# 1 Calculus of Several Variables

Reading: [Simon], Chapter 14, p. 300-312.

## 1.1 Partial Derivatives

Let $f : R^n \to R$. Then for each $x_i$ at each point $x^0 = (x_1^0, ..., x_n^0)$ the $i$th *partial derivative* is defined as

$$\frac{\partial f}{\partial x_i}(x_1^0, ..., x_n^0) = \lim_{h \to 0} \frac{f(x_1^0, ..., x_i^0 + h, ..., x_n^0) - f(x_1^0, ..., x_i^0, ..., x_n^0)}{h}.$$

Notice that here only $i$th variable is changing, the others are treated as constants. Thus the partial derivative $\frac{\partial f}{\partial x_i}(x_1^0, ..., x_n^0)$ is the ordinary derivative of the function $f(x_1^0, ..., x_{i-1}^0, x, x_{i+1}^0, ..., x_n^0)$ of one variable $x$ at the point $x = x_i^0$.

**Examples.** 1. Find partial derivatives for $f(x, y) = 3x^2y^2 + 4xy^3 + 7y$.

For $\frac{\partial f}{\partial x}$ only $x$ is considered as a *variable* and $y$ is treated as a *constant*:

$$\frac{\partial f}{\partial x} = 3 \cdot 2x \cdot y^2 + 4 \cdot 1 \cdot y^3 + 0 = 6xy^2 + 4y^3.$$

For $\frac{\partial f}{\partial y}$ only $y$ is considered as a variable and $x$ is treated as a constant:

$$\frac{\partial f}{\partial y} = 3x^2 \cdot 2y + 4x \cdot 3y^2 + 7 \cdot 1 = 6x^2y + 12xy^2 + 7.$$

2. Find partial derivatives for $f(x, y) = \sqrt{xy}$.

$$\frac{\partial f}{\partial x} = \frac{1}{2\sqrt{xy}} \cdot \frac{\partial\ (xy)}{\partial x} = \frac{y}{2\sqrt{xy}}.$$

$$\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{xy}} \cdot \frac{\partial\ (xy)}{\partial y} = \frac{x}{2\sqrt{xy}}.$$

## 1.2 Competitive and complementary products

If a decrease in demand for one product results in an increase in demand for another product, the two products are said to be *competitive*, or substitute, products (desktop computers and laptop computers are examples of competitive, or substitute, products).

If a decrease in demand for one product results in a decrease in demand for another product, the two products are said to be *complementary* products (computers and printers are examples of complementary products).

Partial derivatives can be used to test whether two products are competitive or complementary or neither. We consider demand functions for two products where the demand for either depends on the prices for both.

Suppose $Q_1 = Q_1(P_1, P_2)$ represents the demand for good 1 in terms of the price $P_1$ of good 1 and price $P_2$ of good 2.

Similarly, suppose $Q_2 = Q_2(P_1, P_2)$ represents the demand for good 2 in terms of the price $P_1$ of good 1 and price $P_2$ of good 2.

The partial derivative $\frac{\partial Q_i}{\partial P_j}$ is the rate of change of demand of good i with respect the price $P_j$, here $i, j = 1, 2$.

In general

$$\frac{\partial Q_1}{\partial P_1} < 0, \quad \frac{\partial Q_2}{\partial P_2} < 0,$$

(why?).

**Test for competitive an complementary products**

$$\begin{array}{cccc}
\frac{\partial Q_1}{\partial P_2} > 0 & and & \frac{\partial Q_2}{\partial P_1} > 0 & competitive \\
\frac{\partial Q_1}{\partial P_2} < 0 & and & \frac{\partial Q_2}{\partial P_1} < 0 & complementary \\
\frac{\partial Q_1}{\partial P_2} \geq 0 & and & \frac{\partial Q_2}{\partial P_1} \leq 0 & neither \\
\frac{\partial Q_1}{\partial P_2} \leq 0 & and & \frac{\partial Q_2}{\partial P_1} \geq 0 & neither
\end{array}$$

**Example.** The weekly demand equations for the sale of butter and margarine in a supermarket are

$$\begin{array}{l}
Q_1(P_1, P_2) = 8,000 - 0.09P_1^2 + 0.08P_2^2 \quad (Butter) \\
Q_2(P_1, P_2) = 15,000 + 0.04P_1^2 - 0.03P_2^2 \quad (Margarine),
\end{array}$$

determine whether the indicated products are competitive, complementary, or neither.

