Reading [SB] Ch. 11, p. 237-250, Ch. 27, p. 750-771.

1 Basis

1.1 Linear Combinations

A *linear combination* of vectors $v_1, v_2, \dots, v_m \in R^n$ with scalar coefficients $\alpha_1, \alpha_2, \dots, \alpha_m \in R$ is the vector

$$\alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + \dots + \alpha_m \cdot v_m.$$

The set of all linear combinations of vectors $v_1, v_2, \dots, v_m \in R^n$ is denoted as

$$L[v_1, v_2, \dots, v_m] = \{\alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + \dots + \alpha_m \cdot v_m, \quad \alpha_i \in R\}.$$

It is evident that $L[v_1, v_2, \dots, v_m] \subset R^n$ is a *subspace*.

Example. For a one single nonzero vector $v \in R^n$

$$L[v] = \{t \cdot v, \ t \in R\}$$

is the line *generated* or *spanned* by v: it passes trough the origin and has direction of v.

Example. For any two nonzero vectors $v, w \in R^n$

$$L[v, w] = \{s \cdot v + t \cdot w, \ s, t \in R\}$$

is either:

the line *generated* (or *spanned*) by v if v and w are collinear, that is if $w = k \cdot v, \ k \in R$,

or is the plane *generated* (or *spanned*) by v and w, which passes trough the origin, if v and w are non-collinear.

Example. For any two non-collinear vectors $v, w \in R^2$

$$L[v, w] = \{s \cdot v + t \cdot w, \ s, t \in R\}$$

is whole R^2.

Example. For any three nonzero vectors $u, v, w \in R^2$ s.t. v and w are non-collinear

$$L[v, w] = L[u, v, w] = R^2.$$

1.2 Linear Dependence and Independence

Definition 1. A sequence of vectors $v_1, v_2, \, ... \, , v_n$ is called *linearly dependent* if one of these vectors is linear combination of others. That is

$$\exists i, \;\; v_i \in L(v_1, \; ... \; , \hat{v_i}, \; ... \; , v_n).$$

Definition 1'. A sequence of vectors $v_1, v_2, \, ... \, , v_m$ is linearly dependent if there exist $\alpha_1, \; ... \; , \alpha_m$ with at last one nonzero α_k s.t.

$$\alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + ... + \alpha_m \cdot v_m = 0.$$

Why these definitions are equivalent?

Example. Any sequence of vectors which contains the zero vector is linearly dependent. (Why?)

Example. Any sequence of vectors which contains two collinear vectors is linearly dependent. (Why?)

Example. Any sequence of vectors of R^2 which consists of more then two vectors is linearly dependent. (Why?)

Example. A sequence consisting of two vectors v_1, v_2 is linearly dependent if and only if these vectors are collinear (proportional), i.e. $v_2 = k \cdot v_1$. (Why?)

Definition 2. A sequence of vectors $v_1, v_2, \, ... \, , v_n$ is called *linearly independent* if it is not linearly dependent.

Definition 2'. A sequence of vectors $v_1, v_2, \, ... \, , v_n$ is called *linearly independent* if non of these vectors is a linear combination of others.

Definition 2". A sequence of vectors $v_1, v_2, \, ... \, , v_n$ is called *linearly independent* if

$$\alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + ... + \alpha_m \cdot v_m = 0$$

is possible only if all α_i-s are zero.

Why these definitions are equivalent?

Example. The vectors $e_1 = (1, 0, 0), \;\; e_2 = (0, 1, 0), \;\; e_3 = (0, 0, 1) \in R^3$ are linearly independent.

Indeed, suppose $\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 = 0$, this means

$$\alpha_1 \cdot (1, 0, 0) + \alpha_2 \cdot (0, 1, 0) + \alpha_3 \cdot (0, 0, 1) =$$
$$(\alpha_1, 0, 0) + (0, \alpha_2, 0) + (0, 0, \alpha_3) = (\alpha_1, \alpha_2, \alpha_3) = (0, 0, 0),$$

thus $\alpha_1 = 0, \;\; \alpha_2 = 0, \;\; \alpha_3 = 0.$

1.2.1 Linear Independence and Systems of Linear Equations

How to check wether a given sequence of vectors $v_1,\ v_2,\ \dots\ ,\ v_m \in R^n$ is linear dependent or independent?

