ISET MATH III Midterm

Problem 1. Find all values of k (if any) for which the 2 variable quadratic form $x^2 + 2kxy + y^2$ is
(a) positive definite; (b) negative definite; (c) positive semidefinite; (d) negative semidefinite; (e) indefinite.

Problem 2. Consider the following constrained quadratic form

$$Q(x, y, z) = x^2 - z^2 + 4xy \quad subject \ \ to \quad 5x + k \cdot y - k \cdot z = 0.$$

(a) For warming up: Determine the definiteness of this constrained quadratic for $k = 1$.

Find all values of k for which the constrained quadratic is
(b) negative definite;
(c) positive definite;
(d) indefinite.

Problem 3. Let $Q(x, y)$ be a 2-variable quadratic form and $Ax + By = 0$ be a linear constraint.

(i) Show that if $Q(x, y)$ is positive definite, then its restriction on any linear constraint $Ax + By = 0$ is also positive definite.

(ii) Show that for indefinite quadratic form $Q(x, y)$ the restriction on a linear constraint $Ax + By = 0$ can be (a) positive, (b) negative, (c) semipositive, (d) seminegative. Namely give an example of one indefinite quadratic form $Q(x, y)$ and indicate various linear constraints $Ax+By = 0$ which realize the possibilities (a), (b), (c), (d).

Problem 4. Formulate a criteria for positive definiteness and for negative definiteness of a quadratic form with n variables and $n-1$ constraints.

Problem 5.

(i) Consider the curve (t^3, t) in R^2. (a) Plot this curve for $-1 \leq t \leq 1$; (b) Identify all the points (if any) where the tangent vector is vertical; (c) Identify all the points (if any) where the tangent vector is horizontal.

(ii) Let $F : R^3 \to R$ be a function given by $F(x, y, z) = xy + yz + xz$ and $\phi : R \to R^3$ be a curve given by $\phi(t) = (x(t) = t^2, y(t) = 1 - t^2, z(t) = 1 - t)$. Calculate $\frac{dF}{dt}$.

Problem 6.

(i) Let $F : R^2 \to R^2$ be the function given by $F(x,y) = \begin{pmatrix} x^2y + y^2 \\ y^3x \end{pmatrix}$. Find the value of Jacobian DF at $(1,2)$.

(ii) Let $f(x,y) = \sqrt{x \cdot y}$. Use gradient to approximate $f(1.01, 3.99)$.

(iii) Consider the function $f(x,y) = y^2e^{3x}$. In which direction should one move from the point $(0,3)$ to increase most rapidly? Express your answer as a vector of length 1.

Problem 7. Sketch the level curves $f(x, y) = 1$ for the following functions

(a) $f(x, y) = max(x, y)$
(b) $f(x, y) = min(x, y)$
(c) $f(x, y) = x \cdot y + 1$
(d) $f(x, y) = x^2 - y^2 + 1$
(e) $f(x, y) = xy - x - y + 2$.

Problem 8. A bat named Bob moves along a path such that his position at time t is $(2t, t^2, 1 + t^2)$ from $t = 0$ until time $t = 2$. At time $t = 2$, he leaves this pat and flies along the tangent line to this path, maintaining the speed he had at time $t = 2$. What will Bob's position be at time $t = 5$?

ADDITIONAL PAPER

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Solutions