1 Distance

Reading

[SB], Ch. 29.4, p. 811-816

A *metric space* is a set S with a given *distance* (or *metric*) function $d(x, y)$ which satisfies the conditions

(a) Positive definiteness $d(x, y) \geq 0, \quad d(x, y) = 0 \;\Leftrightarrow\; x = y$;

(b) Symmetry $d(x, y) = d(y, x)$;

(c) Triangle inequality $d(x, y) + d(y, z) \geq d(x, z)$.

For a given metric function $d(x, y)$:
A *closed ball* of radius r and center $x \in S$ is defined as

$$\bar{B}_r(x) = \{y \in R, \; d(x, y) \leq r\}.$$

An *open ball* of radius r and center $x \in S$ is defined as

$$\bar{B}_r(x) = \{y \in R, \; d(x, y) < r\}.$$

A *sphere* of radius r and center $x \in S$ is defined as

$$S_r(x) = \{y \in R, \; d(x, y) = r\}.$$

Example. Metrics on R^n:

1. *Euclidian metric* $d_E(x, y) = \sqrt{(x_1 - y_1)^2 + \; ... \; + (x_n - y_n)^2}$.

2. *Manhattan metric* (or Taxi Cab metric) $d_M(x, y) = |x_1 - y_1| + \; ... \; + |x_n - y_n|$.

3. *Maximum metric* $d_{max}(x, y) = max(|x_1 - y_1|, \; ... \; , |x_n - y_n|)$.

Some exotic metrics:

4. *Discrete metric* $d_{disc}(x, y) = 0 \;\; if \;\; x = y \;\; and \;\; d_{disc}(x, y) = 1 \;\; if \;\; x \neq y$

5. *British Rail metric* $d_{BR}(x, y) = ||x|| + ||y||$ if $x \neq y$ and $d_{BR}(x, x) = 0$.

6. *Hamming distance.* Let S be the set of all 8 vertices of a cube, in coordinates

$$S = \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (1, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}.$$

Hamming distance between two vertices is defined as the number of positions for which the corresponding symbols are different.

2 Norm

Let V be a vector space, say R^n. A *norm* is defined as a real valued function $||-|| : V \to R,\ v \to ||v||$, which satisfies the following conditions:

(i) positive definiteness $||v|| \geq 0, \quad ||v|| = 0 \ \Leftrightarrow \ v = 0$;

(ii) positive homogeneity or positive scalability $||r \cdot v|| = |r| \cdot ||v||$;

(iii) triangle inequality or subadditivity $||v + w|| \leq ||v|| + ||w||$.

Note that from (ii) follows that $||O|| = 0$ (here $O = (0, \ ... \ , 0)$), indeed, $||O|| = ||0 \cdot x|| = |0| \cdot ||x|| = 0$.

There is the following general *weighted* Euclidian norm on R^n which depends on parameters $a_1, ..., a_n$:

$$||x||_{a_1,...,a_n} = \sqrt{a_1 \cdot x_1^2 + \ ... \ + a_n \cdot x_n^2}.$$

If each $a_i = 1$, then this norm coincides with ordinary *Euclidian norm*

$$||x|| = \sqrt{x_1^2 + \ ... \ + x_n^2}.$$

There is s series of norms which depend on parameter k:

$$||x||_k = \sqrt[k]{|x_1|^k + \ ... \ + |x_n|^k}.$$

The norm $||x||_2$ coincides with Euclidian norm.

2.0.1 From Norm to Metric

Theorem 1 *Any norm $||x||$ induces a metric by $d(x, y) = ||x - y||$.*

Proof. The condition (i) implies the condition (a):
$d(x, y) = ||x - y|| \geq 0$; besides, if $x = y$ then $x - y = O$, thus $d(x, y) = ||x - y|| = ||O|| = 0$; conversely, suppose $d(x, y) = 0$, thus $||x - y|| = 0$, then, according to (i) we obtain $x - y = O$, so $x = y$.

