1 Limits and Open Sets

Reading
[SB], Ch. 12, p. 253-272

1.1 Sequences and Limits in R

1.1.1 Sequences

A *sequence* is a function $N \to R$. It can be written as

$$\{a_i,\ i = 1, 2, ...\} = \{a_1, a_2, a_3, ..., a_n, ...\} \ .$$

A sequence is called *bounded* if there exists $M \in R$ such that $|a_n| < M$ for each n.

A sequence is *increasing* if $a_n < a_{n+1}$ for each n.

A sequence is *decreasing* if $a_n > a_{n+1}$ for each n.

A sequence is *monotonic* if it is either increasing or decreasing.

Examples.

The sequence $\{1, 2, 3, ..., a_n = n, ...\}$ is increasing, bounded below, and not bounded.

The sequence $\{-1, -2, -3, ..., a_n = -n, ...\}$ is decreasing, bounded from above, and not bounded.

The sequence $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ..., a_n = \frac{1}{n}, ...\}$ is decreasing and bounded.

The sequence $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ..., a_n = \frac{n}{n+1}, ...\}$ is increasing and bounded.

The sequence $\{1, -1, 1, -1, 1, -1, ..., a_n = (-1)^{n+1}, ...\}$ is not monotonic but is bounded.

The sequence $\{-1, 2, -3, 4, -5, 6, ..., a_n = (-1)^n n, ...\}$ is neither monotonic nor bounded.

1.1.2 Limits

A sequence $\{a_i\}$ *converges to the limit* a, notation

$$\lim_{n \to \infty} a_n = a$$

or $a_n \to a$, if for any $\epsilon > 0$ exists N such that for $n > N$ each a_n is in the interval $(a - \epsilon, a + \epsilon)$.

A sequence $\{a_i\}$ has *accumulation point* a if for any $\epsilon > 0$ *infinitely many* elements of the sequence are in the interval $(a-\epsilon, a+\epsilon)$.

A sequence can have several accumulation points but at most one limit.

If a is an accumulation point of $\{a_n\}$ then this sequence has a subsequence convergent to a.

Any convergent sequence is bounded.

Any bounded sequence has an accumulation point.

Any monotonic and bounded sequence is convergent.

Examples.

The sequence $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ..., a_n = \frac{1}{n}, ...\}$ converges to $\lim_{n\to\infty} a_n = 0$.

The sequence $\{-1, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, ..., a_n = \frac{(-1^n)}{n}, ...\}$ converges to $\lim_{n\to\infty} a_n = 0$.

The sequence $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ..., a_n = \frac{n}{n+1}, ...\}$ converges to $\lim_{n\to\infty} a_n = 1$.

The sequence $\{0, \frac{1}{2}, 0, \frac{1}{4}, 0, \frac{1}{6},\ ..., a_n = \frac{(-1)^n+1}{2n}, ...\}$ converges to 0.

The sequence $\{1, -1, 1, -1, 1, -1, ..., a_n = (-1)^{n+1}, ...\}$ does not converge but it has two accumulation points -1 and 1.

The sequence $\{\frac{-1}{2}, \frac{2}{3}, \frac{-3}{4}, ..., a_n = \frac{(-1)^n n}{n+1}, ...\}$ does not converge but it has two accumulation points -1 and 1. Its subsequence which consists of terms with odd indices (the negative terms) converges to -1. Its subsequence which consists of terms with even indices (the positive terms) converges to 1.

The sequence $\{1, 2, 0, 1, 2, 0, ..., a_n = res(n:3), ...\}$ does not converge but it has three accumulation points $0, 1$ and 2.

The sequence $\{1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, ...\}$ does not converge but it has infinitely many accumulation points $1, 2, 3, ...$

1.1.3 Algebraic Properties of Limits

If $\lim x_n = x, \quad \lim y_n = y$, then

$$\begin{aligned} &\lim(x_n \pm y_n) = x \pm y;\\ &\lim(x_n \cdot y_n) = x \cdot y;\\ &\lim \tfrac{x_n}{y_n} = \tfrac{x}{y} \end{aligned}$$

if $y_n \neq 0,\ y \neq 0$.

