1 Linear Programming

A linear programming problem is the problem of maximizing (or minimizing) a linear function subject to linear constraints. The constraints may be equalities or inequalities.

1.1 Example

Problem. There are two ports P_1 and P_2 where $s_1 = 10$ and $s_2 = 30$ tons of some commodity are stored respectively. The demand of the supermarket M in this commodity is minimum 20 tons. The transportation of 1 ton from P_1 to M costs \$3 and from P_2 to M costs \$5. The problem is to meet the market requirements at minimum transportation cost.

Let the amount of commodity shipped from P_1 and P_2 to M be y_1 and y_2 respectively. Then we must minimize the *objective function*

$$3y_1 + 5y_2$$

subject of the following constraints

$$\begin{aligned}
&y_1 + y_2 \geq 20\\
&y_1 \leq 10,\\
&y_2 \leq 30,\\
&y_1 \geq 0,\ \ y_2 \geq 0.
\end{aligned}$$

Using Maple

$>$with(Optimization):
$>$ LPSolve($3x + 5y, \{x + y >= 20, x <= 10, y <= 30, x >= 0, y >= 0\}$);
$[80, [x = 10, y = 10]]$

1.2 Graphical Method

To solve this problem first we must find the *feasible set*, that is the set of all points $(x, y) \in R^2$ which satisfy all above constraints. For this we must graph the solution of *each* of these inequalities separately and the final feasible set will be the intersection of all these solution sets.

This might be:
(a) Empty set, in this case the problem is called *unfeasible* and has no solution;

(b) A bounded *polygon*. In this case the problem is *feasible*. To solve the problem compute the coordinates of all *corner points* of the feasible polygon, substitute the coordinates of the corner points into the objective function to see which gives the optimal value.

(c) An unbounded polygon. In this case a solution may or may not exist.

1.2.1 How to graph the solution of a linear inequality

Solution of the inequality $3x - 4y \leq 12$:

Step 1. First sketch the line $3x - 4y = 12$.

Find the y-intercept: $x = 0$ and y can be solved from $3 \cdot 0 - 4y = 12,\ y = -3$, so the y-intercept is the point $(0, -3)$.

Find the x-intercept: $y = 0$ and x can be solved from $3 \cdot x - 4 \cdot 0 = 12,\ x = 4$, so the x-intercept is $(4, 0)$.

This two points are enough to sketch the graph of $3x - 4y = 12$.

This graph divides the plane into two regions, the *upper* one and the *lower* one. Which one is the solution of our inequality $3x - 4y \leq 12$?

Step 2. Choose the origin $(0, 0)$ as the *test point* (since it is not on the line). Substituting $x = 0, y = 0$ in the inequality gives $3 \cdot 0 - 4 \cdot 0 \leq 12$. Since this is a true statement, $(0, 0)$ is in the solution set, so the solution set consists of all points on the same side as $(0, 0)$.

Choose some another test point if $(0, 0)$ is on the line.

1.2.2 How to find corner points

To find the corner points we must just solve the systems of linear equations.

For example the corner point which corresponds to two constrains $ax + by \leq c$ and $px + qy \leq r$ is the solution of the system

$$\begin{cases} ax + by = c \\ px + qy = r. \end{cases}.$$

1.3 Standard Form of Linear Programming Problem

Maximize a linear function (called *objective function*)

$$f(x_1, ..., x_n) = c_1 x_1 + ... + c_n x_n$$

subject if following constraints:

$$\begin{array}{l} a_{11}x_1 + ... + a_{1n}x_n \leq b_1 \\ \dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + ... + a_{mn}x_n \leq b_m \\ \\ x_1 \geq 0, \ \dots \ x_n \geq 0. \end{array}$$

In matrix form: Maximize $c^T x$ subject of $Ax \leq b, \quad x \geq 0$.
Remark 1. An *equality* constraint

$$a_{11}x_1 + ... + a_{1n}x_n = b_1$$

can be converted to *inequality* by introducing two inequality constraints

$$\begin{array}{l} a_{11}x_1 + ... + a_{1n}x_n \leq b_1, \\ a_{11}x_1 + ... + a_{1n}x_n \geq b_1 \end{array}$$

or, equivalently,

$$\begin{array}{l} a_{11}x_1 + ... + a_{1n}x_n \leq b_1, \\ -a_{11}x_{-}... - a_{1n}x_n \leq -b_1. \end{array}$$

So we can assume that all constraints are given inequalities.

Remark 2. Sometimes it is more convenient to work with equality constraints. An *inequality* constraint

$$a_{11}x_1 + ... + a_{1n}x_n \leq b$$

can be converted to an *equality* constraint by adding a new nonnegative variable, w, which is called a *slack variable*,

$$\begin{aligned} &a_{11}x_1 + ... + a_{1n}x_n + w = b, \\ &w \geq 0. \end{aligned}$$

So we can assume that all constraints are given equalities.