The condition (ii) implies (b):

$$d(y, x) = ||y - x|| = ||(-1) \cdot (x - y)|| = |(-1)| \cdot ||x - y|| = ||x - y|| = d(x, y).$$

The condition(iii) implies (c):

$$d(x, y) + d(y, z) = ||x - y|| + ||y - z|| \geq ||x - y + y - z|| = ||x - z|| = d(x, z).$$

2.0.2 From Metric to Norm

Conversely, *some* metrics on R^n, which fit with the vector space structure determine a norm.

Theorem 2 *Suppose a metric $d(u,v)$ is given on a vector space V, and assume that the following two additional conditions are satisfied*

(d) translation invariance $d(u,v) = d(u+w, v+w)$,

and

(e) homogeneity $d(ku, kv) = |k| \cdot d(u,v)$.

Then $||v|| := d(v,O)$ is a norm.

Proof. The condition (a) implies the condition (i): $||v|| = d(v,O) \geq 0$; besides, suppose $||v|| = 0$, then $d(v,O) = 0$, thus, according to (i) we obtain $v = O$.

The condition (e) implies the condition (ii):

$$||k \cdot v|| = d(k \cdot v, O) = d(k \cdot v, k \cdot O) = |k| \cdot d(v,O) = |k| \cdot ||v||.$$

The condition (d) implies the condition (iii):

$$||v|| + ||w|| = d(v,O) + d(w,O) = d(v+w, O+w) + d(w,O) =$$
$$d(v+w, w) + d(w,O) \geq d(v+w, O) = ||v+w||.$$

Examples.

The above metrics 1,2,3 satisfy the properties (d) and (e) (**prove this!**), thus they determine the following norms on R^n: for a vector $x = (x_1, ..., x_n)$

1'. Euclidian norm $||x||_E = \sqrt{(x_1)^2 + \ ... \ + (x_n)^2}$.
2'. Manhattan norm $||x||_M = |x_1| + \ ... \ + |x_n|$.
3'. Maximum norm $||x||_{max} = max(|x_1|, \ ... \ , |x_n|)$.

Note that $||x||_E = ||x||_2$, $||x||_M = ||x||_1$ and in some sense $||x||_{max} = ||x||_\infty$.

4'. The discrete metric d_{disc} does not induce a norm.

Indeed, take $v \neq O$, then $||2 \cdot v|| = d(2 \cdot v, O) = 1 \neq 2 = 2 \cdot d(v,O) = 2 \cdot ||v||$.

2.1 Metric and Norm Induce Topology*

Any metric produces the notion of open ball. In its turn a notion of open ball produces the notion of open set, i.e.induces a *topology*.

Since any norm determines a metric, so it induces a topology too.

2.1.1 Equivalence of Norms*

Two norms $||x||$ and $||x||'$ are called equivalent if there exist two positive scalars a and b such that

$$a \cdot ||x|| \leq ||x||' \leq b \cdot ||x||.$$

This is an equivalence relation on the set of all possible norms on R^n.

If two norms are equivalent, then they induce the same notions of open sets (same topology). In particular, if a sequence $\{a_n\}$ converges to the limit a with respect to the norm $|| - ||$ then this sequence converges to the same limit with respect to the equivalent norm $|| - ||'$.

The three metrics $||v||_{max}, \; ||v||_E, \; ||v||_M$ are equivalent. This is a result of following geometrical inequalities

$$\begin{array}{l} ||v||_{max} \leq ||v||_E \leq ||v||_M; \\ ||v||_E \leq \sqrt{2}||v||_{max}; \\ ||v||_M \leq 2||v||_{max}; \\ ||v||_M \leq 2||v||_E. \end{array}$$

So all these three metrics induce the same topology.

3 Ordering

Reading

[Debreu], Ch.1.4, p.7-9

The set of real numbers R is *ordered*: $x > y$ if the difference $x - y$ is positive.