In Italy, Russia and France, the *squeeze* theorem is also known as the two carabinieri theorem, two militsioner theorem, two gendarmes theorem, or two policemen and a drunk theorem:

If $\lim_{n\to\infty} a_n = u = \lim_{n\to\infty} b_n$ and $a_n \leq c_n \leq b_n$ then $\lim_{n\to\infty} c_n = u$.

Example

$$\lim_{n\to\infty} \tfrac{2n+1}{n+3} = \lim_{n\to\infty} \tfrac{\frac{2n+1}{n}}{\frac{n+3}{n}} = \lim_{n\to\infty} \tfrac{2+\frac{1}{n}}{1+\frac{3}{n}} = \tfrac{\lim_{n\to\infty}(2+\frac{1}{n})}{\lim_{n\to\infty}(1+\frac{3}{n})} = \tfrac{\lim_{n\to\infty} 2+\lim_{n\to\infty}\frac{1}{n}}{\lim_{n\to\infty} 1+\lim_{n\to\infty}\frac{3}{n}} = \tfrac{2+0}{1+0} = 2.$$

Example

$\lim_{n\to\infty} \frac{n!}{n^n} =?$

$$0 < \frac{n!}{n^n} = \frac{n\cdot(n-1)\cdot(n-2)\cdot\ \dots\ \cdot 2\cdot 1}{n\cdot n\cdot n\cdot\ \dots\ \cdot n\cdot n} < \frac{n\cdot n\cdot n\cdot\ \dots\ \cdot n\cdot 1}{n\cdot n\cdot n\cdot\ \dots\ \cdot n\cdot n} = \frac{1}{n}$$

so $0 = lim_{n\to\infty} 0 < lim_{n\to\infty}\frac{n!}{n^n} < \frac{1}{n} = 0$, thus $lim_{n\to\infty}\frac{n!}{n^n} = 0$.

1.1.4 Some Special Limits

1. $lim_{n\to\infty}\frac{ln\ n}{n} = 0$.
2. $lim_{n\to\infty}\sqrt[n]{n} = 1$.
3. $lim_{n\to\infty} a^{\frac{1}{n}} = 1\ for\ a > 0$.
4. $lim_{n\to\infty} a^n = 0\ for\ |a| < 1$.
5. $lim_{n\to\infty}(1 + \frac{1}{n})^n = e$.

1.2 Sequences and Limits in R^m

Let $x \in R^m$ and $r \in R$. The *open ball* with center x and radius r is defined as

$$B_r(x) = \{y \in R^m,\ ||y - x|| < r\}.$$

The *closed* ball is defined as

$$\bar{B}_r(x) = \{y \in R^m,\ ||y - x|| \le r\}.$$

A sequence in R^m is a function $N \to R^m$.

A sequence $\{v^1, v^2, ..., v^i, ...;\ \ v^i \in R^n\}$ is called *bounded* if there exists $M \in R$ such that $||v^i|| < M$ for each i. In other words $v^i \in B_M(O)$: each term of the sequence belongs to some ball centered in the origin O with radius M.

A sequence $\{v^k\}$ of vectors is *convergent* and has the *limit* v (notation $\lim_{n\to\infty} v^n = v$) if for any $\epsilon > 0$ there exists N such that for $n > N$ each v^n is in the open ball $B_\epsilon(v)$.

A sequence in R^m converges if and only if all m sequences of its components converge in R. In other words, let

$$v^k = (v_1^k, v_2^k,\ \dots\ , v_m^k)$$

be the kth term of a sequence $\{v^k,\ \ k = 1, 2, 3, ...\}$ written in coordinates, then

$$\lim_{k\to\infty} v^k = \lim_{k\to\infty}(v_1^k, v_2^k,\ \dots\ , v_m^k) = (\lim_{k\to\infty} v_1^k, \lim_{k\to\infty} v_2^k,\ \dots\ , \lim_{k\to\infty} v_m^k).$$

