[SB], Ch. 13, p. 273-299

1 Functions

A function (map, transformation) from the set X (domain) to the set Y (codomain, or target)

$$f : X \to Y$$

is a rule that assigns to each element $x \in X$ one element $f(x) \in Y$.

The image of f is the set of all elements $y \in Y$ that correspond to *some* x:

$$Im\ f = \{y \in Y, y = f(x)\}.$$

For an element $y \in Y$ its preimage $f^{-1}(y)$ is the set of all elements $x \in X$ such that $f(x) = y$.

More generally, let $V \subset Y$ be a subset of target. The preimage of V is defined as

$$f^{-1}(V) = \{x \in X, f(x) \in V\}.$$

Example. For the function $f : R \to R$ defined by $f(x) = x^2$

$$Im\ f = [0, +\infty), \quad f^{-1}(4) = \{-2, 2\}, \quad f^{-1}(0) = \{0\}, \quad f^{-1}(-9) = \emptyset,$$

$$f^{-1}([0, 9]) = [-3, +3], \quad f^{-1}((2, 9)) = (-3, -\sqrt{2}) \bigcup (\sqrt{2}, 3).$$

1.0.1 Functions $R^n \to R$

In the first miniterm we studied elementary calculus which deals with functions of a single variable. However, most functions which arise in economics involve more than one variable.

Examples.

1. The area of a rectangle with dimensions x and y is a function of two variables $S : R^2 \to R$ given by *quadratic* function

$$S(x, y) = xy.$$

The perimeter of this rectangle is a *linear* function of two variables $P : R^2 \to R$ given by

$$P(x, y) = 2x + 2y.$$

2. The volume of a box with dimensions $x,\ y,\ z$ is a function of three variables $V : R^3 \to R$ given by *cubical* function

$$V(x, y, z) = xyz.$$

The area of the surface is a *quadratic* function of three variables

$$S(x, y, z) = 2xy + 2xz + 2yz.$$

3. The *amount* A is a function of three variables: P*-principal*, r*-annual rate*, t*-time* in years. The function $A : R^3 \to R$ is given by

$$A(P, r, t) = P(1 + rt).$$

4. For a *demand functions* $q = f(p)$ the quantity demanded q is a function of one variable: its own price p.

In reality the demanded quantity depends also on the prices of other goods in the market and on income y :

$$q_1 = f(p_1, p_2, y).$$

A concrete example is the *constant elasticity demand function*

$$q_1 = f(p_1, p_2, y) = k_1 p_1^{a_{11}} p_2^{a_{12}} y^{b_1},$$

where $a_{11},\ a_{12},\ b_1$ are elasticities.

5. Another example of multivariable function in economics is *production function*. Consider a firm which uses n inputs to produce a single output. For $i = 1, ..., n$, let x_i denote the amount of input i. The vector $(x_1, ..., x_n)$ is called an *input bundle*. The firm's production function assigns to each input bundle $(x_1, ..., x_n)$ the amount of output $y = f(x_1, ..., x_n)$.

6. One more example is a *utility function*. Consider an economy with k commodities. Let x_i denote the amount of commodity i. The vector $(x_1, ..., x_k) \in R^k$ is called a *commodity bundle*.

Suppose two bundles $x = (x_1, ..., x_k)$ and $x' = (x'_1, ..., x'_k)$ are given.

Is it possible to say which from these two bundles is preferable?

There is clear ordering on R, we know that $5 > 3,\ 7 > 1$. Also we can say $(5, 3) > (2, 1)$ in R^2, but what about $(5, 3)$ and $(3, 5)$? There is no canonical ordering (preference) on R^n for $n > 1$. Often the preference depends on the context of the problem. Good way to introduce some preference relation on R^n is so called *utility function*.

A *utility function* is a function $u : R^k \to R$ which assigns to a commodity bundle $(x_1, ..., x_k)$ a number $u(x_1, ..., x_k)$ which measures the consumer's degree of satisfaction or utility with the given commodity bundle. Utility function determines preferences: a commodity bundle $x = (x_1, ..., x_k)$ is *preferred* to another bundle $x' = (x'_1, ..., x'_k)$ if

$$u(x_1, ..., x_k) > u(x'_1, ..., x'_k),$$

and x and x' are called *indifferent* if $u(x_1, ..., x_k) = u(x'_1, ..., x'_k)$.