But what about the ordering on the plane R^2? Well, we can say that the vector $(5, 7) \in R^n$ is "bigger" than the vector $(1, 2)$, but how can we compare the vectors $(1, 2)$ and $(2, 1)$?

Unfortunately (or fortunately) we do not have a *canonical* ordering on R^n for $n > 1$. It is possible to consider various notions of ordering suitable for each particular problem.

3.1 Preorderings and Orderings

A *partial preordering* on a set X is a relation $x \geq y$ which satisfies the following conditions

(i) *reflexivity*: $\forall x \in X,\ x \geq x$;

(ii) *transitivity*: $\forall x, y, z \in X,\ x \geq y,\ y \geq z\ \Rightarrow x \geq z$.

A preordering is called *total* if additionally it satisfies

(iii) *totality*: $\forall x, y \in X$ either $x \geq y$ or $y \geq x$.

A (total) preordering is called (total) *ordering* if it satisfies additionally the condition

(iv) *antisymmetricity*: $x \geq y,\quad y \geq x\quad \Rightarrow x = y$.

The above defined four notions: partial (total) (pre)ordering can be observed by the following diagram

$$\begin{array}{ccccc} & & i, ii & & \\ & & \textit{partial preordering} & & \\ i, ii, iii & \nearrow & & \nwarrow & i, ii, iv \\ \textit{total preordering} & & & & \textit{partial ordering} \\ & \nwarrow & & \nearrow & \\ & & \textit{total ordering} & & \\ & & i, ii, iii, iv & & \end{array}$$

where an arrow indicates implication.

Partially ordered sets are called *posets*.

3.1.1 Preorderings on R^2

1. Norm preordering:

$$v = (x, y) \geq v' = (x', y') \quad if \quad ||v|| = \sqrt{x^2 + y^2} \geq ||v'|| = \sqrt{x'^2 + y'^2}.$$

This is a total preordering. Why *"pre"*?

2. Product ordering: $(a, b) \leq (c, d)$ if $a \leq c$ and $b \leq d$. *This is a partial ordering.* Why *"partial"*?

3. Lexicographical ordering: $(a, b) \leq (c, d)$ if and only if $a < c$, but if $a = c$ then $b \leq d$. *This is a total ordering.* Why *"total"*?

3.1.2 Other Examples

1. The set of natural numbers N of course is ordered by the usual ordering "$m \geq n$ if $m - n$ is nonnegative".

2. There exists on N also the following *partial ordering* "$m \geq n$ if m is divisible by n" ("n divides m", notation $n|b$). For example $6 \geq 2,\ 6 \geq 3$, but 6 and 4 are not comparable (thus "partial").

3. Let S be a set, the set of all its subsets is denoted by 2^S. Let us introduce on 2^S the following relation: for arbitrary subsets $A \subseteq S,\ B \subseteq S$ we say $B \leq A$ if $B \subseteq A$. This is a partial ordering. Why *"partial"*?

4. Consider on R^3 the following relation:

$$(x, y, z) \geq (a, b, c)$$

if $x \geq a$ and $y \geq b$. This is partial preordering. Why *"partial"* and why *"pre"*?

3.1.3 Indifference Relation

Each preordering $\geq$ defines *indifference relation*:

$$x \sim y \ \ if \ \ x \geq y \ \ and \ \ y \geq x.$$

Theorem 3 *The relation $x \sim y$ is an* equivalence relation*.*

Proof. We show that the relation $x \sim y$ satisfies the axioms of equivalence:
(1) Reflexivity $x \sim x$;
(2) Symmetricity $x \sim y \ \Rightarrow \ y \sim x$;

(3) Transitivity $x \sim y,\ y \sim z\ \Rightarrow\ x \sim z$.
Indeed,
(1) Since of (i) $x \geq x$, thus $x \sim x$.
(2) Suppose $x \sim y$, then $x \geq y$ and $y \geq x$, thus $y \sim x$.
(3) Suppose $x \sim y$, this implies $x \geq y$ and $y \geq x$, and suppose $y \sim z$, this implies $y \geq z$ and $z \geq y$. Then since of (ii) we have

$$x \geq y,\ y \geq z\ \Rightarrow x \geq z$$

and

$$z \geq y,\ y \geq x\ \Rightarrow z \geq x,$$

thus $x \sim z$.