**Solution.** $\frac{\partial Q_1}{\partial P_2} = 0.16P_2 > 0, \quad \frac{\partial Q_2}{\partial P_1} = 0.08P_1 > 0$, so competitive.

## 1.3 Elasticity

### 1.3.1 One variable case

For a demand function $Q = Q(P)$ the quantity $\frac{dQ}{dP}$ measures the price sensitivity of $Q$.

More subtle tool to measure the price sensitivity of $Q$ is *price elasticity* which is defined as

$$\epsilon(P) = \frac{\%\ chnge\ in\ demand}{\%\ chnge\ in\ price} = \frac{\frac{\Delta Q}{Q} \cdot 100}{\frac{\Delta P}{P} \cdot 100} = \frac{\frac{\Delta Q}{Q}}{\frac{\Delta P}{P}} = \frac{P}{Q} \cdot \frac{\Delta Q}{\Delta P} \approx \frac{P}{Q} \cdot \frac{dQ}{dP}.$$

In general $\epsilon(P) \leq 0$ (why?)

**Theorem.** $\epsilon(P) = \frac{d\ ln\ Q}{d\ ln\ P}$.
**Proof.**

$$\frac{d\ \ln\ Q}{d\ \ln\ P} = \frac{\frac{1}{Q} \cdot \frac{dQ}{dP}}{\frac{1}{P}} = \frac{P}{Q} \cdot \frac{dQ}{dP} = \epsilon(P).$$

The demand at $P$ is *elastic* if $\epsilon(P) \in (\infty, -1)$.

The demand at $P$ is *of unit elasticity* if $\epsilon(P) = -1$.

The demand at $P$ is *inelastic* if $\epsilon(P) \in (-1, 0)$.

If $\epsilon(P) = -1$ (unit elasticity), a 10% increase in price produces a 10% decrease in demand.

If $\epsilon(P) = -4$ (elastic), a 10% increase in price produces a 40% decrease in demand.

If $\epsilon(P) = -0.25$ (inelastic), a 10% increase in price produces a 2.5% decrease in demand.

**Example.** Suppose $Q(P) = 120 - 20P$. Then the derivative is constant $\frac{dQ}{dP} = -20$, but

$$\epsilon(P) = \frac{P}{Q} \cdot \frac{dQ}{dP} = \frac{-20P}{120 - 20P}.$$

The point of unit elasticity is the solution of the equation $\epsilon(P) = -1$. This gives $P = 3$.

The interval of elasticity is the solution of inequality $\epsilon(P) < -1$. This gives $P \in (3, 6)$.

The interval of inelasticity is the solution of inequality $\epsilon(P) > -1$. This gives $P \in (0, 3)$.

### 1.3.2 Elasticity and Revenue

If the demand function is given by $Q(P)$ the total revenue at price $P$ is

$$R(P) = P \cdot Q(P)$$

.

Let us first find a critical point of $Q(P)$

$$Q'(P) = Q(P) + PQ'(P) = 0, \ \ PQ'(P) = -Q(P), \ \ \frac{PQ'(P)}{Q(P)} = -1, \ \ \epsilon(P) = -1,$$

so a critical point is the unit elasticity point. But is it a max?

**Once more:*** Let us calculate the marginal revenue

$$R'(P) = (P \cdot Q(P))' = Q(P) + P \cdot Q'(P) = \\ Q(P) \cdot (1 + \tfrac{P}{Q} \cdot \tfrac{dQ}{dP}) = Q(P) \cdot (1 + \epsilon(P)).$$

Consider the following cases:

**1. The demand is inelastic** i.e. $-1 < \epsilon(P) < 0$. In this case $1+\epsilon(P) > 0$, thus $R'(P) = Q(P) \cdot (1 + \epsilon(P)) > 0$, so the revenue is increasing.

