1 Linear Transformations

1.1 Linear Function $R \to R$

A linear function $f : R \to R$ is a function which satisfies two conditions

$$f(x + x') = f(x) + f(x'), \quad x, x' \in R;$$
$$f(c \cdot x) = c \cdot f(x), \quad c,\ x \in R.$$

Such a function has the form

$$f(x) = k \cdot x,$$

where $k \in R$ is some *scalar*.

1.2 Linear Function $R^n \to R$

A linear function $f : R^n \to R$ is a function which satisfies two conditions

$$f(v + w) = f(v) + f(w), \quad v,\ w \in R^n;$$
$$f(c \cdot v) = c \cdot f(v), \quad v \in R^n, \quad c \in R.$$

Such a function has the form

$$f(v) = k_1 \cdot x_1 + \ \dots\ + k_n \cdot x_n,$$

where $v = (x_1,\ \dots\ , x_n), \quad k = (k_1,\ \dots\ , k_n)$.

Thus any linear function $f : R^n \to R$ has the form

$$f(v) = k \cdot v$$

where $k \in R^n$ is considered as a *vector*.

1.3 Linear Function $R^n \to R^m$

A *linear function* $f : R^n \to R^m$ is a function which satisfies two conditions

$$f(v + w) = f(v) + f(w), \quad v,\ w \in R^n;$$
$$f(c \cdot x) = c \cdot f(x)\ v \in R^n, \quad c \in R.$$

Such a function has the form

$$f(v) = (a_{11} \cdot x_1 + \ \dots\ + a_{1n} \cdot x_n,\ \dots\ , a_{m1} \cdot x_1 + \ \dots\ + a_{mn} \cdot x_n) \in R^m.$$

Thus any linear function $f : R^n \to R^m$ has the form

$$f(v) = A \cdot v$$

where A is some matrix

$$A = \begin{pmatrix} a_{11} & ... & a_{1n} \\ ... & & \\ a_{m1} & ... & a_{mn} \end{pmatrix}.$$

A linear function $f : R^2 \to R^2$ is determined by a matrix $A = \begin{pmatrix} a_{11} & a_{1,2} \\ a_{21} & a_{22} \end{pmatrix}$,

$$f(x_1, x_2) = \begin{pmatrix} a_{11} & a_{1,2} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11} \cdot x_1 + a_{12} \cdot x_2 \\ a_{21} \cdot x_1 + a_{22} \cdot x_2 \end{pmatrix}.$$

From this expression easily follows that

$$f \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}, \quad f \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix},$$

so the column vectors of the matrix A are images of basis vectors $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Theorem 1 *Suppose $f : R^n \to R^m$ is a linear map. Suppose also that the images of the basis vectors*

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ ... \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ ... \\ 0 \end{pmatrix}, \quad ... \quad , \quad e_n = \begin{pmatrix} 0 \\ 0 \\ ... \\ 1 \end{pmatrix}$$

are the column vectors

$$f(e_1) = \begin{pmatrix} a_{11} \\ a_{21} \\ ... \\ a_{m1} \end{pmatrix}, \quad f(e_2) = \begin{pmatrix} a_{12} \\ a_{22} \\ ... \\ a_{m2} \end{pmatrix}, \quad ... \quad , \quad f(e_n) = \begin{pmatrix} a_{n1} \\ a_{n2} \\ ... \\ a_{nm} \end{pmatrix}.$$

Then

$$A = \begin{pmatrix} a_{11} & ... & a_{1n} \\ ... & & \\ a_{m1} & ... & a_{mn} \end{pmatrix}.$$

is the matrix of f.

Example 1. Let $f : R^2 \to R^2$ be the linear map which is *rotation* of the plane by 90° clockwise. Find $f(2, 3)$.

The values of basis vectors are

$$f(1,0) = (0,-1), \quad f(0,1) = (1,0),$$

so the matrix of this linear map is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Thus

$$f(2,3) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}.$$

Example 2. Let $g : R^2 \to R^2$ be the linear map which is the *expansion* 2 times. Let us find it's matrix.

The values of basis vectors are

$$g(1,0) = (2,0), \quad g(0,1) = (0,2),$$

so the matrix of this linear map is $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

Example 3. Let $g : R^2 \to R^2$ be the linear map which is the *unequal expansion* in two perpendicular directions: 2 times in direction x and 3 times in direction y. Let us find it's matrix.