1.0.2 Functions $R^m \to R^n$

A function $F : R^m \to R^n$ in fact is a collection of n real valued functions $\{f_i : R^m \to R,\ i = 1, 2, ..., n\}$:

$$F(x_1, ..., x_m) = (f_1(x_1, ..., x_m), ..., f_n(x_1, ..., x_m)).$$

Examples.
1. If the firm uses three inputs to produce two outputs, we need two separate production functions $q_1 = f_1(x_1, x_2, x_3)$ and $q_2 = f_2(x_1, x_2, x_3)$. In this case, we can write $q = (q_1, q_2) \in R^2$ as an output bundle for this firm and summarize the firm's activities by a function $F : R^3 \to R^2$:

$$F(x_1, x_2, x_3) = (f_1(x_1, x_2, x_3), f_2(x_1, x_2, x_3)).$$

2. The *constant elasticity demand function* for two goods looks as

$$Q(p_1, p_2, y) = (k_1 p_1^{a_{11}} p_2^{a_{12}} y^{b_1}, k_2 p_1^{a_{21}} p_2^{a_{22}} y^{b_2}).$$

1.1 Special Kinds of Functions

1.1.1 Linear Function $R^k \to R^m$

A *linear function* $f : R^k \to R^m$ is a function that preserves the vector space structure

$$f(x + y) = f(x) + f(y), \quad f(kx) = kf(x).$$

Such a function is determined by a $m \times k$ matrix

$$A = \begin{pmatrix} a_{11} & ... & a_{1k} \\ ... & ... & ... \\ a_{m1} & ... & a_{mk} \end{pmatrix}$$

and $f(x) = A \cdot x$ where $x \in R^k$ and $f(x) \in R^m$ are written as column vectors. If you remember the i-th column of A is the column vector $f(e_i)$ where e_i is the i-th ort.

Examples.

1. A linear function $f : R \to R$ has the form

$$f(x) = ax.$$

2. A linear function $f : R^n \to R$ has the form

$$f(x_1, ..., x_n) = a_1x_1 + ... + a_nx_n,$$

in fact this is the inner product

$$f(x_1, ..., x_n) = (a_1, ..., a_n) \cdot (x_1, ..., x_n).$$

3. A linear function $f : R^2 \to R^2$ is determined by a matrix $\begin{pmatrix} a_{11} & a_{1,2} \\ a_{21} & a_{22} \end{pmatrix}$,

$$f(x_1, x_2) = \begin{pmatrix} a_{11} & a_{1,2} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

From this expression easily follows that $f(1, 0) = (a_{11}, a_{21})$ and $f(0, 1) = (a_{12}, a_{22})$, so the column vectors of the matrix are images of basis vectors $(1, 0)$ and $(0, 1)$ (orts).

4. Let $f : R^2 \to R^2$ be the linear map which is rotation of the plane by $90°$ clockwise. Thus

$$f(1, 0) = (0, -1), \quad f(0, 1) = (1, 0),$$

so the matrix of this linear map is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

1.1.2 Quadratic Forms

A *quadratic function* $f : R \to R$ has the form $f(x) = a \cdot x^2$. Generalization of this notion to two variables is the *quadratic form*

$$Q(x_1, x_2) = a_{11}x_1^2 + a_{12}x_1x_2 + a_{21}x_2x_1 + a_{22}x_2^2.$$

Here each term has degree 2 (the sum of exponents is 2 for all summands).