So, if $\lim_{k\to\infty} v_i^k = v_i$ then

$$\begin{array}{ccccccc} v^1 & = & (v_1^1, & v_2^1, & \ldots & , v_m^1) \\ v^2 & = & (v_1^2, & v_2^2, & \ldots & , v_m^2) \\ \ldots & & \ldots & \ldots & \ldots & \ldots \\ v^k & = & (v_1^k, & v_2^k, & \ldots & , v_m^k) \\ \ldots & & \ldots & \ldots & \ldots & \ldots \\ \downarrow & & \downarrow & \downarrow & & \downarrow \\ v & = & (v_1, & v_2, & \ldots & , v_m) \end{array}$$

Examples

The sequence of vectors of R^2 $\{v^1, v^2, ...\}$ with $v^n = (2-\frac{1}{n}, 3+\frac{1}{n})$ converges to the vector $v = (2, 3)$.

The sequence $(x^n, y^n) = (\frac{(-1)^n+1}{2n}, \frac{(-1)^{n+1}+1}{2n})$ converges to the origin $(0, 0)$. Try to plot some points of this sequence.

1.3 Open Sets

A point $x \in S \subset R^n$ is an *interior point* of S if there exists an open ball $B_\epsilon(x)$ centered at x, that is contained in S.

A subset $S \subset R^m$ is called *open* if each $x \in S$ is an interior point.

Examples.

The interval $(0, 1) = \{x \in R,\ 0 < x < 1\} \subset R$ is open set.

A one point set in R^n is not open.

A line in R^2 is not open.

An open ball $B_r(x) = \{y \in R^m,\ ||y - x|| < r\}$ is an open subset of R^m, but a closed ball $\bar{B}_r(x) = \{y \in R^m,\ ||y - x|| \le r\}$ is not.

1.3.1 Main properties of open sets in R^n

1. The empty set $\emptyset$ and whole R^n are open.

2. Any union of open sets is open.

3. The *finite* intersection of open sets is open.

Here the requirement "finite" is essential: the *infinite* intersection of open sets

$$\{U_n = (-\frac{1}{n}, \frac{1}{n}),\ n = 1, 2, 3, ...\}$$

is the one point set $\{0\}$ which is not open.

Similar example in R^2: the *infinite* intersection of open balls

$$\{U_n = (B_{\frac{1}{n}}(0,0),\ n = 1, 2, 3, ...\}$$

is the one point set $\{(0,0)\}$ which is not open.

Actually these three conditions define a *topological space*, but forget it.

1.3.2 Interior

Interior $int(S)$ of a set $S \subset R^m$ is defined as

1. The set of all interior points of S.
2. $int(S)$ is the union of all open subsets contained in S.
3. $int(S)$ is the largest open subset of S.

All these conditions are equivalent.

Examples.

In R: $int\ [0,1] = int\ (0,1] = int\ [0,1) = int\ (0,1) = (0,1)$.

The interior of a closed ball $\bar{B}_r(x)$ is the open ball $B_r(x) = \{y \in R^m,\ ||y-x|| < r\}$.

The interior of a line in R^2 is empty.

1.4 Closed Sets

A point x is said to be a *limit point* of a subset $S \subset R^n$ if every open ball around x contains at least one point of S distinct from x.

A point x is a limit point of $S \subset R^n$ if and only if there exists a sequence in $S \setminus \{x\}$ that converges to x, i.e. $\exists\ \{s_1, s_2, ..., s_n, ...\ ;\quad s_i \in S \setminus\ \{x\}\}$ with $lim_{n\to\infty} s_n = x$.

Three equivalent definitions of closed set:

1. A subset $S \subset R^m$ is called closed if it contains all its limit points.
2. $S \subset R^m$ is closed if, whenever $\{x_n\}$ is a *convergent* sequence completely contained in S, its limit is also contained in S.
3. A set S is closed if its *complement* $S^c = R^m - S$ is open.

Examples.

The segment

$$[0,1] = \{x \in R,\ 0 \geq x \geq 1\} \subset R$$

is a closed set.

A line in R^2 is closed.

