ISET MATH Final IV

Problem 1. Assume $F : R^2 \to R$ is homogeneous of degree 2 and $f : R \to R$ is homogenous of degree 3. Prove that the composition $f \circ F$ homogenous and indicate its degree of homogeneity.

Solution

$(f \circ F)(tx, ty) = f(F(tx, ty)) = f(t^2 F(x, y)) = t^6 f(F(x, y)) = t^6 (f \circ F)(x, y).$

Problem 2. Assume $F : R^2 \to R$ is homogeneous of degree n, $f : R \to R$ is homogenous of degree p, and $g : R \to R$ is homogenous of degree q. Let's consider the function $G(x, y)$ given by $G(x, y) = F(f(x), g(x))$. Investigate the homogeneity of G.

Solution

If $p = q$ G is homogeneous of degree np:

$$G(tx, ty) = \\ F(f(tx), g(ty)) = F(t^p f(x), t^p g(x)) = t^{pn} F(f(x), g(x)) = \\ t^{pn} G(x, y).$$

Problem 3. Suppose $F(x, y)$ is homogenous of degree k. Consider the function $G(x, y) = MRS(F(x, y))$ (marginal rate of substitution). Show that G is homogenous and indicate its degree.

Solution

$deg\ G = 0$:

$$G(tx, ty) = MRS\ F(tx, ty) = -\frac{F_x(tx,ty)}{F_y(tx,ty)} = -\frac{t^{k-1} F_x(x,y)}{t^{k-1} F_y(x,y)} = -\frac{F_x(x,y)}{F_y(x,y)} = G(x, y).$$

Problem 4. It is known that $F(x, y)$ is homogeneous of degree 2. Besides, the restriction of F for fixed value $y = 2$ is a one variable function given by $F(x, 2) = x^3$.

(a) Calculate $F(3, 3)$.

(b) Calculate the slope of the tangent line to the level curve of F which passes trough the point $(3, 3)$ at this point.

(c) Calculate the slope of the tangent line to the level curve of F which passes trough the point $(2, 2)$ at this point.

Solution

$$F(x, y) = F(\tfrac{y}{2} \cdot \tfrac{2x}{y}, \tfrac{y}{2} \cdot 2) = \tfrac{y^2}{4} F(\tfrac{2x}{y}, 2) = \tfrac{y^2}{4} \cdot \tfrac{8x^3}{y^3} = \tfrac{2x^3}{y}.$$

Particularly $F(3, 3) = 18$.

$MRS\ F(x, y) = -\frac{F_x(x,y)}{F_y(x,y)} = -\frac{3y}{x}$.

Particularly $MRS\ F(3, 3) = 3 = MRS\ F(2, 2)$.

Problem 5. About monotonic (increasing) transformations of concave functions. Justify your answer with proofs and/or counterexamples.

(a) Show that concave monotonic transformation of a concave function is concave.

(b) Show that a monotonic transformation of a concave function is not necessarily concave.

(c) What can you say about a monotonic transformation of a concave function?

Solution

(a) Let $f : R^n \to R$ be a concave function and $g : R \to R$ be concave and increasing, then

$$(g \circ f)(1-t)x + ty) =$$
$$g(f((1-t)x + ty)) \geq g((1-t)f(x) + tf(y)) \geq (1-t)g(f(x)) + tg(f(y)) =$$
$$(1-t)(g \circ f)(x)) + t(g \circ f)(y)),$$

here the first inequality holds since f is concave and g is increasing, and the second inequality holds since g is concave.

(b) No, take $f(x) = x$ (cocncave) and $g(x) = x^3$ (increasing). The composition $(g \circ f)(x) = x^3$ is not concave.

(c) A monotonic transformation gf of a concave function f is itself quasiconcave: Take any $K \in R$. Since g is monotonic, there exists $K' \in R$ such that $K = g(K')$. Then

$$U_K(gf) = \{x,\ gf(x) \geq K\} = \{x,\ gf \geq g(K')\} = \{x,\ f(x) \geq K'\} = U_{K'}(f)$$

is a convex set.

Problem 6. Can a Cobb-Douglas function

$$f(x, y) = cx^a y^b, \quad a, b, c > 0, \quad x > 0,\ y > 0$$

(a) ever be concave?
(b) ever be convex?
Justify your answer with proofs and/or counterexamples.