The *indifference set* (or orbit) of an element $x \in X$ is defined as

$$I(x) = \{y \in X,\ x \sim y\}.$$

Since indifference relation is an equivalence, the indifference sets form a *partition* of X.

Examples.

1. If the starting relation $\geq$ is an *ordering* then $x \sim y$ if and only if $x = y$. So the indifference sets are one point sets: $I(x) = \{x\}$.

2. For the norm preordering indifference sets are spheres centered at the origin: $I(x) = S_{|x|}(O)$.

3.1.4 Strict Preordering

Each preoredering $\geq$ induces the *strict preordering* $>$ defined by: $x > y$ if $x \geq y$ but not $y \geq x$. Equivalently $x > y$ if $x \geq y$ and not $x \sim y$.

If a starting preoredering $\geq$ is an ordering, then $x > y$ is defined as $x \geq y$ and $x \neq y$.

3.2 Maximal and Greatest

Let S be a partially preordered set.

An element $x \in S$ is called *maximal* if there exists no $y \in S$ such that $y > x$.

An element $x \in S$ is called *minimal* if there exists no $y \in S$ such that $y < x$.

An element $x \in S$ is called *greatest* if $x \geq y$ for all $y \in S$.

An element $x \in S$ is called *least* if $x \leq y$ for all $y \in S$.

Theorem 4 *If S is an ordered set, then a greatest (least) element is unique.*

Proof. Suppose x and x' are greatest elements. Then $x \geq x'$ since x is greatest, and $x' \geq x$ since x' is greatest. Thus, since S is an ordered set, we get $x = x'$.

Theorem 5 *A greatest element is maximal.*

Proof. Suppose $x \in S$ is greatest, that is $x \geq y$ for all $y \in S$, but not maximal, that is $\exists y \;\; s.t. \;\; y > x$. By definition of $>$ this means that $y \geq x$ but not $x \geq y$. The last contradicts to $x \geq y$.

Theorem 6 *If a preordering is total, then a maximal element is greatest.*

Proof. Suppose $x \in S$ is maximal, that is there exists no $y \in S$ such that $y > x$. Let us show that x is greatest, that is $x \geq z$ for each z. Indeed, since of totality ether $x \geq z$ or $z \geq x$. Suppose that x is not greatest, that is $x \geq z$ is not correct. Then $z \geq x$, but this, together with negation of $x \geq z$, implies $z > x$, which contradicts to maximality of x.

So when the preordering is *total*, there is no difference between maximal and greatest. Similarly for minimal and least.

Examples.

1. The set $\{1, 2, 3, 4, 5, 6\}$ ordered by the partial ordering "divisible by" has three maximal elements $4, 5, 6$, no greatest element, one minimal element 1 and one least element 1:

$$\begin{array}{ccc} & 6 & \\ & \uparrow & \\ 4 & | & \\ \uparrow & | & \\ 2 & 3 & 5 \\ \nwarrow & \uparrow & \nearrow \\ & 1 & \end{array}$$

2. Let S be the set of all 8 vertices of a cube, in coordinates

$$S = \{(0,0,0), (0,0,1), (0,1,0), (1,0,0), (0,1,1), (1,0,1), (1,1,0), (1,1,1)\}.$$

Hamming ordering on S is defined as follows: $v \geq w$ if v contains more 1-s than w.

The least (minimal) element here is $(0, 0, 0)$ and greatest (maximal) element is $(1, 1, 1)$.