A quadratic form of three variables looks as

$$f(x_1, x_2, x_3) = \\ a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + \\ a_{21}x_2x_1 + a_{22}x_2^2 + a_{23}x_2x_3 + \\ a_{31}x_1x_3 + a_{32}x_3x_2 + a_{33}x_3^3.$$

A general quadratic form of n variables is a real-valued function $Q : R^n \to R$ of the form

$$\begin{array}{rccccccccc} Q(x_1, x_2, ..., x_n) = & a_{11}x_1^2 & + & a_{12}x_1x_2 & + & ... & + & a_{1n}x_1x_n + \\ & a_{21}x_2x_1 & + & a_{22}x_2^2 & + & ... & + & a_{2n}x_2x_n + \\ & ... & ... & ... & ... & & & \\ & a_{n1}x_nx_1 & + & a_{n2}x_nx_2 & + & ... & + & a_{nn}x_n^2 \end{array}$$

In short $Q(x_1, x_2, ..., x_n) = \sum_{i,j}^n a_{ij}x_ix_j$.

As we see a quadratic form is determined by the matrix

$$A = \begin{pmatrix} a_{11} \; ... \; a_{1n} \\ .... \\ a_{n1} \; ... \; a_{nn} \end{pmatrix}.$$

1.1.3 Matrix Representation of Quadratic Forms

Let $Q(x_1, x_2, ..., x_n) = \sum_{i,j}^n a_{ij}x_ix_j$ be a quadratic form with matrix A. Easy to see that

$$Q(x_1, ..., x_n) = (x_1, ..., x_n) \cdot \begin{pmatrix} a_{11} \; ... \; a_{1n} \\ .... \\ a_{n1} \; ... \; a_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ .. \\ x_n \end{pmatrix}.$$

Equivalently $Q(x) = x^T \cdot A \cdot x$.

Example. The quadratic form $Q(x_1, x_2, x_3) = 5x_1^2 - 10x_1x_2 + x_2^2$ whose symmetric matrix is $A = \begin{pmatrix} 5 & -5 \\ -5 & 1 \end{pmatrix}$ is the product of three matrices

$$(x_1, x_2, x_3) \cdot \begin{pmatrix} 5 & -5 \\ -5 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

1.1.4 Symmetrization of matrix

The quadratic form $Q(x_1, x_2, x_3) = 5x_1^2 - 10x_1x_2 + x_2^2$ can be represented by each of following 2×2 matrix

$$\begin{pmatrix} 5 & -2 \\ -8 & 1 \end{pmatrix}, \quad \begin{pmatrix} 5 & -3 \\ -7 & 1 \end{pmatrix}, \quad \begin{pmatrix} 5 & -5 \\ -5 & 1 \end{pmatrix}$$

the last one is *symmetric*: $a_{ij} = a_{ji}$.

Theorem 1 *Any quadratic form can be represented by symmetric matrix.*

Indeed, if $a_{ij} \neq a_{ji}$ we replace them by new $a'_{ij} = a'_{ji} = \frac{a_{ij}+a_{ji}}{2}$, this does not change the corresponding quadratic form. Generally the symmetrized matrix A' in fact is $A' = \frac{A+A^T}{2}$.

1.1.5 Polynomials

A *monomial* is a function $f : R^k \to R$ of the form

$$f(x_1, ..., x_k) = cx_1^{a_1} \cdot ... \cdot x_k^{a_k},$$

the sum $a_1 + ... + a_k$ is called the *degree* of monomial.

A *polynomial* is the finite sum of monomials. The degree of polynomial is the highest degree of it's monomials.

1.1.6 Continuous Functions

A function $F : R^k \to R^M$ is continuous at x_0 if whenever a sequence $\{x_n\}$ converges to x_0, the sequence $\{F(x_n)\}$ converges to $F(x_0)$.

1.2 General Notions About Functions

1.2.1 Surjections, Injections, Bijections

A function $f : X \to Y$ is called *surjective* (onto) if for each $y \in Y$ there exists $x \in X$ such that $f(x) = y$.

A function $f : X \to Y$ is called *injective* (one-to-one) if

$$f(x_1) = f(x_2) \quad \Rightarrow \quad x_1 = x_2.$$

A function is called *bijection* if it is a surjection and injection simultaneously.

In other words:
f is a surjection if the equation $f(x) = y$ has *at least* one solution;
f is an injection if the equation $f(x) = y$ has *at most* one solution.
f is bijection if the equation $f(x) = y$ has *exactly* one solution.