Let

$$A = \begin{pmatrix} v_{11} & v_{21} & \dots & v_{m1} \\ v_{12} & v_{22} & \dots & v_{m1} \\ \dots & \dots & \dots & \dots \\ v_{1n} & v_{2n} & \dots & v_{mn} \end{pmatrix},$$

be the matrix whose columns are v_j's.

Theorem 1 *A sequence of vectors $v_1,\ v_2,\ \dots\ ,\ v_m$ is linear independent iff the homogenous system $A\alpha = 0$ has only zero solution $\alpha = (0,\ \dots\ ,0)$.*

Example. Determine whether the sequence of vectors is linearly dependent

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix},\quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix},\quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

Solution. We must check whether the equation

$$c_1 \cdot v_1 + c_2 \cdot v_2 + c_3 \cdot v_3 = 0$$

has non-all-zero solution for c_1, c_2, c_3. In coordinates this equation looks as a system

$$\begin{cases} 1 \cdot c_1 & + & 1 \cdot c_2 & + & 0 \cdot c_3 & = & 0 \\ 0 \cdot c_1 & + & 0 \cdot c_2 & + & 0 \cdot c_3 & = & 0 \\ 1 \cdot c_1 & + & 0 \cdot c_2 & + & 1 \cdot c_3 & = & 0 \\ 0 \cdot c_1 & + & 1 \cdot c_2 & + & 1 \cdot c_3 & = & 0 \end{cases}.$$

The matrix of the system

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

has maximal rank 3. So there are no free variables, and the system has only zero solution. Thus this sequence of vectors is linearly independent.

Example. Determine whether the sequence of vectors is linearly dependent

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix},\quad v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix},\quad v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solution. We must check whether the equation

$$c_1 \cdot v_1 + c_2 \cdot v_2 + c_3 \cdot v_3 = 0$$

has non-all-zero solution for c_1, c_2, c_3. In coordinates this equation looks as a system

$$\begin{cases} 1 \cdot c_1 & + & 1 \cdot c_2 & + & 1 \cdot c_3 & = & 0 \\ 0 \cdot c_1 & + & 0 \cdot c_2 & + & 0 \cdot c_3 & = & 0 \\ 1 \cdot c_1 & - & 1 \cdot c_2 & + & 0 \cdot c_3 & = & 0 \\ 0 \cdot c_1 & + & 0 \cdot c_2 & + & 0 \cdot c_3 & = & 0 \end{cases}.$$

The matrix of the system

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

has the rank 2. So there is free variable, and the system has non-zero solutions too. Thus this sequence of vectors is linearly dependent.

Theorem 2 *A set of vectors $v_1, v_2, \ ... \ , v_k$ in R^n with $k > n$ is linearly dependent.*

Proof. We look at a nonzero solution $c_1, \ ... \ , c_k$ of the equation

$$c_1 \cdot v_1 + \ ... \ + c_k \cdot v_k = 0,$$

or, equivalently, of the system

$$\begin{cases} v_{11} \cdot c_1 & + ... + & v_{k1} \cdot c_k & = & 0 \\ v_{12} \cdot c_1 & + ... + & v_{k2} \cdot c_k & = & 0 \\ ... & ... & ... & ... & ... & ... \\ v_{1n} \cdot c_1 & + ... + & v_{kn} \cdot c_k & = & 0 \end{cases}.$$

This homogenous system has k variables and n equations. Then $rank \leq n < k$, so there definitely are free variables, consequently there exists nonzero solution $c_1, \ ... \ , c_k$.

Theorem 3 *A set of vectors $v_1, v_2, \ ... \ , v_n$ in R^n is linearly independent iff*

$$det(v_1 \ v_2 \ ... \ v_n) \neq 0.$$

Proof. We look at a nonzero solution for $c_1, \ ... \ , c_n$ of the equation

$$c_1 \cdot v_1 + \ ... \ + c_n \cdot v_n = 0.$$

The system which corresponds to this equation has n variables and n equations and is homogenous. So it has a non-all-zero solutions iff its determinant is zero.