The values of basis vectors are

$$g(1,0) = (2,0), \quad g(0,1) = (0,3),$$

so the matrix of this linear map is $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

Example 4. Let $p : R^2 \to R^2$ be the *projection* on x axes: $f(x,y) = (x,0)$. Let us find it's matrix.

The values of basis vectors are

$$p(1,0) = (1,0), \quad p(0,1) = (0,0),$$

so the matrix of this linear map is $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Example 5. Let $h : R^2 \to R^2$ be the linear map which is the *reflection* with respect to y axes. Let us find it's matrix.

The values of basis vectors are

$$g(1,0) = (-1,0), \quad g(0,1) = (0,1),$$

so the matrix of this linear map is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

Theorem 2 *A linear map $F : R^n \to R^n$ given by a matrix A is bijective if and only if $det(A) \neq 0$.*

Try to prove this!

2 Eigenvalues and Eigenvectors

Let

$$A = \begin{pmatrix} a_{11} & ... & a_{1n} \\ ... & ... & ... \\ a_{n1} & ... & a_{nn} \end{pmatrix}$$

be a matrix, which, as we know, defines a linear map $F : R^n \to R^n$ defined by $F(x) = A \cdot x$.

A scalar $\lambda \in R$ and a nonzero vector $x \in R^n$ are called respectively *eigenvalue* and *eigenvector* of A if

$$A \cdot x = \lambda \cdot x.$$

This actually means that the linear map F changes the magnitude of x but not its direction,

Note that if x is an eigenvector corresponding to an eigenvalue λ then kx is an eigenvector too: $A \cdot (kx) = kA \cdot x = k\lambda x = \lambda(kx)$.

The *specter* of A (denoted by $spec(A)$) is defined as the set of all eigenvalues $\lambda_1, ..., \lambda_k$ of A.

Eigenspace corresponding to an eigenvalue λ is defined as the subspace spanned by all eigenvectors corresponding to this eigenvalue.

The *geometric degree* of an eigenvalue λ is defined as the dimension of its eigenspace.

Let us observe examples 1-6 from previous section.

Example 1. Rotation $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. No eigenvalues and eigenvectors. Check!

Example 2. Expansion 2 times, $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Eigenvector $\lambda = 2$, eigenvector - any nonzero vector, eigenspace - whole R^2. Check!

Example 3. Unequal expansion $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. Eigenvalues $\lambda_1 = 2, \quad \lambda_2 = 3$, corresponding eigenvectors $v_1 = (1, 0), \quad v_2 = (0, 1)$. Check!

Example 4. Projection on x-axes $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Eigenvalues $\lambda_1 = 1, \;\; \lambda_2 = 0$, corresponding eigenvectors $v_1 = (1, 0), \;\; v_2 = (0, 1)$. Check!

Example 5. Reflection about the y-axes $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Eigenvalues $\lambda_1 = -1, \;\; \lambda_2 = 1$, corresponding eigenvectors $v_1 = (1, 0), \;\; v_2 = (0, 1)$. Check!

Example 6. Horizontal shear $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Eigenvalue $\lambda = 1$, corresponding eigenvector $v_1 = (1, 0)$. Check!

2.0.1 How to Find Eigenvalues and Eigenvectors

These can be found solving the matrix equation $A \cdot x = \lambda \cdot x$, equivalently $(A - \lambda I)x = 0$, which in its turn is the system

$$\begin{cases} (a_{11} - \lambda)x_1 & + & a_{12}x_2 & + & ... & + & a_{1n}x_n = 0 \\ a_{21}x_1 & + & (a_{22} - \lambda)x_2 & + & ... & + & a_{2n}x_n = 0 \\ ... & ... & ... & ... & & & \\ a_{n1}x_1 & + & a_{n2}x_2 & + & ... & + & (a_{nn} - \lambda)x_n = 0 \end{cases}.$$

This is homogenous system so it has a nonzero solution if and only if its determinant $|A - \lambda I|$ (which is called *characteristic polynomial* of A) is zero, so $|A - \lambda I| = 0$.