A closed ball $\bar{B}_r(x)$ is a closed subset of R^m.
An open ball $B_r(x)$ is not a closed subset of R^m.

There are two subsets of R^n which are simultaneously open and closed, these are the empty set $\emptyset$ and the whole space R^n.

1.4.1 Main Properties of Closed Sets

1. The empty set $\emptyset$ and whole R^n are closed.
2. Any intersection of closed sets is closed.
3. The *finite* union of closed sets is closed.
   Here the requirement "finite" is essential: the *infinite* union of closed sets

$$\{F_n = [-1 + \frac{1}{n}, 1 - \frac{1}{n}],\ n = 1, 2, 3, ...\}$$

is the open interval $(-1, 1)$ which is not closed.

1.4.2 Closure

The closure $cl(S)$ of a subset $S \subset R^m$ is defined as:

1. The union of S and all its limit points.
2. As the intersection of all closed sets containing S.
3. The smallest closed set which contains S.

All these conditions are equivalent.

Examples.
$cl\ (0, 1) = cl\ [0, 1) = cl\ (0, 1)] = cl\ [0, 1] = [0, 1]$.

The closure of an open ball $B_r(x)$ is the closed ball $\bar{B}_r(x)$.

1.4.3 Some Properties of Interior and Closure operators

Some properties

$$cl(cl(S)) = cl(S), \quad int(int(S)) = int(S);$$
$$cl(S \bigcup T) = cl(S) \bigcup cl(T), \quad int(S \bigcap T) = int(S) \bigcap int(T);$$
$$int(S) \subset S \subset cl(S).$$

Generally

$$cl(int(S)) \neq cl(S), \quad int(cl(S)) \neq int(S),$$

$$cl(S\bigcap T) \neq cl(S)\bigcap cl(T), \quad int(S\bigcup T) \neq int(S)\bigcup int(T).$$

Try to construct conterexamples.

1.4.4 Boundary of a Set

A point x is in the *boundary* of a set S if every open ball about x contains both points in S and points in the complement S^c.

The boundary ∂S of a set $S \in R^n$ is defined as the set of boundary points of S.

$\partial S = cl\ S \setminus intS = cl\ S \cap\ cl\ S^c$.

Examples.

The boundary of the sets $(0,1),\ [0,1),\ (0,1],\ [0,1]$ is two point set $\{0,1\}$.

The boundary of an open ball $B_r(x)$, as well as of an closed ball $\bar{B}_r(x)$ is the *sphere*

$$S_r(x) = \{y \in R^m,\ ||y-x|| = r\}.$$

1.5 Compact Sets

Suppose $f : S \to R$ be a *continuous* function defined on a set S. What properties of the set S can guarantee that f achieves its maximum (minimum) on S?

This is not the openness: If $S = (0,1)$ is an *open* interval then the function $f(x) = \frac{1}{x}$ does not achieve maximum on S.

This is not closeness: If $S = R$ (a *closed* set) then the function $f(x) = x^2$ does not achieve maximum on $S = R$.

The existence of maximum guarantees the following condition: S must be bounded and closed. This leads to the following notion.

A set $S \subset R^m$ is called *compact* if it is both closed and bounded.

Theorem. A continuous function $F : S \to R$ on a nonempty compact set S attains its supremum.

Exercises

1. Write the n-th term for these sequences and indicate which of these sequences are bounded, convergent, have accumulation points:
(a) $\{2, 4, 6, 8, ...\}$. (b) $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\}$. (c) $\{1, \frac{1}{2}, 4, \frac{1}{8}, , ...\}$.
(d) $\{0, -\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, ...\}$. (e) $\{-1, 1, -1, 1, -1, ...\}$. (f) $\{\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, ...\}$.

2. Show that any convergent sequence is bounded.

3. Give an example of a sequence with two accumulation points 0 and 1.

4. Give an example of nonconvergent sequence with one accumulation point.

5. Give an example of a sequence in R whose accumulation points are all natural numbers.

6. Give 4 examples of sequences in R^2 each converges to the point $(1, 1)$.

7. Evaluate first 5 terms of the sequence $\{a_n = \frac{1}{n}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^n \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in R^2\}$. Is this sequence bounded? convergent?