1.3 Span

Let $v_1, \ \dots, v_k$ be a sequence of m vectors from R^n.

The set of all linear combinations of these vectors

$$L[v_1, \ \dots, v_k] = \{\alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + \dots + \alpha_k \cdot v_k, \ \alpha_1, \ \dots, \alpha_k \in R\}$$

is called the set **generated** (or **spanned**) by the vectors $v_1, \ \dots, v_k$.

Example. The vectors $v_1 = (1, 0, 0), \ v_2 = (0, 1, 0)$ span the xy plane (the plane given by the non-parameterized equation $z = 0$) of R^3. Indeed, any point $p = (a, b, 0)$ of this plane is the following linear combination

$$av_1 + bv_2 = a(1, 0, 0) + b(0, 1, 0) = (a, 0, 0) + (0, b, 0) = (a, b, 0).$$

Example. The vectors $v_1 = (1, 2), \ v_2 = (3, 4)$ span whole R^2. Indeed, let's take any vector $v = (a, b)$. Our aim is to find c_1, c_2 s.t.

$$c_1 \cdot v_1 + c_2 \cdot v_2 = v.$$

In coordinates this equation looks as a system

$$\left\{ \begin{array}{ccccc} c_1 \cdot 1 & + & c_2 \cdot 3 & = & a \\ c_1 \cdot 2 & + & c_2 \cdot 4 & = & b \end{array} \right. .$$

The determinant of this system $\neq 0$, so this system has a solution for each a and b.

Example. Different sequences of vectors can span the same sets. For example R^2 is spanned by each of the following sequences:

(a) $v_1 = (1, 0), \ v_2 = (0, 1)$;
(b) $v_1 = (-1, 0), \ v_2 = (0, 1)$;
(c) $v_1 = (1, 1), \ v_2 = (0, 1)$;
(d) $v_1 = (1, 2), \ v_2 = (2, 1)$;
(e) $v_1 = (1, 0), \ v_2 = (0, 1), \ v_3 = (2, 3)$.

Check this!

For a given sequence of vectors $v_1, \ \dots, v_k \in R^n$ form the $n \times k$ matrix whose columns are v_i's:

$$A = \begin{pmatrix} v_{11} & v_{21} & \dots & v_{k1} \\ v_{12} & v_{22} & \dots & v_{k1} \\ \dots & \dots & \dots & \\ v_{1n} & v_{2n} & \dots & v_{kn} \end{pmatrix},$$

here $v_i = (v_{i1}, v_{i2}, \ \dots, v_{in})$.

Theorem 4 *Let $v_1, \dots, v_k \in R^n$ be a sequence of vectors. A vector $b \in R^n$ lies in the space $L(v_1, \dots, v_k)$ if and only if the system $A \cdot c = b$ has a solution.*

Proof. Evident: $A \cdot c = b$ means $c_1 v_1 + \dots + c_k v_k = b$.

Corollary 1 *A sequence of vectors $v_1, \dots, v_k \in R^n$ spans R^n if and only if the system $A \cdot c = b$ has a solution for any vector $b \in R^n$.*

Corollary 2 *A sequence of vectors $v_1, \dots, v_k \in R^n$ with $k \leq n$ can not span R^n.*

Proof. In this case the matrix A has less columns than rows. Choosing appropriate b we can make $rank(A|b) > rank(A)$ (how?), this makes the system $A \cdot c = b$ non consistent for this b.

1.4 Basis and Dimension

A sequence of vectors $v_1, \dots, v_n \in R^n$ forms a *basis* of R^n if
(1) they are linearly independent;
(2) they span R^n.

Example. The vectors

$$e_1 = (1, 0, \dots, 0),\ e_2 = (0, 1, \dots, 0),\ \dots, e_n = (0, 0, \dots, 1)$$

form a basis of R^n.