So, the eigenvalues can be found from the *characteristic equation* $|A - \lambda I| = 0$ that is

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} - \lambda & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} - \lambda \end{vmatrix} = 0.$$

Algebraic degree of an eigenvalue $\lambda^* \in Spec(A)$ is defined as its multiplicity in characteristic polynomial: $AlgDeg(\lambda) = k$ if $|A - \lambda I| = (\lambda - \lambda^*)^k \cdot Q(\lambda)$ where $Q(\lambda)$ is some polynomial.

The algebraic degree of an eigenvalue λ is more or equal to its geometric degree.

Example. Find the eigenvalues for the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Solution. The characteristic equation looks as

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0.$$

Calculating this determinant we obtain

$$(1-\lambda)^3 - 3(1-\lambda) + 2 = 0, \quad \lambda^3 - 3\lambda^2 = 0, \quad \lambda^2(\lambda - 3) = 0,$$

thus $\lambda_1 = 0,\ \lambda_2 = 3$. The algebraic degree of $\lambda_1 = 0$ is 2, and of $\lambda_2 = 3$ is 1.

2.0.2 How to Find Eigenvectors

Eigenvectors corresponding to the eigenvalue λ can be found solving the matrix equation

$$(A - \lambda I)x = 0$$

which is equivalent to the system

$$\left\{ \begin{array}{ccccccc} (a_{11}-\lambda)x_1 & + & a_{12}x_2 & + & ... & + & a_{1n}x_n = 0 \\ a_{21}x_1 & + & (a_{22}-\lambda)x_2 & + & ... & + & a_{2n}x_n = 0 \\ ... & ... & ... & ... & & & \\ a_{n1}x_1 & + & a_{n2}x_2 & + & ... & + & (a_{nn}-\lambda)x_n = 0 \end{array} \right. .$$

Since λ is an eigenvalue the determinant of this system is zero. Thus this homogenous system *has* nonzero solutions.

2.1 Examples

Example. Find an eigenvector x corresponding to the eigenvalue $\lambda = 3$ of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

from the previous example.
Solution. We can find x from the matrix equation $(A - 3 \cdot I) \cdot x = 0$ which as a system of linear equations looks as

$$\left\{ \begin{array}{ccccc} (1-3)x_1 & + & x_2 & + & x_3 = 0 \\ x_1 & + & (1-3)x_2 & + & x_3 = 0 \\ x_1 & + & x_2 & + & (1-3)x_3 = 0 \end{array} \right| ,$$

$$\left\{ \begin{array}{ccccc} -2x_1 & + & x_2 & + & x_3 = 0 \\ x_1 & - & 2x_2 & + & x_3 = 0 \\ x_1 & + & x_2 & - & 2x_3 = 0 \end{array} \right| .$$

Rank of the determinant of this system is 2: a nonzero minor is

$$\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = -5.$$

Thus we can ignore the third equation and the system is equivalent to

$$\left\{ \begin{array}{ccccc} -2x_1 & + & x_2 & = -x_3 \\ x_1 & + & -2x_2 & = -x_3 \end{array} \right| .$$

Here

$$\Delta = 3, \quad \Delta_1 = \begin{vmatrix} -x_3 & 1 \\ -x_3 & -2 \end{vmatrix} = 3x_3, \quad \Delta_2 = \begin{vmatrix} -2 & -x_3 \\ 1 & -x_3 \end{vmatrix} = 3x_3,$$

thus

$$x_1 = \frac{3x_3}{3} = x_3, \quad x_2 = \frac{3x_3}{3} = x_3.$$

So (x_3, x_3, x_3) is a general solution of our system with exogenous variable x_3. Taking this variable $x_3 = 1$ we obtain the eigenvector $x = (1, 1, 1)$. As we see the geometric degree of eigenvalue $\lambda = 3$ is 1, as well as its algebraic degree.

