f(x, y) = x^2 + y^2$ subject of $g(x, y) = x^2 + 2y^2 \leq 4$.

(d) Suppose that the constraint in (c) is changed to 4.01. Use the Lagrange multiplier to estimate new minimal and maximal values of f.

Solution

(a) min: $(x = 0, y = \pm\sqrt{2}, \lambda = \frac{1}{2})$, $f(x = 0, y = \pm\sqrt{2}) = 2$; max: $(x = \pm 2, y = 0, \lambda = 1)$, $f(x = \pm 2, y = 0) = 4$.

(b) If we change the constraint 4 to 4.01 the minimal value will change by $\lambda \cdot 0.01 = \frac{1}{2} \cdot 0.01 = 0.005$, and the maximal value will change approximately by $\lambda \cdot 0.01 = 1 \cdot 0.01 = 0.01$. So the new min is 2.005 and the new max is 4.01.

(c) min: $(x = 0, y = 0, \lambda = 0)$, $f((x = 0, y = 0) = 0$; max: $(x = \pm 2, y = 0, \lambda = 1)$, $f(x = \pm 2, y = 0) = 4$.

(d) If we change the constraint 4 to 4.01 the minimal value will not change: $\lambda \cdot 0.01 = 0 \cdot 0.01 = 0$, and the maximal value will change approximately by $\lambda \cdot 0.01 = 1 \cdot 0.01 = 0.01$. So the new min is 0 and the new max is 4.01.

Problem 3.
Consider the problem

$$min \quad f(x,y) = (x-2)^2 + (y-2)^2 \quad s.t. \quad x + y = k.$$

(a) For which values of k is the constraint nonbinding?

(b) For which values of k is the constraint binding?

Solution.

$L(x,y) = (x-2)^2 + (y-2)^2 - \lambda(x+y-k)$
KKT

$$\begin{aligned}
&2x - 4 - \lambda = 0 \\
&2y - 4 - \lambda = 0 \\
&\lambda(x + y - k) = 0 \\
&\lambda \leq 0, \quad x + y \leq k.
\end{aligned}$$

1. $\lambda = 0 \;\Rightarrow\; x = 2,\; y = 2,\; 2 + 2 - k \leq 0,\; 4 \geq k.$

2. $x + y - k = 0 \;\Rightarrow\; x = \frac{k}{2},\; y = \frac{k}{2},\; \lambda = k - 4 \leq 0,\; k \leq 4.$

(a) $k \leq 4.$

(b) $k \geq 4.$

Problem 4. Solve he following problems, do not forget the second order conditions, indicate the optimal values too.

(a) $min \quad (x-8)^2 + y^2 \quad s.t. \quad x + y \leq 6.$

(b) $min \quad (x-8)^2 + y^2 \quad s.t. \quad x + y \leq 6, \quad y \geq 0.$

(c) $min \quad (x-8)^2 + y^2 \quad s.t. \quad x + y \leq 6, \quad x \leq 0.$

Solution

(a) $x = 7, \; y = -1, \lambda = 2, \; f = 2$

(b) $x = 6, \; y = 0, \lambda_1 = -4, \; \lambda_2 = -4, \; f = 4$

(c) $x = 0, \; y = 0, \lambda_1 = -16, \; \lambda_2 = -4, \; f = 64$