Indeed, firstly they are linearly independent since the $n \times n$ matrix

$$(e_1 \ \ e_2 \ \dots \ \ e_n)$$

is the identity, thus it's determinant is $1 \neq 0$.

Secondly, they span R^2: any vector $v = (x_1, \dots, x_n)$ is the following linear combination

$$v = x_1 \cdot e_1 + \dots + x_n \cdot e_n.$$

A basis $v_1, \dots, v_n \in R^n$ is called *orthogonal* if $v_i \cdot v_j = 0$ for $i \neq j$. This means that all vectors are perpendicular to each other: $v_i \cdot v_j = 0$ for $i \neq j$.

An orthogonal basis $v_1, \dots, v_n \in R^n$ is called *orthonormal* if $v_i \cdot v_i = 1$. This means that each vector of this basis has the length 1. In other words: $v_i \cdot v_j = \delta_{i,j}$ where δ_{ij} is famous Kroneker's symbol

$$\delta_{ij} = \begin{cases} 1 & if \quad i = j \\ 0 & if \quad i \neq j \end{cases}.$$

The basis $e_1, \dots, e_n$ is orthonormal.

Theorem 5 *Any two non-collinear vectors of* R^2 *form a basis.*

For example $e_1 = (1,0),\ e_2 = (0,1)$ is a basis. Another basis is, say $e_1' = (1,0),\ e_2' = (1,1)$.

Theorem 6 *Any basis of* R^n *contains exactly* n *vectors.*

Why? Because more than n vectors are linearly dependent, and less than n vectors can not span R^n.

The dimension of a vector space is defined as the number of vectors in its basis. Thus

$$dim\ R^n = n.$$

Theorem 7 *Let* $v_1,\ \dots ,v_k \in R^n$ *and* A *be the matrix whose columns are* v_j*'s:*

$$A = \begin{pmatrix} v_{11} & v_{21} & \dots & v_{n1} \\ v_{12} & v_{22} & \dots & v_{n1} \\ \dots & \dots & \dots & \\ v_{1n} & v_{2n} & \dots & v_{nn} \end{pmatrix}.$$

Then the following statements are equivalent

(a) $v_1,\ \dots ,v_n$ *are linearly independent;*

(b) $v_1,\ \dots ,v_n$ *span* R^n*;*

(c) $v_1,\ \dots ,v_n$ *is a basis of* R^n*;*

(d) $\det\ A \neq 0$.

Example. R^3 is *3 dimensional*: we have here a basis consisting of 3 vectors

$$v_1 = (1,0,0),\ v_2 = (0,1,0),\ v_3 = (0,0,1)$$

Generally, the dimension of R^n is n: it has a basis consisting of n elements

$$e_1 = (1,0,\ \dots ,0),\ e_2 = (0,1,\ \dots ,0),\ \dots ,e_n = (0,0,\ \dots ,1).$$

1.5 Subspace

A subset $V \in R^n$ is called *subspace* if V is *closed* under vector operations - summation and scalar multiplication, that is:

$$v,w \in V,\ c \in R \quad \Rightarrow \quad v + w \in V,\ c \cdot v \in V.$$

Example. The line $x(t) = t \cdot (2,1)$, that is all multiples of the vector $v = (2,1)$ which passes trough the origin is a subspace. But the line $x(t) = (1,1) + t \cdot (2,1)$ is not.

Theorem 8 *Let $w_1, \ ... \ , w_k \in R^n$ be a sequence of vectors. Then the set of all linear combinations*

$$L[w_1, \ ... \ , w_k] \subset R^n$$

is a subspace.

Why?
Example. The *subspace* of R^3

$$\{(a, b, 0), \ a, b \in R\},$$

which is the xy plane, has dimension 2:

$$v_1 = (1, 0, 0), \ v_2 = (0, 1, 0)$$

is its basis.
Example. Similarly, the *subspace* of R^3

$$\{(a, 0, 0), \ a \in R\},$$

which is the y line, has dimension 1:

$$v_1 = (1, 0, 0)$$

is its basis.