Example. Find an eigenvector x corresponding to the eigenvalue $\lambda = 0$ of the same matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

from the previous example.
Solution. We can find x from the matrix equation $(A - 0 \cdot I) \cdot x = 0$ which as a system of linear equations looks as

$$\left\{ \begin{array}{ccccc} (1-0)x_1 & + & x_2 & + & x_3 = 0 \\ x_1 & + & (1-0)x_2 & + & x_3 = 0 \\ x_1 & + & x_2 & + & (1-0)x_3 = 0 \end{array} \right| ,$$

$$\left\{ \begin{array}{ccccc} x_1 & + & x_2 & + & x_3 = 0 \\ x_1 & + & x_2 & + & x_3 = 0 \\ x_1 & + & x_2 & + & x_3 = 0 \end{array} \right| .$$

Rank of the determinant of this system is 1, and its general solution is

$$(x_1 = -x_2 - x_3, x_2, x_3)$$

with exogenous variable x_2, x_3. Taking this variables $x_2 = 1, x_3 = 0$ we obtain the eigenvector $v = (-1, 1, 0)$, and taking this variables $x_2 = 0, x_3 = 1$ we obtain the eigenvector $v = (-1, 0, 1)$. As we see the geometric degree of eigenvalue $\lambda = 0$ is 2, as well as its algebraic degree.

Example. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}.$$

Solution. The characteristic equation of the matrix A looks as

$$A = \begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = 0 \ \ , \lambda^2 - 5\lambda + 4 = 0.$$

The roots of this equation, that is the eigenvalues are $\lambda_1 = 1, \quad \lambda_2 = 4$.

The eigenvectors can be found solving the system of equations

$$\begin{cases} (2-\lambda)x_1 + & 2x_2 = 0 \\ x_1 + & (3-\lambda)x_2 = 0 \end{cases}$$

For $\lambda = 1$:

$$\left.\begin{cases} (2-1)x_1 + & 2x_2 = 0 \\ x_1 + & (3-1)x_2 = 0 \end{cases}\right| , \quad \left.\begin{matrix} x_1 + & 2x_2 = 0 \\ x_1 + & 2x_2 = 0 \end{matrix}\right|$$

$$x_1 + 2x_2 = 0, x_1 = 2x_2,$$

thus the solution depending on the free parameter x_2 is $(2x_2, x_2)$. Taking, say, $x_2 = 1$ we obtain the eigenvector $v_1 = (2, 1)$.

For $\lambda = 4$:

$$\left.\begin{cases} (2-4)x_1 + & 2x_2 = 0 \\ x_1 + & (3-4)x_2 = 0 \end{cases}\right| , \quad \left.\begin{matrix} -2x_1 + & 2x_2 = 0 \\ x_1 - & x_2 = 0 \end{matrix}\right|$$

$$x_1 - x_2 = 0, \quad x_1 = x_2,$$

thus the solution depending on the free parameter x_2 is (x_2, x_2). Taking, say, $x_2 = 1$ we obtain the eigenvector $v_1 = (1, 1)$.

Example. Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

(horizontal shear).

Then $|A - \lambda I| = (1-\lambda)^2$ thus there is one eigenvalue $\lambda = 1$ of multiplicity 2. Eigenvectors are solutions of the system

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

that is

$$\begin{cases} 0 \cdot x + 1 \cdot y = 0 \\ 0 \cdot x + 0 \cdot y = 0 \end{cases}.$$

The solution of this system is $(x, 0)$, the x-axes, so the geometric multiplicity of $\lambda = 1$ is 1, so it is less then its algebraic multiplicity.

Example. Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $|A - \lambda I| = (1 - \lambda)^2$ thus there is one eigenvalue $\lambda = 1$ of multiplicity 2. Eigenvectors are solutions of the system

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

that is

$$\begin{cases} 0 \cdot x + 0 \cdot y = 0 \\ 0 \cdot x + 0 \cdot y = 0 \end{cases}.$$

The solution of this system is (x, y), the whole R^2 so the geometric multiplicity of $\lambda = 1$ is 2, so it equals to its algebraic multiplicity.

2.1.1 Viett Theorem

Theorem 3 *Suppose an $n \times n$ matrix A has n eigenvalues $\lambda_1, ..., \lambda_n$. Then*

(i) The determinant of the matrix A equals to the product of eigenvalues

$$|A| = \lambda_1 \cdot ... \cdot \lambda_n;$$

(ii) The trace of a matrix A, i.e., the sum of the elements on the main diagonal, equals to the sum of eigenvalues of A

$$tr(A) = a_{11} + ... + a_{nn} = \lambda_1 + ... + \lambda_n.$$

Example. Find the eigenvalues of the matrix $A = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$.