1.2.2 Composition of Functions

Suppose $f : X \to Y$ and $g : Y \to Z$. The *composition* $g \circ f : X \to Z$ is defined by $g \circ f(x) = g(f(x))$.

Example

The function $h : R^2 \to R$, $h(x, y) = (x^2y)^3 + x^2y$ is the composition $h = f \circ g : R^2 \xrightarrow{g} R \xrightarrow{f} R$ with $g(x, y) = x^2y$ and $f(z) = z^3 + z$.

But not only: $h = F \circ G : R^2 \xrightarrow{G} R^2 \xrightarrow{F} R$ with $G(x, y) = (x^2, y)$ and $F(u, v) = (uv)^3 + uv$.

1.2.3 Inverse Function

When $f : X \to Y$ is *bijective*, there is an *inverse* function $g : Y \to X$ which assigns to $y \in Y$ the unique element $g(y) = x$ such that $f(x) = y$.

It is clear that $g \cdot f(x) = x$ and $f \cdot g(y) = y$ for arbitrary x and y in this case.

More explicitly, let

$$f : X \to Y \quad g : Y \to X.$$

g is left inverse of f iff $g \circ f = id_X$.
g is right inverse of f iff $f \circ g = id_Y$.
g is inverse of f iff $g \circ f = id_X$ and $f \circ g = id_Y$.

f is injective iff it has a left inverse.
f is surjective iff it has a right inverse.
f is bijective iff it has the inverse.

Example

Consider the function given by $f(x) = \sqrt{x-1}$.
Domain: $x - 1 \geq 0, \quad x \geq 1, \quad x \in [1, +\infty]$.
Range: $y \geq 0, \quad y \in [0, +\infty)$.
So $f : [1, +\infty) \to [0, +\infty)$ is surjective. Is it injective? Yes:
Suppose $f(x_1) = f(x_2)$, i.e. $\sqrt{x_1 - 1} = \sqrt{x_2 - 1}$, then squaring both sides $x_1 - 1 = x_2 - 1$ thus $x_1 = x_2$.
Inverse: solve x from $y = f(x)$:
$y = \sqrt{x-1}, \quad y^2 = x - 1, \quad x = y^2 + 1$ so the inverse function is $g(y) = y^2 + 1$.

Exercises

1. Draw a significant number of level curves and the graphs of the following functions:

$a)\ z = x^2 + y^2;\quad b)\ z = -y^2 - x^2;\quad c)\ z = x^2 - y^2;\quad d)\ z = x \cdot y;$
$e)\ z = y^2;\quad f)\ z = x^2;\quad g)\ z = (y - x)^2;\quad h)\ z = (x - y)^2.$

2. Sketch each of the following parameterized curves:

$a)\ f(t) = (4 - 2t, 1 + t);\quad b)\ f(t) = (t^2, t^2 + 2); c)\ f(t) = (\sqrt{t}, 1 - t).$

3. Write the following linear functions in matrix form
(a) $f : R^3 \to R$ given by $f(x_1, x_2, x_3) = 2x_1 - 3x_2 + 5x_3$.
(b) $f : R^2 \to R^?$ given by $f(x_1, x_2) = (2x_1 - 3x_2, x_1 - 4x_2, x_1)$.
(c) $f : R^? \to R^?$ given by $f(x_1, x_2, x_3) = (x_1 - x_3, 2x_1 + 3x_2 - 6x_3, x_3 - 2x_2)$.

4. Write the following quadratic functions in matrix form
(a) $x_1^2 - 2x_1x_2 + x_2^2$;
(b) $5x_1^2 - 10x_1x_2 - x_2^2$
(c) $x_1^2 + 2x_2^2 + 3x_3^2 + 4x_1x_2 - 6x_1x_3 + 8x2x_3$.

5. For each of the following functions, what is the domain and image of f? Which of them are one-to-ones (injective)? For those that are one-to one, write the inverse function. Which of them are onto (surjective) on R?