1.5.1 How to find the dimension and the basis of $L(v_1, \ ... \ , v_k)$?

Let $v_1, \ ... \ , v_k \in R^n$ be a sequence of vectors from R^n, and $L(v_1, \ ... \ , v_k) \subset R^n$ be the corresponding subspace. How can we find the dimension and basis of this subspace?

Let

$$A = \begin{pmatrix} v_{11} & v_{21} & \dots & v_{k1} \\ v_{12} & v_{22} & \dots & v_{k1} \\ \dots & \dots & \dots & \\ v_{1n} & v_{2n} & \dots & v_{kn} \end{pmatrix},$$

be the matrix whose columns are v_j's. Let r be the rank of this matrix and M be a corresponding main $r \times r$ minor. Then the dimension of $L(v_1, \ ... \ , v_k)$ is r and its basis consists of those v_i-s, who intersect M (why?).

1.6 Conclusion

Let $v_1, v_2, \ ... \ , v_k$ be a sequence of vectors from R^n and let

$$A = \begin{pmatrix} v_{11} & v_{21} & \dots & v_{k1} \\ v_{12} & v_{22} & \dots & v_{k1} \\ \dots & \dots & \dots & \\ v_{1n} & v_{2n} & \dots & v_{kn} \end{pmatrix},$$

be the matrix whose columns are v_j's. Let r be the rank of this matrix. The following table shows when this sequence is linearly independent or spans R^n depending on value of k:

	$k < n$	$k = n$	$n < k$
independent	$r = k$	$r = n$	*no*
spans R^n	*no*	$r = n$	$r = n$

2 Spaces Attached to a Matrix

Let

$$A = \begin{pmatrix} a_{11} & ... & a_{1n} \\ ... & ... & ... \\ a_{m1} & ... & a_{mn} \end{pmatrix}$$

be an $m \times n$ matrix. There are three vector spaces attached to A: the column space $Col(A) \subset R^m$, the row space $Row(A) \subset R^n$ and the null space $Null(A) \subset R^n$.

2.1 Column Space

The column space $Col(A)$ is defined as a subspace of R^m spanned by column vectors of A, that is

$$Col(A) = L(\begin{pmatrix} a_{11} \\ a_{21} \\ ... \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ ... \\ a_{m2} \end{pmatrix}, \ ... \ , \begin{pmatrix} a_{n1} \\ a_{n2} \\ ... \\ a_{nm} \end{pmatrix}).$$

Theorem 9

$$dim\ Col(A) = rank\ A.$$

2.1.1 How to Find a Basis of $Col(A)$

Just find a basic minor of A. Then all the columns that intersect this minor form a basis of $Col(A)$.
Example. Find a basis of $Col(A)$ for

$$A = \begin{pmatrix} 2 & 3 & 1 & 4 \\ 2 & 3 & 7 & 9 \\ 2 & 3 & 13 & 14 \end{pmatrix}.$$

Solution. Calculation shows that a basic minor here can be chosen as $\begin{pmatrix} 1 & 4 \\ 7 & 9 \end{pmatrix}$. So the basis of $Col(A)$ consists of last two columns

$$\begin{pmatrix} 1 \\ 7 \\ 13 \end{pmatrix}, \begin{pmatrix} 4 \\ 9 \\ 14 \end{pmatrix}.$$

Of course we can choose as a basic minor $\begin{pmatrix} 3 & 7 \\ 3 & 13 \end{pmatrix}$. In this case we obtain a basis of $Col(A)$ consisting of second and third columns $\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 7 \\ 1 \end{pmatrix}$.

2.1.2 The Role of Column Space

(a) The system $A \cdot x = b$ has a solution for a particular $b \in R^m$ if b belongs to column space $Col(A)$.

(b) The system $A \cdot x = b$ has a solution for every $b \in R^m$ if and only if $rank\ A$ equals of number of equations m.