Solution. The matrix is clearly singular (degenerate, $|A| = 0$). Therefore $\lambda_1 = 0$ is an eigenvalue (why?). By the trace rule $\lambda_1 + \lambda_2 = 2 + 2 = 4$, thus $\lambda_2 = 4$.

2.2 Linearly Independent Eigenvectors

Theorem 4 *The eigenvectors of the matrix A corresponding to the different eigenvalues are linearly independent.*

More precisely, suppose $\lambda_1,\ \lambda_2,\ ...\ \lambda_k$ are eigenvalues of A and $\lambda_i \neq \lambda_j$ for all $i \neq j$, and suppose $v_1,\ ...\ , v_k$ are corresponding eigenvectors, then they are linearly independent.

Let us check it for $k = 2$. We assume $\lambda_1 \neq \lambda_2$ and $Av_1 = \lambda_1 v_1$, $Av_2 = \lambda_2 v_2$. Suppose v_1, v_2 are linearly dependent, say $v_2 = mv_1$, then $A \cdot v_2 = A \cdot kv_1 = mA \cdot v_1 = m\lambda_1 v_1$, on the other hand side $A \cdot v_2 = \lambda_2 v_2 = \lambda_2 m v_1$, thus $m(\lambda_1 - \lambda_2)v_1 = 0$, this contradicts to $\lambda_1 \neq \lambda_2$.

Corollary 1 *Suppose an $n \times n$ matrix A has n different eigenvalues $\lambda_1, \ldots, \lambda_n$. Then the corresponding eigenvectors $x^{(1)}, \ldots, x^{(n)}$ form a (eigen)basis.*

2.3 Representation of a Matrix in Terms of Eigenvalues and Eigenvectors

Suppose an $n \times n$ matrix A has n eigenvalues $\lambda_1, \ldots, \lambda_n$ and

$$x^{(1)} = \begin{pmatrix} x_1^{(1)} \\ \ldots \\ x_n^{(1)} \end{pmatrix}, \ldots, x^{(n)} = \begin{pmatrix} x_1^{(n)} \\ \ldots \\ x_n^{(n)} \end{pmatrix}$$

are the corresponding *linearly independent* eigenvectors. Form two matrixes, first the diagonal matrix whose diagonal elements are eigenvalues and the second the matrix whose columns are eigenvectors

$$\Lambda = \begin{pmatrix} \lambda_1 & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & \lambda_n \end{pmatrix}, \quad S = \begin{pmatrix} x_1^{(1)} & \ldots & x_1^{(n)} \\ \ldots & \ldots & \ldots \\ x_n^{(1)} & \ldots & x_n^{(n)} \end{pmatrix}.$$

Note that since of Theorem 4 the matrix S is invertible.

Theorem 5 $A = S \cdot \Lambda \cdot S^{-1}$.

Example. Find a 3×3 matrix A which eigenvalues and eigenvectors are:

$$\begin{aligned} \lambda_1 = 3, \quad & x^{(1)} = (-3, 2, 1)^T, \\ \lambda_2 = -2, \quad & x^{(2)} = (-2, 1, 0)^T \\ \lambda_3 = 1, \quad & x^{(3)} = (-6, 3, 1)^T. \end{aligned}$$

Solution. $\Lambda = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad S = \begin{pmatrix} -3 & -2 & -6 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix}$. Then

$$A = S \cdot \Lambda \cdot S^{-1} = \begin{pmatrix} -3 & -2 & -6 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -3 & -2 & -6 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix}^{-1},$$

which can be directly calculated.

Example. Find the matrix A^{100}, where $A = \begin{pmatrix} 41 & -30 \\ 56 & -41 \end{pmatrix}$.

Solution. First find eigenvalues and eigenvectors. The solution of the characteristic equation gives

$$A = \begin{vmatrix} 41-\lambda & -30 \\ 56 & -41-\lambda \end{vmatrix}, \quad \lambda^2 - 1 = 0, \quad \lambda_1 = 1, \quad \lambda_2 = -1.$$