**2. The demand is elastic** i.e. $\epsilon(P) < -1$. In this case $1 + \epsilon(P) < 0$, thus $R'(P) = Q(P) \cdot (1 + \epsilon(P)) < 0$, so the revenue is decreasing.

**3. The demand has unit elasticity** i.e. $\epsilon(P) = -1$. In this case $1 + \epsilon(P) = 0$, thus $R'(P) = Q(P) \cdot (1 + \epsilon(P)) = 0$, so the revenue is at maximum.

### 1.3.3 Multivariable case

Suppose $Q_1(P_1, P_2, I)$ is the demand function for good 1 in terms of the price $P_1$ of good 1 and price $P_2$ of good 2 and the income $I$.

The *own price elasticity* is defined by:

$$\epsilon_{Q_1,P_1} = \frac{\%\ \textit{chnge in demand}}{\%\ \textit{chnge in price}} = \frac{\frac{\Delta Q_1}{Q_1}}{\frac{\Delta P_1}{P_1}} = \frac{P_1}{Q_1} \cdot \frac{\Delta Q_1}{\Delta P_1} \approx \frac{P_1}{Q_1} \cdot \frac{\partial Q_1}{\partial P_1}.$$

Similarly, the *cross price elasticity* is

$$\epsilon_{Q_1,P_2} = \frac{P_2}{Q_1} \cdot \frac{\partial Q_1}{\partial P_2},$$

and the *income elasticity* is

$$\epsilon_{Q_1,I} = \frac{I}{Q_1} \cdot \frac{\partial Q_1}{\partial I}.$$

## 1.4 The Total Derivative - Linear Approximation

Recall that for a function of one variable $f(x)$ the derivative

$$f'(x^*) = \frac{df}{dx}(x^*)$$

allows to approximate $f(x^* + \Delta\ x)$ as a linear function of $\Delta\ x$

$$f(x^* + \Delta\ x) \approx f(x^*) + f'(x^*) \cdot \Delta\ x.$$

Similarly, for a function of two variables $F(x, y)$ the partial derivatives $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ allow to approximate $F(x, y)$ in the neighborhood of a given point $(x^*, y^*)$:

$$F(x^* + \Delta x, y^* + \Delta y) \approx F(x^*, y^*) + \frac{\partial F}{\partial x}(x^*, y^*) \cdot \Delta x + \frac{\partial F}{\partial y}(x^*, y^*) \cdot \Delta y.$$

For a function of $n$ variables similar linear approximation looks as

$$F(x_1^* + \Delta x_1, ..., x_n^* + \Delta x_n) \approx$$
$$F(x_1^*, ..., x_n^*) + \tfrac{\partial F}{\partial x_1}(x_1^*, ..., x_n^*) \cdot \Delta x_1 + ... + \tfrac{\partial F}{\partial x_n}(x_1^*, ..., x_n^*) \cdot \Delta x_n.$$

The expression

$$dF = \frac{\partial F}{\partial x_1}(x_1^*, ..., x_n^*) \cdot dx_1 + ... + \frac{\partial F}{\partial x_n}(x_1^*, ..., x_n^*) \cdot dx_n$$

is called the *total differential*, and it approximates the actual change $\Delta F = F(x_1^* + \Delta x_1, ..., x_n^* + \Delta x_n) - F(x_1^*, ..., x_n^*)$.