(c) If $A \cdot x = b$ has a solution for every b, then

$$number\ of\ equations\ =\ rank\ A \leq number\ of\ variables.$$

2.2 Row Space

The row space $Row(A)$ is defined as a subspace of R^n spanned by row vectors of A, that is

$$Row(A) = L(w_1,\ w_2,\ \dots,w_m)$$

where $w_1,\ \dots,w_m$ are the row vectors of A:

$$w_1 = (a_{11},\ \dots,a_{1n})$$
$$\dots$$
$$w_m = (a_{m1},\ \dots,a_{mn}).$$

Theorem 10

$$dim\ Row(A) = rankA.$$

So the dimensions of the column space $Col(A)$ and the row space $Row(A)$ both equal to $rank\ A$.

2.2.1 How to Find a Basis of $Row(A)$

Just find a basic minor of A. Then all the rows that intersect this minor form a basis of $Row(A)$

Example. Find a basis of $Row(A)$ for

$$A = \begin{pmatrix} 2 & 3 & 1 & 4 \\ 2 & 3 & 7 & 9 \\ 2 & 3 & 13 & 14 \end{pmatrix}.$$

Calculation shows that a basic minor here can be chosen as $\begin{pmatrix} 1 & 4 \\ 7 & 9 \end{pmatrix}$. So the basis of $Row(A)$ consists of first two rows $(2,3,1,4),\quad(2,3,7,9)$.

2.3 Null-space

Previous two attached spaces $Col(A)$ and $Row(A)$ are defined as subspaces **generated** by some vectors.

The third attached space $Null(A)$ is defined as the **set** of all solutions of the system $A \cdot x = 0$, i.e.

$$Null(A) = \{x = (x_1, \dots, x_n) \in R^n, \quad A \cdot x = 0\}.$$

But is this **set** a **subspace**? Yes, yes! But why?

Theorem 11 *A subset $Null(A)$ is a subspace.*

Proof. We must show that $Null(A)$ is closed with respect to addition and scalar multiplication. Indeed, suppose $x, x' \in Null(A)$, that is $A \cdot x = 0,\ A \cdot x' = 0$. Then

$$A(x + x') = A(x) + A(x') = 0 + 0 = 0.$$

Furthermore, let $x \in Null(A)$ and $c \in R$. Then

$$A \cdot (c \cdot x) = c \cdot (A \cdot x) = c \cdot 0 = 0.$$

2.3.1 How to Find a Basis of $Null(A)$

To find a basis of null-space $Null(A))$ just solve a system $A \cdot x = 0$, that is express basic variables in terms of free variables. As we know there are $r = rank(A)$ **basic** variables, say

$$x_1,\ x_2,\ \dots, x_r,$$

and consequently $n - r$ **free** variables, in this case

$$x_{r+1},\ x_{r+2},\ \dots, x_n.$$

Express the basic variables in terms of free variables, and find $n-r$ following particular solutions particular solutions which form a basis of $Null(A)$

$$v_1 = \begin{pmatrix} x_1^1 \\ x_2^1 \\ \dots \\ x_r^1 \\ 1 \\ 0 \\ \dots \\ 0 \\ 0 \end{pmatrix},\ v_2 = \begin{pmatrix} x_1^2 \\ x_2^2 \\ \dots \\ x_r^2 \\ 0 \\ 1 \\ \dots \\ 0 \\ 0 \end{pmatrix},\ \dots, v_{n-r-1} = \begin{pmatrix} x_1^{n-r-1} \\ x_2^{n-r-1} \\ \dots \\ x_r^{n-r-1} \\ 0 \\ 0 \\ \dots \\ 1 \\ 0 \end{pmatrix},\ v_{n-r} = \begin{pmatrix} x_1^{n-r} \\ x_2^{n-r} \\ \dots \\ x_r^{n-r} \\ 0 \\ 0 \\ \dots \\ 0 \\ 1 \end{pmatrix}.$$

Example. Find a basis for the null-space of the matrix

$$A = \begin{pmatrix} 1 & -1 & 3 & -1 \\ 1 & 4 & -1 & 1 \\ 3 & 7 & 1 & 1 \\ 3 & 2 & 5 & -1 \end{pmatrix}.$$