Furthermore, solving the suitable systems we obtain corresponding eigenvectors $x^{(1)} = (3,4)^T$, $x^{(2)} = (5,7)^T$. Thus $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $S = \begin{pmatrix} 3 & 5 \\ 4 & 7 \end{pmatrix}$.
Then

$$A^{100} = (S \cdot \Lambda \cdot S^{-1}) \cdot (S \cdot \Lambda \cdot S^{-1}) \cdot \ldots \cdot (S \cdot \Lambda \cdot S^{-1}) = S \cdot \Lambda^{100} \cdot S^{-1} =$$
$$\begin{pmatrix} 3 & 5 \\ 4 & 7 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{100} \cdot \begin{pmatrix} 3 & 5 \\ 4 & 7 \end{pmatrix}^{-1} =$$
$$\begin{pmatrix} 3 & 5 \\ 4 & 7 \end{pmatrix} \cdot \begin{pmatrix} 1^{100} & 0 \\ 0 & (-1)^{100} \end{pmatrix} \cdot \begin{pmatrix} 7 & -5 \\ -4 & 3 \end{pmatrix} =$$
$$\begin{pmatrix} 3 & 5 \\ 4 & 7 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 7 & -5 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

2.4 Similar Matrices

Two matrices A and B are called similar if there exists an invertible matrix S such that $B = S^{-1} \cdot A \cdot S$.

Theorem 6 *Similarity of matrices is an equivalence relation.*

Theorem 7 *If A and B are similar, then*
(i) $|A - \lambda I| = |B - \lambda I|$;
(ii) $spec(A) = spec(B)$;
(iii) $|A| = |B|$;
(iv) $rank(A) = rank(B)$;
(iii) $tr(A) = tr(B)$.

2.5 Diagonalization of a Matrix

A square matrix A is called diagonalizable if it is *similar* to a diagonal matrix, i.e. if there exists an invertible matrix S such that $S^{-1} \cdot A \cdot S$ is a diagonal matrix.

Theorem 8 *If an $n \times n$ matrix A has n different eigenvalues then it is diagonalizable.*

Indeed, as we already know in this case $A = S \cdot \Lambda \cdot S^{-1}$. Then, multiplying this equality by S^{-1} and S respectively from right and left we obtain

$$S^{-1} \cdot A \cdot S = S^{-1} \cdot (S \cdot \Lambda \cdot S^{-1}) \cdot S = \Lambda,$$

which is diagonal matrix.

Thus *the existence of* n *distinct eigenvalues* is a *sufficient* condition for diagonalizability, but not necessary:

Example. The identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is already diagonal, nevertheless it has two equal eigenvalues $\lambda_1 = \lambda_2 = 1$. By the way, any vector $v \in R^2$ is an eigenvector.

Furthermore, there are nondiagonalizable matrixes:

Example. The matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has two equal eigenvalues $\lambda_1 = \lambda_2 = 1$ and the corresponding eigenvector is $v = (1, 0)$, so in this case the algebraic degree is 2 and the geometric degree is 1 (see above). This matrix is not diagonalizable.

Example. The matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has no real eigenvalues, consequently no eigenveqtors. This matrix is not diagonalizable.

Which $n \times n$ matrices are diagonlizable?
1. Matrices with n distinct eigenvalues.
2. Matrices with n linearly independent eigenvectors.
3. Symmetric matrices ($A = A^t$).

Let us prove the last proposition for a 2×2 symmetric matrix

$$A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}.$$

First let us prove that A has only real eigenvalues:

$$|A - \lambda I| = \begin{vmatrix} a-\lambda & b \\ b & d-\lambda \end{vmatrix} = (a-\lambda)\cdot(d-\lambda) - b^2 = \lambda^2 - (a+d)\cdot\lambda + ad - b^2 = 0,$$

the discriminant of this quadratic equation $D = (a-d)^2 + 4b^2 \geq 0$, thus the characteristic quadratic equation has only real roots.

Consider two cases.

1. Suppose we have a multiple root $\lambda_1 = \lambda_2$, it happens when $D = 0$, that is if $a = d,\ b = 0$, in this case $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ is already a diagonal matrix.

2. Now assume that $\lambda_1 \neq \lambda_2$. By Theorem above two distinct real eigenvalues guarantee the diagonalizability.

Exercises

1. Let $\begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix}$.

(a) Check that $\lambda = 2$ is an eigenvalue of A.

(b) Check that $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a corresponding eigenvector of A.

(c) Find all eigenvalues and corresponding eigenvectors of A.