3. In the partially ordered set 2^S the least (minimal) element his the empty set and greatest (maximal) element is S.

3.3 Utility Function

A real valued function $U : X \to R$ is said to *represent* a preordering $\geq$ if

$$\forall x, y \in X,\ x \geq y \Leftrightarrow\ U(x) \geq U(y).$$

In economics a preordering $\geq$ is called *preference preordering* and a representing function U is called *utility function*.

The norm preordering:

$$v = (x, y) \geq v' = (x', y') \quad if \quad ||v|| = \sqrt{x^2 + y^2} \geq ||v'|| = \sqrt{x'^2 + y'^2}.$$

is represented by the utility function

$$U(x, y) = \sqrt{x^2 + y^2},$$

or by the function $2U(x, y) = 2\sqrt{x^2 + y^2}$, or by $U^2(x, y) = x^2+y^2$, etc. These functions differ but all of them have the same indifference sets.

3.3.1 Equivalent Utility Functions

A given preordering can be represented by various functions. Two utility functions are called *equivalent* if they have same indifferent sets.

A *monotonic transformation* of an utility function U is the composition $g \circ U(x) = g(U(x))$ where g is a strictly monotonic function.

It is clear that an utility function U and any its monotonic transformation $g \circ U$ represent the same or opposite preordering, so they are equivalent.

Example. The functions

$$3xy + 2, \quad (xy)^3, \quad (xy)^3 + xy, \quad e^{xy}, \quad ln\ x + ln\ y$$

all are monotonic transformations of the function xy: the corresponding monotonic transformations are respectively

$$3z + 2, \quad z^3, \quad z^3 + z, \quad e^z, \quad ln\ z.$$

Exercises

1. Draw the balls $\bar{B}_1((0,0))$, $\bar{B}_1(1,1)$, $\bar{B}_2(1,1)$ and $\bar{B}_3(1,1)$ for each of the following metrics

Euclidian metric $d_E(x,y) = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$.
Manhattan metric $d_M(x,y) = |x_1-y_1|+|x_2-y_2|$.
Maximum metric $d_{max}(x,y) = max(|x_1-y_1|,|x_2-y_2|)$.
British Rail metric $d_{BR}(x,y) = ||x||+||y||$.
Discrete metric $d_{disc}(x,y) = 0 \;\; if \;\; x=y \;\; and \;\; d(x,y)=1 \;\; if \;\; x\neq y$

2. Show that the discrete metric d_{disc} does not induce a norm.

3. For a vector $v=(x,y)\in R^2$ let us define $||v||_{min} = min(|x|,|y|)$. Is this a norm?

4. Does the British rail metric $d_{BR}(x,y)$ satisfy the conditions
(d) translation invariance $d(u,v) = d(u+w, v+w)$,
and
(e) homogeneity $d(ku,kv) = |k|\cdot d(u,v)$?
Does d_{BR} induce a *norm* $||x||_{BR} = d_{BR}(x,O)$?

5. Give examples of (a) partial preordering, (b) total preordering, (c) partial ordering, (d) total ordering.

6. Is the relation defined on R^2 by

$$(x,y)\geq(x',y') \Leftrightarrow x\geq x', \;\; y\geq y'$$

a (a) partial preordering? (b) total preordering? (c) partial ordering? (d) total ordering?

7. What can you say about indifference sets of an ordering?

8. Draw indifference sets $I(0,0,0)$, $I(1,1,1)$, $I(2,2,2)$ in R^3 for the preordering

$$(x,y,z)\geq(x',y',z') \Leftrightarrow ||(x,y,z)||_E \geq ||(x',y',z')||_E$$

.

9. Draw indifference sets $I(0,0)$, $I(1,1)$, $I(2,2)$ in R^2 for the preordering defined by Manhattan norm

$$(x,y)\geq(x',y') \Leftrightarrow ||(x,y)||_M \geq ||(x',y')||_M$$

.