Solution. First solve the homogenous system

$$\begin{cases} x_1 - x_2 + 3x_3 - 1x_4 = 0 \\ x_1 + 4x_2 - x_3 + x_4 = 0 \\ 3x_1 + 7x_2 + x_3 + x_4 = 0 \\ 3x_1 + 2x_2 + 5x_3 - x_4 = 0 \end{cases}.$$

Computation gives $rank\ A = 2$, so $dim\ Null(A) = 4 - rank\ A = 4 - 2 = 2$, and the solution gives

$$x_1 = -2.2x_3 + 0.6x_4, \quad x_2 = 0.8x_3 - 0.4x_4, \quad x_3 = x_3, \quad x_4 = x_4$$

So the general solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2.2x_3 + 0.6x_4 \\ 0.8x_3 - 0.4x_4 \\ x_3 \\ x_4 \end{pmatrix}$$

Substituting $x_3 = 1,\ x_4 = 0$ we obtain the first basis vector of null space $v_1 = \begin{pmatrix} -0.22 \\ 0.8 \\ 1 \\ 0 \end{pmatrix}$.

Now substituting $x_3 = 0,\ x_4 = 1$ we obtain the second basis vector of null space $v_2 = \begin{pmatrix} 0.6 \\ -0.4 \\ 0 \\ 1 \end{pmatrix}$.

So the basis of $Null(A)$ is $\begin{pmatrix} -0.22 \\ 0.8 \\ 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0.6 \\ -0.4 \\ 0 \\ 1 \end{pmatrix}$.

2.4 Fundamental Theorem of Linear Algebra

The column space of A, spanned by n column vectors, and the row space of A, spanned by m row vectors, have the same dimension equal to $rankA$.

The *Fundamental Theorem of Linear Algebra* describes the dimension of the third subspace attached to A:

Theorem 12 *dim Null(A)+rank A=n.*

2.5 Solutions of Systems of Linear Equations

We already know how to express all solutions of homogenous system $A\cdot x = 0$: just find a basis of $Null(A)$

$$v_1, \quad v_2, \quad ... \quad , v_{n-r},$$

then any solution, since it is an element of $Null(A)$, is a linear combination

$$x = \alpha_1 v_1 + \ ... \ + \alpha_{n-r} v_{n-r}.$$

Now turn to non-homogenous systems.

Let $A \cdot x = b, \ x \in R^n, \ b \in R^m$ be a system of linear equations and $A \cdot x = 0$ be the corresponding homogenous system.

Theorem 13 *Let c be a particular solution of $A \cdot x = b$. Then, every other solution c' of $A \cdot x = b$ can be written as $c' = c + w$ where w is a vector from $Null(A)$, that is a solution of homogenous system $A \cdot x = 0$.*

Proof. Since c and c' are solutions, we have $A \cdot c = b, \ A \cdot c' = b$. Let's define $w = c' - c$. Then

$$A \cdot w = A \cdot (c' - c) = A \cdot c' - A \cdot c = b - b = 0,$$

so $w = c' - c$ is a solution of $A \cdot x = 0$. Thus $c' = c + w$.

According to this theorem in order to know *all solutions* of $A \cdot x = b$ it is enough to know *one particular solution* of $A \cdot x = b$ and *all solutions* of $A \cdot x = 0$. Then any solution is given by

$$\{c + \alpha_1 \cdot v_1 + ... + \alpha_{n-r} \cdot v_{n-rank\ A}\}.$$

But how to find one particular solution of $A \cdot x = b$? Just take (for example) the following free variables $x_{r+1} = 0, \ x_{r+2} = 0, \ ... \ , \ x_n = 0$ and solve $x_1, \ ... \ , x_r$.