**Example.** Consider the Cobb-Douglas production function $Q = 4K^{\frac{3}{4}}L^{\frac{1}{4}}$.
For $K = 10000,\ L = 625$ the output is $Q = 20000$. We want to use marginal analysis to estimate (a) $Q(10010, 625)$, (b) $Q(10000, 623)$, (c) $Q(10010, 623)$.
Step 1. The partial derivatives of $Q$ are:
the *marginal product of capital* is $\frac{\partial Q}{\partial K} = 3K^{-\frac{1}{4}}L^{\frac{1}{4}}$
the *marginal product of labor* is $\frac{\partial Q}{\partial K} = 3K^{\frac{3}{4}}L^{-\frac{3}{4}}$.
Step 2. Calculate these partial derivatives on $(10000, 625)$:
$\frac{\partial Q}{\partial K}(1000, 625) = 1.5, \quad \frac{\partial Q}{\partial L}(1000, 625) = 8.$
Step 3. (a) $Q(10010, 625) = Q(1000, 625) + \frac{\partial Q}{\partial K}(1000, 625) \cdot 10 = 20000 + 1.5 \cdot 10 = 20015.$
(b) $Q(10000, 623) = Q(1000, 625) + \frac{\partial Q}{\partial L}(1000, 625) \cdot (-2) = 20000 + 8 \cdot (-2) = 19984.$
(c) $Q(10000, 623) = Q(1000, 625) + \frac{\partial Q}{\partial K}(1000, 625) \cdot 10 + \frac{\partial Q}{\partial L}(1000, 625) \cdot (-2) = 20000 + 1.5 \cdot 10 + 8 \cdot (-2) = 19999.$

**Exercises**

1. Compute the partial derivatives of the following functions

$$a)\ 4x^2y-3xy^3+6x;\quad b)\ xy;\quad c)\ xy^2;\quad d)\ e^{2x+3y};$$
$$e)\ \frac{x+y}{x-y};\quad f)\ 3x^2y-7x\sqrt{y};\quad g)\ (x^2-y^3)^3;\quad h)\ \sqrt{2x-y^2};$$
$$i)\ ln(x^2+y^2);\quad j)\ y^2e^{xy^2};\quad k)\ \frac{x^2-y^2}{x^2+y^2}.$$

2. Find an example of a function $f(x,y)$ such that $\frac{\partial f}{\partial x}=3$ and $\frac{\partial f}{\partial y}=2$. How many such a functions are there?

3. The daily demand equations for the sale of brand A coffee and brand B coffee in a supermarket are

$$Q_1(P_1,P_2)=200-5P_1+4P_2\quad (Brand\ A)$$
$$Q_2(P_1,P_2)=300+2P_1-4P_2\quad (Brand\ B),$$

determine whether the indicated products are competitive, complementary, or neither.

4. The monthly demand equations for the sale of ski and ski boots in a sporting goods store are

$$Q_1(P_1,P_2)=800-0.004P_1^2-0.003P_2^2\quad (Skie)$$
$$Q_2(P_1,P_2)=600-0.003P_1^2-0.002P_2^2\quad (Skie\ boots),$$

determine whether the indicated products are competitive, complementary, or neither.

5. The monthly demand equations for the sale of tennis rackets and tennis balls in a sporting goods store are

$$Q_1(P_1,P_2)=500-0.5P_1-P_2^2\quad (Rackets)$$
$$Q_2(P_1,P_2)=10,000-8P_1^2-100P_2^2\quad (Balls),$$

determine whether the indicated products are competitive, complementary, or neither.

6. A demand function is given by $Q(P)=967-25P$.
a) Find the elasticity.
b) At what price the elasticity of demand equals to 1?
c) At what prices the demand is elastic?
d) At what prices the demand is inelastic?
e) At what price is the revenue maximumal?

f) At a price of $P = 20$, will a small increase in price cause the total revenue to increase or decrease?

7. Suppose that you have been hired by OPEC as an economic consultant for the world demand for oil. The demand function is

$$Q(P) = \frac{600}{(p+25)^2},$$

where $Q$ is measured in millions of barrels of oil per day at a price of $p$ dollars per barrel.

a) Find the elasticity.

b) Find the elasticity at a price of $10 per barrel, stating whether the demand is elastic or inelastic.

c) Find the elasticity at a price of $25 per barrel, stating whether the demand is elastic or inelastic.

d) Find the elasticity at a price of $30 per barrel, stating whether the demand is elastic or inelastic.

e) At what price is the revenue a maximum?

f) What quantity of oil will be sold at the price that maximizes revenue?

g) At a price of $30 per barrel, will a small increase in price cause the total revenue to increase or decrease?