10. Draw indifference sets $I(0,0),\ I(1,1),\ I(2,2)$ in R^2 for the preordering defined by maximum norm

$$(x,y) \geq (x',y') \Leftrightarrow ||(x,y)||_{max} \geq ||(x',y')||_{max}$$

.

11. Suppose a set S has two greatest elements x and x'. Show that $x \sim x'$.

12. Find (draw) two sets

$$S = \{(x,y) \in R^2,\ (x,y) \leq (1,1)\}, \quad T = \{(x,y) \in R^2,\ (1,1) \leq (x,y)\}$$

where $\leq$ assumes the *product ordering* of R^2: $(x,y) \leq (x',y')$ if $x \leq x',\ y \leq y'$.

13. Find (draw) two sets

$$S = \{(x,y) \in R^2,\ (x,y) \leq (1,1)\}, \quad T = \{(x,y) \in R^2,\ (1,1) \leq (x,y)\}$$

where $\leq$ assumes the *lexicographical ordering* of R^2.

14. Find maximal, minimal, greatest, least elements of the set $S = \{2,3,4,5,6,12\}$ with respect of the ordering "$a \leq b$ if $a|b$" (a divides b).

15. Find maximal, minimal, greatest, least elements of the set $S = \{(x,y),\ 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$ with respect to the product ordering of R^2.

16. Find maximal, minimal, greatest, least elements of the set $S = \{(x,y),\ x^2+y^2 \leq 1\}$ with respect to the product ordering of R^2.

17. Find maximal, minimal, greatest, least elements of the set $S = \{(x,y),\ x^2+y^2 \leq 1, x \geq 0,\ y \geq 0\}$ with respect to the product ordering of R^2.

18. For each of the functions

$$(a)\ \ 3xy+2, \quad (b)\ \ (xy)^2, \quad (c)\ \ (xy)^3+xy, \quad (d)\ \ e^{xy}, \quad (e)\ \ ln\ x + ln\ y$$

(which are equivalent to xy) identify the level sets which correspond to the level sets $xy = 1$ and $xy = 4$. For example to the level set $xy = 1$ corresponds the level set $3xy + 2 = 5$ for the function (a).

19. Which of the following functions are equivalent to xy? For those which are, what monotonic transformation provides this equivalence?
(a) $7x^2y^2+2$, (b) $ln\ x + ln\ y + 1$, (c) x^2y, (d) $x^{\frac{1}{3}}y^{\frac{1}{3}}$.

Homework
Exercises 3, 10, 13, 17, 19.

Short Summary
Metric and Norm

Axioms

	Metric		Norm
a	$d(x,y) \geq 0$	i	$\|\|v\|\| \geq 0$
	$d(x,y) = 0 \;\Leftrightarrow\; x = y$;		$\|\|v\|\| = 0 \;\Leftrightarrow\; v = 0$;
b	$d(x,y) = d(y,x)$;	ii	$\|\|r \cdot v\|\| = \|r\| \cdot \|\|v\|\|$;
c	$d(x,y) + d(y,z) \geq d(x,z)$;	iii	$\|\|v + w\|\| \leq \|\|v\|\| + \|\|w\|\|$.

From Norm to Metric: $d(x,y) = ||x - y||$.
From Metric to Norm: $||v|| := d(v, O)$ if $d(x,y)$ additionally satisfies $d(u,v) = d(u+w, v+w)$ and $d(ku, kv) = |k| \cdot d(u,v)$.
Examples of Metrics.