$a)\ f(x) = 3x - 7;\quad b)\ f(x) = x^2 - 1;\quad c)\ f(x) = e^x;$
$d)\ f(x) = x^3 - x;\quad e)\ f(x) = \frac{x}{x^2+1};\quad f)\ f(x) = x^3;$
$g)\ f(x) = \frac{1}{x};\quad h)\ f(x) = \sqrt{x - 1};\quad i)\ f(x) = ln\ x.$

5. For each of the following function, write h as a composition of two functions f and g:

$a)\ h(x) = log(x^2 + 1);\quad b)\ h(x) = (sin\ x)^2;$
$c)\ h(x) = (cos\ x^3, sin\ x^3);\quad d)\ h(x, y) = (x^2y)^3 + x^2y.$

6. Evaluate the integrals using two methods, that is $dxdy$ and $dydx$
$\int_{y=1}^{2} \int_{x=0}^{3} (1 - 8xy)dxdy$
$\int\int_R (4 - x - y)dxdy$ where $R = \{(x, y),\ x \in [0, 1], y \in [0, 2]\}$
$\int\int_R y^2 x dxdy$ where R is the area between the graphs of $y = x^2$ and $y = x$.

Homework

13.1(b), 13.11(c), 13.12(c), 13.23(i) from [SB],
Evaluate the integrals
$\int\int_R (4 - x - y)dxdy$ where $R = \{(x, y),\ x \in [-1, 1], y \in [0, 2]\}$
$\int\int_R y^2 x dxdy$ where R is the area between the graphs of $y = x^2$ and $y = 2x$.

Short Summary
Maps

For a function (map) $f : X \to Y$:
Image $Im\ f = \{y \in Y, y = f(x)\}$.
Preimage $f^{-1}(V) = \{x \in X, f(x) \in V\}$.

A Function $F : R^m \to R^n$: $F(x_1, ..., x_m) = (f_1(x_1, ..., x_m), ..., f_n(x_1, ..., x_m))$.

A **linear function** $F : R^k \to R^m$:

$$F(x) = \begin{pmatrix} a_{11} & ... & a_{1k} \\ ... & ... & ... \\ a_{m1} & ... & a_{mk} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ ... \\ x_k \end{pmatrix}. \quad Here \quad \begin{pmatrix} a_{1i} \\ ... \\ a_{mi} \end{pmatrix} = F(e_i).$$

Quadratic Forms

$$Q(x_1, x_2, ..., x_n) = \sum_{i,j}^{n} a_{ij} x_i x_j = (x_1, ..., x_n) \cdot \begin{pmatrix} a_{11} \ ... \ a_{1n} \\ .... \\ a_{n1} \ ... \ a_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ .. \\ x_n \end{pmatrix} = x^T \cdot A \cdot x$$

A is symmetric. If not, take its symmetrization $A' = \frac{A+A^T}{2}$.

Monomial of degree $a_1 + ... + a_k$: $f(x_1, ..., x_k) = c x_1^{a_1} \cdot ... \cdot x_k^{a_k}$.
A **polynomial** is the finite sum of monomials. The degree of polynomial is the highest degree of it's monomials.

A function $F : R^k \to R^M$ is **continuous** at x_0 if $\lim_{n\to\infty} x_n = x_0 \Rightarrow \lim_{n\to\infty} F(x_n) = F(x_0)$.

Surjections, Injections, Bijections
A function $f : X \to Y$ is;
Surjective if $\forall y \in Y\ \exists x \in X\ s.t.\ f(x) = y$.
Injective if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.
Bijective if it is a surjection and injection simultaneously.

In other words:
f is a surjection if $f(x) = y$ has *at least* one solution;
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Composition of Functions
For $f : X \to Y$ and $g : Y \to Z$ the *composition* $gf : X \to Z$ is defined by $g \cdot f(x) = g(f(x)$.

Inverse Function
Let $f : X \to Y \quad g : Y \to X$.
g is left inverse of f iff $g \circ f = id_X$.
g is right inverse of f iff $f \circ g = id_Y$.

g is inverse of f iff $g \circ f = id_X$ and $f \circ g = id_Y$.

f is injective iff it has a left inverse.
f is surjective iff it has a right inverse.
f is bijective iff it has the inverse.