2. Find the eigenvalues and eigenvectors for the matrix $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 5 & 0 \\ 3 & 0 & 2 \end{pmatrix}$.

3. Suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a Markov matrix, that is $a + c = 1,\ b + d = 1$. Show that $\lambda = 1$ is it's eigenvector.

4. Find eigenvalues of an upper-triangular matrix $\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$.

5. For each of the following matrix A find diagonal matrix Λ and invertible matrix S so that $A = S \cdot \Lambda \cdot S^{-1}$

$$(a)\ \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}. \qquad (b)\ \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}.$$

$$(c)\ \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}. \qquad (d)\ \begin{pmatrix} 4 & -2 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Exercises 23.1-23.7, 23.15.

Homework

1. Exercise 23.2

2. Show that a 2×2 symmetric matrix $\begin{pmatrix} a & b \\ b & d \end{pmatrix}$ has *real* eigenvalues. In which case it has just one eigenvalue?

3. Show that a 2×2 symmetric matrix $\begin{pmatrix} a & b \\ b & d \end{pmatrix}$ has *two orthogonal* eigenvectors (hint: in the case of two eigenvalues $\lambda_1 \neq \lambda_2$ consider the inner product $Av_1 \cdot v_2$ and use $Av_1 \cdot v_2 = v_1 \cdot A^T v_2$, in the case $\lambda_1 = \lambda_2$ characterize A).

4. Show that each symmetric 2×2 matrix $\begin{pmatrix} a & b \\ b & d \end{pmatrix}$ can be diagonalized by an orthogonal matrix P.

5. Find a and b for which two vectors $v_1 = (\frac{\sqrt{2}}{2}, a)$ and $v_2 = (b, \frac{\sqrt{2}}{2})$ form an orthnormal basis of R^2.

Summary

Linear map $f: R^n \to R^m$: $f(v+w) = f(v) + f(w)$, $f(c \cdot x) = c \cdot f(x)$.

$f(v) = A \cdot v$ where A is a matrix whose columns are $f(e_1), \dots, f(e_n) \in R^m$.

$f: R^n \to R^n$ is bijective iff $det(A) \neq 0$

$\lambda \in R$ and a nonzero vector $x \in R^n$ are called respectively **eigenvalue and eigenvector** of A if $A \cdot x = \lambda \cdot x$.

spec(A) is the set of all eigenvalues of A.

Eigenspace of λ: the subspace spanned by all its eigenvectors.

The **geometric degree** of λ is dim of its eigenspace.

Eigenvalues of A are solutions of **characteristic equation** $det(A - \lambda I) = 0$.

Eigenvectors of eigenvalue λ are solutions of $(A - \lambda I)v = 0$.

Algebraic degree of $\lambda^* \in spec(A)$ is its multiplicity in $det(A - \lambda I) = 0$.

Algebraic degree $\geq$ geometric degree.

Viett Theorem: If A has n eigenvalues $\lambda_1, \dots, \lambda_n$ then $|A| = \lambda_1 \cdot \dots \cdot \lambda_n$ and $tr(A) = a_{11} + \dots + a_{nn} = \lambda_1 + \dots + \lambda_n$.

If $\{\lambda_1, \lambda_2, \dots \lambda_k\} = spec(A)$ and $i \neq j \Rightarrow \lambda_i \neq \lambda_j$ then corresponding eigenvectors $v_1, \dots, v_k$ are **lin. indep.**

If A has n different eigenvalues, then corresponding eigenvectors form **eigenbasis**.

If A has n eigenvalues $\lambda_1, \dots, \lambda_n$ and eigenbasis $(x^{(1)}, \dots, x^{(n)})$ then $A = S\Lambda S^{-1}$ or $\Lambda = S^{-1}AS$ where $\Lambda = \begin{pmatrix} \lambda_1 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \lambda_n \end{pmatrix}$ $S = \begin{pmatrix} x_1^{(1)} & \dots & x_1^{(n)} \\ \dots & \dots & \dots \\ x_n^{(1)} & \dots & x_n^{(n)} \end{pmatrix}$.

A and B are **similar** if $B = S^{-1} \cdot A \cdot S$. In this case $|A - \lambda I| = |B - \lambda I|$, $spec(A) = spec(B)$, $|A| = |B|$, $rank(A) = rank(B)$, $tr(A) = tr(B)$.