1.3.1 Fundamental Theorem of Linear Programming

Each of the constraints of a standard LP problem defines a closed half-space of R^n, which is a convex set. The whole feasible region, as the intersection of convex sets, is a convex set too.

Given a convex set S, point x_0 is an *extreme point*, if each line segment that lies completely in S and contains point x_0, has x_0 as an end point of the line segment.

Theorem 1 *If a LP problem has a solution then there exists an extreme point of the feasible region which is optimal.*

1.3.2 The Transportation Problem

There are I ports, or production plants, $P_1, ..., P_I$, that supply a certain commodity, and there are J markets, $M_1, ..., M_J$, to which this commodity must be shipped. Port P_i possesses an amount s_i of the commodity ($i = 1, 2, ..., I$), and market M_j must receive the amount r_j of the commodity ($j = 1, ..., J$). Let b_{ij} be the cost of transporting one unit of the commodity from port P_i to market M_j. The problem is to meet the market requirements at minimum transportation cost.

Let y_{ij} be the quantity of the commodity shipped from port P_i to market M_j. The total transportation cost is

$$\sum_{i=1}^{I} \sum_{j=1}^{J} b_{ij} y_{ij}.$$

The amount sent from port P_i is $\sum_{j=1}^{J} y_{ij}$ and since the amount available at port P_i is s_i, we must have

$$\sum_{j=1}^{J} y_{ij} \leq s_i, \quad i = 1, 2, ..., I.$$

The amount sent to market M_j is $\sum_{i=1}^{I} y_{ij}$, and since the amount required there is r_j, we must have

$$\sum_{i=1}^{I} y_{ij} \geq r_j, \quad j = 1, 2, ..., J.$$

It is assumed that we cannot send a negative amount from P_i to M_j, we have

$$y_{ij} \geq 0, \quad for \;\; i = 1, ..., I \;\; and \;\; j = 1, ..., J.$$

Our problem is: minimize

$$\sum_{i=1}^{I} \sum_{j=1}^{J} b_{ij} y_{ij}.$$

subject to

$$\sum_{j=1}^{J} y_{ij} \leq s_i, \quad i = 1, 2, ..., I.$$

$$\sum_{i=1}^{I} y_{ij} \geq r_i, \quad j = 1, 2, ..., J.$$

$$y_{ij} \geq 0, \quad i = 1, ..., I, \;\; j = 1, ..., J.$$

Exercises

Solve the following LP problems using graphical method and check the solution using Maple.

1. Find $x,\ y$ to minimize $x+y$ subject of the constraints

$$\begin{aligned} &x+2y \geq 3 \\ &2x+y \geq 5 \\ &y \geq 0. \end{aligned}$$

2. Find $x_1,\ x_2$ to maximize x_1+x_2 subject of the constraints

$$\begin{aligned} &x_1+2x_2 \leq 4 \\ &4x_1+2x_2 \leq 12 \\ &-x_1+x_2 \leq 1 \\ &x_1 \geq 0,\ x_2 \geq 0. \end{aligned}$$

3. Consider the LP problem below. The CVH Company of Leeds is a revenue maximizing firm. It has chosen to focus on the production and sale of two goods: A and B. Good A requires 1 hour of skilled labor and 4 kilos of metal. It is a standard product and retails for \$40. Good B is a higher quality version of good A. It needs 2 hours of skilled labor and demands 3 kilos of metal. It sells for a slightly higher price of \$50 per unit. Under current operating conditions, the firm is prepared to hire 40 hours worth of labor each day and it has a long term contract with a supplier who can deliver 120 kilos of metal each day. Your task is to construct a simple LP and graph to solve.

4. MFG is small car manufacturer producing two models of handmade cars. These cars are prestige vehicles and command a very high price. The two models are well known throughout the specialist car market as X and Y. Both car X and car Y are very profitable. Car X produces \$2,000 of profit for each vehicle produced whilst Car Y produces \$3,000. the firm is motivated purely by profit. The company cannot produce as many cars as it would like. It cannot produce more than 15 cars in a year because they all have to be hand finished. For operational reasons, there are also limitations on the

combinations of cars X and Y which can be manufactured in a given year. It is not possible given the current technology for more than 3 model Y cars to be produced for each model X car. This can create a headache for MFG.

(a) Write in words MFGs objective function

(b) Translate the objective function into a simple expression using X to show the number of model X and Y to denote the number of model Y vehicles that can be manufactured in a year.

(c) How many constraints are there to this problem? Work out what the expressions are for these constraints.

(d) Construct the linear program. How many of each should be produced?