8. A computer software store determines the following demand function for a new videogame: $Q(P) = \sqrt{200 - P^3}$ where $Q(P)$ is the number of videogames sold per day when the price is $P$ dollars per game.

a) Find the elasticity.

b) Find the elasticity at a price $3.

c) At $P = \$3$, will a small increase in price cause total revenue to increase or decrease?

9. The demand function $Q_1 = K_1 P_1^{a_{11}} P_2^{a_{12}} I^{b_1}$ is called a *constant elasticity demand function*. Compute three elasticities (own price, cross price, and income) and show that they are all constants.

10. Compute the partial derivatives and elasticities of the Cobb-Douglas production function $q = kx_1^{a_1}x_2^{a_2}$ and of the Constant Elasticity of Substitution (CES) production function $q = k(c_1 x_1^{-a} + c_2 x_2^{-a})^{-\frac{h}{a}}$.

11. Consider the production function $Q = 9L^{\frac{2}{3}}K^{\frac{1}{3}}$.

a) What is the output when $L = 1000$ and $K = 216$?

b) Use marginal analysis to estimate $Q(998, 216)$ and $Q(100, 217.5)$.

c) Compute directly these two values to three decimal places and compare these values with previous estimates.

d) How big $\Delta L$ be in order for the difference between $Q(1000 + \Delta L, 216)$ and its linear approximation

$$Q(1000, 216) + \frac{\partial Q}{\partial L}(1000, 216) \cdot \Delta L,$$

to differ by more than two units?

12. Consider the constant elasticity demand function $Q = 6p_1^{-2}p_2^{\frac{3}{2}}$ Suppose current prices are $p_1 = 6$ and $p_2 = 9$.

a) What is the current demand for $Q$?

b) Use differentials to estimate the change in demand as $p_1$ increases by 0.25 and $p_2$ decreases by 0.5.

c) Similarly, estimate the change in demand when poth prices increase by 0.2.

d) Estimate the total demand for situations b and c and compare this estimates with the actual demand.

13. A firm has the Cobb-Douglas production function $y = 10x_1^{\frac{1}{3}}x_2^{\frac{1}{2}}x_3^{\frac{1}{6}}$. Currently it is using the input bundle $(27, 16, 64)$.

a) How much is producing?

b) Use differentials to approximate its new output when $x_1$ increases to 27.1, $x_2$ decreases to 15.7, and $x_3$ remains the same.

c) Compare the answer in part b with the actual output.

d) Do b and c for $\Delta x_1 = \Delta x_2 = 0$ and $\Delta x_3 = -0.4$.

14. Use differentials to approximate each of the following:

$$a)\ \ f(x, y) = x^4 + 2x^2y^2 + xy^4 + 10y \ \ at \ \ x = 10.36 \ \ and \ \ y = 1.04;$$
$$b)\ \ f(x, y) = 6x^{\frac{2}{3}}y^{\frac{1}{2}} \ \ at \ \ x = 998 \ \ and \ \ y = 101.5;$$
$$c)\ \ f(x, y) = \sqrt{x^{\frac{1}{2}} + y^{\frac{1}{3}} + 5x^2} \ \ at \ \ x = 4.2 \ \ and \ \ y = 1.02.$$

15. Use calculus and no calculator to estimate the output given by the production function $Q = 3K^{\frac{2}{3}}L^{\frac{1}{3}}$ when

a) $K = 1000$ and $L = 125$;

b) $K = 998$ and $L == 128$.

16. Estimate $\sqrt{(4.1)^3 - (2.95)^3 - (1.02)^3}$.

Exercises 14.1-14.10 from [SB].

Homework: Exercises 1j, 1k, 5, 7, 13