8. Show that the interval $(0, 1)$ is open finding formula for ϵ for each $x \in (0, 1)$.

9. Show that open balls are open and closed balls are closed.

10. Show that any open set is a union of open balls.

11. Give an example of nonempty set with empty interior.

12. Show that one point set is closed.

13. Show that any finite set is closed.

14. For each of the following subsets of of R^2, draw the set, state whether it is open, closed, compact, or neither, and justify your answer
a) $\{(x, y),\ -1 < x < +1,\ y = 0\}$,
b) $\{(x, y),\ -1 < x < +1\}$,
c) $\{(x, y)$, x and y are integers$\}$,
d) $\{(x, y),\ x + y = 1\}$,
e) $\{(x, y),\ x + y < 1\}$,
f) $\{(x, y),\ x = 0\ or\ y = 0\}$.

15. Which subset of R^2 is open and closed simultaneously?

16. Prove that a closed subset of a compact set is compact.

17. Prove that each finite set is compact.

18. Prove that the intersection of compact sets is compact and that the finite union of compact sets is compact. Show that the infinite union of compact sets need not be compact.

Short Summary
Topology

Limit

A sequence $\{a_1, a_2, ..., a_n, ...\}$ is:
Increasing if for $\forall n\ a_n < a_{n+1}$.
Bounded if $\exists M\ s.t.\ \forall n\ ||a_n|| < M$.
Convergent to a, i.e. $\lim_{n\to\infty} a_n = a$ if $\forall \epsilon > 0\ \exists N\ s.t.\ n > N\ \Rightarrow\ ||a - a_n|| < \epsilon$.
Has an accumulation point a if $\exists$ subsequence convergent to a.

Open Sets

Open ball: $B_\epsilon(x) = \{y,\ d(x, y) < \epsilon\}$.
Interior point $x \in S$: $\exists \epsilon > 0\ s.t.\ B_\epsilon(x) \subset S$.
Open set $S \subset R^n$: $\forall\ x \in S$ is an interior point of S.

Properties of open sets:
1. $\emptyset$ and R^n are open.
2. If each S_α is open, then $\cup S_\alpha$ is open.
3. If S_1 and S_2 are open then $S_1 \cap \cap S_2$ is open.

Interior of S is the maximal open subset of S, i.e. $int\ S = \{x,\ \exists \epsilon > 0\ s.t.\ B_\epsilon(x) \subset S\}$.

Closed Set

Limit point of a set $S \subset R^n$ is x if $\forall \epsilon\ \exists s \in S,\ s \neq x,\ s.t.\ s \in B_\epsilon(x)$.
$x \in S$ is a limit point iff $\exists$ a sequence $\{s_1, s_2, ..., s_n \in S, ...\}$ s.t. $\lim_{n\to\infty} s_n = x$.

Three equivalent definitions of closed set:
Def. 1. $S \subset R^m$ is closed if it contains all its limit points.
Def. 2. $S \subset R^m$ is closed if $s_n \in S, \lim_{n\to\infty} s_n = x\ \Rightarrow\ x \in S$.
Def. 3. S is closed if its *complement* $S^c = R^m - S$ is open.

Properties of closed sets:
1. $\emptyset$ and R^n are closed.
2. If each S_α is closed, then $\cap S_\alpha$ is closed.
3. If S_1 and S_2 are closed then $S_1 \cup S_2$ is closed.

Closure of S is the minimal closed set containing S, i.e. $cl\ S = S \cup \{all\ limit\ points\ of\ S\}.int\ S \subset S \subset cl\ S$.
Boundary of S: $\{b,\ \forall \epsilon\ \exists s \in S\ and\ t \in S^c\ s.t. s \in B_\epsilon(x) \ni t\}$. In fact this is $cl\ S \cap\ cl\ S^c$.
$S \subset R^m$ is **compact** if it is both closed and bounded.
A continuous function $F : S \to R$ on a compact S attains its **supremum**.