Solution

Recall the principal minors of Hessian:

$$M_1 = a(a-1)cx^{a-2}y^b,\ \ M_1' = b(b-1)cx^a y^{b-2},\ \ M_2 = abcx^{2a-2}y^{2b-2}(1-(a+b)).$$

(b) f is convex if and only if $M_1 \geq 0, \quad M_1' \geq 0, \quad M_2 \geq 0$. Answer: No

$$\left.\begin{array}{l} M_1 \geq 0 \ \Rightarrow \ a \geq 1 \\ M_1' \geq 0 \ \Rightarrow \ b \geq 1 \end{array}\right\} a + b \geq 2 \ \Rightarrow \ 1 - (a+b) < 0 \ \Rightarrow \ M_2 < 0.$$

Problem 7. Let $A = (1, 1),\ B = (3, 3) \in R^2$. Draw the "segment" $[A, B] = \{(x, y) \in R^2,\ \ A \leq (x, y) \leq B\}$ where $\leq$ denotes

(a) Norm preordering;

(b) Product ordering;

(c) Lexicographical ordering.

Problem 8. For a vector $v = (x, y) \in R^2$ let us define $||v||_{min} = min(|x|, |y|)$. Is this a norm? Justify your answer with proofs and/or counterexamples.

Solution

No: Take the nonzero vector $u = (0, 1)$, then $||u||_{min} = 0$, this contradicts to axiom $||u|| \ \Leftrightarrow\ u = 0$.

Problem 9. Give an example of a distance (metrics) on R^2 which determines a norm by $||a|| := d(a, 0)$. Justify your answer with proofs and/or counterexamples.

Solution

The Euclidian metric d_E, the manhattan metric d_M, and the maximum metric d_{max} satisfy needed additional conditions. Let us check just one:

$$d_M(k(a,b),)k(c,d)) = d_M((ka,kb),)(kc,kd)) = |ka-kc|+|kb-kd| =$$
$$|k(a-c)|+|k(b-d)| = |k||a-c|+|k||b-d| =$$
$$|k|(|a-c|+|b-d| = |k|d_M((a,b),)(c,d)).$$

Problem 10. Give an example of a distance on R^2 which does not determine a norm by $||a|| := d(a, 0)$. Justify your answer with proofs and/or counterexamples.

Solution

d_{disc} does not satisfy homogeneity $d(ku, kv) = |k| \cdot d(u, v)$: Indeed, take $v \neq O$, then $||2 \cdot v|| = d(2 \cdot v, 2 \cdot O) = 1 \neq 2 = 2 \cdot d(v, O)$.

Problem 11. Give TWO examples of ordered sets which have minimal but not least elements

(a) one example in R^2 where the product ordering is assumed;

(b) another example in N where the divisibility ordering is assumed.

Solution

(a) Say $S_1^E((0,0))$ or $\bar{B}_1^E((0,0))$ with respect to product ordering, (b) the set $\{2,3,6\}$ with respect to divisibility ordering.

Problem 12. Solve the *limited growth* equation $y' = k(M-y),\ k > 0$ using two methods: (a) separation of variables, (b) Integrating factor.

Solution

(a) $\int \frac{dy}{M-y} = \int kdt, \quad \int \frac{d(M-y)}{M-y} = \int kdt, \quad ln(M-y) = kt + C, \quad y = M + Ce^{kt}.$

(b) $y' + ky = kM, \quad I = e^{\int kdt} = e^{kt}, \quad y = \frac{\int e^{kt} kM dt}{e^{kt}} = \frac{Me^{kt}+C}{e^{kx}} = M + e^{kx}.$

Problem 13. Give an example of an equation that can be solved using the integrating factor but can not be solved using separation of variables. Solve this equation.

Problem 14. Give an example of an equation that can be solved using separation of variables but can not be solved using the integrating factor. Solve this equation.

Problem 15. Assume that the demand and supply functions are given by $Q_d(p) = 10 - p$ and $Q_s(p) = 2 + p$ respectively. Assume also that the prise changes in time, i.e. $p = p(t)$ and at the starting moment $t = 0$ it is $p(0) = 2$. Besides, assume that the rate of change of prise satisfies $p\,'(t) = Q_d - Q_s$.

(a) Calculate the equilibrium prise.

(b) Show that when $t \to \infty$ the prise $p(t)$ tends to equilibrium prise.

(c) Find the time t when the difference between the equilibrium price and the price $p(t)$ becomes 0.1.

Solution

$p'(t) = 8-2p(t), \quad \frac{dp}{8-2p} = dt, \quad \int \frac{dp}{8-2p} = \int dt, \quad -\frac{1}{2}\int \frac{d(8-2p}{8-2p} = \int dt, \quad \frac{1}{2} ln(8-2p) = t+K, \quad 8-2p = Ce^{-2t}, \quad p = 4-Ce^{2t}; \quad p(0) = 2, \quad C = 2; p(t) = 4-2e^{2t}.$

$lim_{t\to\infty}(4-2e^{-2t}) = 4.\ 4 - e^{-2t} = 4 - 0.1, \quad e^{-2t} = \frac{1}{10}, \quad t := \frac{1}{2} ln\ 20.$