1. *Euclidian metric* $d_E(x,y) = \sqrt{(x_1 - y_1)^2 + \ \dots\ + (x_n - y_n)^2}$.
2. *Manhattan metric* (or Taxi Cab metric) $d_M(x,y) = |x_1 - y_1| + \ \dots\ + |x_n - y_n|$.
3. *Maximum metric* $d_{max}(x,y) = max(|x_1 - y_1|, \ \dots\ , |x_n - y_n|)$.
4. *Discrete metric* $d_{disc}(x,y) = 0 \ \ if \ \ x = y \ \ and \ \ d_{disc}(x,y) = 1 \ \ if \ \ x \neq y$
5. *British Rail metric* $d_{BR}(x,y) = ||x|| + ||y||$ if $x \neq y$ and $d_{BR}(x,x) = 0$.

Examples of Norms

1. $||x||_{a_1,\dots,a_n} = \sqrt{a_1 \cdot x_1^2 + \ \dots\ + a_n \cdot x_n^2}$.
   If each $a_i = 1$ this norm coincides with *Euclidian norm*

$$||x||_E = \sqrt{x_1^2 + \ \dots\ + x_n^2}.$$

2. Manhattan norm $||x||_M = |x_1| + \ \dots\ + |x_n|$.
3. Maximum norm $||x||_{max} = max(|x_1|, \ \dots\ , |x_n|)$.
4. The k-norm $||x||_k = \sqrt[k]{|x_1|^k + \ \dots\ + |x_n|^k}$. Particularly $||x||_E = ||x||_2$, $||x||_M = ||x||_1$ and in some sense $||x||_{max} = ||x||_\infty$.

Short Summary
Orderings

Axioms

(i) *reflexivity*: $\forall x \in X,\ x \geq x$;
(ii) *transitivity*: $\forall x, y, z \in X,\ x \geq y,\ y \geq z\ \Rightarrow x \geq z$.
(iii) *totality*: $\forall x, y \in X$ either $x \geq y$ or $y \geq x$.
(iv) *antisymmetricity*: $x \geq y,\ \ y \geq x\ \ \Rightarrow x = y$.

$$
\begin{array}{ccccc}
 & & i, ii & & \\
 & & \textit{partial preordering} & & \\
i, ii, iii & \nearrow & & \nwarrow & i, ii, iv \\
\textit{total preordering} & & & & \textit{partial ordering} \\
 & \nwarrow & & \nearrow & \\
 & & \textit{total ordering} & & \\
 & & i, ii, iii, iv & &
\end{array}
$$

Examples

1. Norm total preordering on R^2:

$$v = (x, y) \geq v' = (x', y') \quad if \quad ||v|| = \sqrt{x^2 + y^2} \geq ||v'|| = \sqrt{x'^2 + y'^2}.$$

2. Product partial ordering on R^2: $(a, b) \leq (c, d)$ if $a \leq c$ and $b \leq d$.
3. Lexicographical total ordering on R^2: $(a, b) \leq (c, d)$ if and only if $a < c$, but if $a = c$ then $b \leq d$.
4. Standard total ordering on N: "$m \geq n$ if $m - n$ is nonnegative".
5. Divisibility partial ordering on N: $m \geq n$ if $n|b$.
6. Standard partial ordering on 2^S: $B \leq A$ if $B \subseteq A$.
7. Partial preordering on R^3: $(x, y, z) \geq (a, b, c)$ if $x \geq a$ and $y \geq b$.

Indifference Relation: $x \sim y\ \ if\ \ x \geq y\ \ and\ \ y \geq x$. The indifference set (orbit) of x: $I(x) = \{y \in X,\ x \sim y\}$. For an ordering $x \sim y$ iff $x = y$ and $I(x) = \{x\}$.

Strict Preordering: $x > y$ if $x \geq y$ but not $y \geq x$.

Greatest and Maximal.
$x \in S$ is **maximal** if there exists no $y \in S$ s.t. $y > x$.
$x \in S$ is **greatest** if $x \geq y$ for all $y \in S$.
Greatest is always maximal.
If a preordering is total, then maximal is greatest.
If S is an ordered set, then a greatest element is unique.

A **utility** function $f : S \to R$ determines a total (pre) ordering $x \leq y$ if $f(x) \leq f(y)$.