Example. Express general solution of the system

$$\left\{ \begin{array}{ccccccccc} x_1 & - & x_2 & + & 3x_3 & - & 1x_4 & = & 1 \\ x_1 & + & 4x_2 & - & x_3 & + & x_4 & = & 6 \\ 3x_1 & + & 7x_2 & + & x_3 & + & x_4 & = & 13 \\ 3x_1 & + & 2x_2 & + & 5x_3 & - & x_4 & = & 8 \end{array} \right. .$$

Solution. We already know general solution of corresponding homogenous system $A \cdot x = 0$: a basis of $Null(A)$ is

$$v_1 = \begin{pmatrix} -0.22 \\ 0.8 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0.6 \\ -0.4 \\ 0 \\ 1 \end{pmatrix},$$

so the general solution of homogenous system is given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \alpha_1 \cdot \begin{pmatrix} -0.22 \\ 0.8 \\ 1 \\ 0 \end{pmatrix} + \alpha_2 \cdot \begin{pmatrix} 0.6 \\ -0.4 \\ 0 \\ 1 \end{pmatrix}.$$

Now we need one particular solution of non-homogenous system. Take $x_3 = 0,\ x_4 = 0$, we obtain

$$\begin{cases} x_1 - x_2 = 1 \\ x_1 + 4x_2 = 6 \end{cases}.$$

This gives $x_1 = 2,\ x_2 = 1$. So a particular solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Finally, the general solution of nonhomogenous system is given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_1 \cdot \begin{pmatrix} -0.22 \\ 0.8 \\ 1 \\ 0 \end{pmatrix} + \alpha_2 \cdot \begin{pmatrix} 0.6 \\ -0.4 \\ 0 \\ 1 \end{pmatrix}.$$

2.6 Orthogonal Complement

For a subspace $V \subset R^n$ its orthogonal complement $V^\perp \subset R^n$ is defined as the set of all vectors $w \in R^n$ that are orthogonal to every vector from V, i.e.

$$V^\perp = \{w \in R^n,\ v \cdot w = 0\ for\ \forall\ v \in V\}.$$

Proposition 1 *For any subspace $V \subset R^n$*
(a) $V^\perp$ is a subspace.
(b) $V \bigcap V^\perp = \{0\}$.
(c) $dim\ V + dim\ V^\perp = n$.
(d) $(V^\perp)^\perp = V$.
(e) Suppose $V,\ W \in R^n$ are subspaces, $din\ V + din\ W = n$ and for each $v \in V,\ w \in W$ one has $v \cdot w = 0$. Then $W = V^\perp$.

Proof of (a). 1. Suppose $w \in V^\perp$, i.e. $w \cdot v = 0\ for\ \forall v \in V$. Let us show that $kw \in V^\perp$. Indeed

$$kw \cdot v = k(w \cdot v) = k \cdot 0 = 0.$$

2. Suppose $w, w' \in V^\perp$, i.e. $w \cdot v = 0,\ w' \cdot v = 0\ for\ \forall v \in V$. Let us show that $w + w' \in V^\perp$. Indeed

$$(w + w') \cdot v = w \cdot v + w' \cdot v = 0 + 0 = 0.$$

Theorem 14 *For a matrix A*
(a) $Row(A)^{\perp} = Null(A)$.
(b) $Col^{\perp} = Null A^T$.

Example. In R^3, the orthogonal complement to xy plane is the z-axes. Prove it!

Exercises

Exercises from [SB]
11.2, 11.3, 11.9, 11.10, 11.12, 11.13, 11.14
27.1, 27.2, 27.3, 27.4, 27.5, 27.6, 27.7, 27.8, 27.10
27.12, 27.13, 27.14, 27.17

Homework

1. Exercise 11.12.

2. Show that the vectors from 11.14 (b) do not span R^3: present at last one vector which is NOT their linear combination.

3. Show that the vectors from 11.14 (b) are linearly dependent: find their linear combination with non-all-zero coefficients which gives the zero vector.

4. Show that if $v \in Row(A),\ w \in Null(A)$ then $v \cdot w = 0$. Actually this proofs $Row(A)^{\perp} = Null(A)$.

5. Exercise 27.10 (d).

Summary

Let $v_1,\ v_2,\ ...\ ,\ v_m \in R^n$ and $A = \begin{pmatrix} v_{11} & v_{21} & ... & v_{m1} \\ ... & ... & ... & ... \\ v_{1n} & v_{2n} & ... & v_{mn} \end{pmatrix}$ be the matrix whose columns are v_j's.

Linear Combinations: $L[v_1, v_2,\ ...\ , v_m] = \{\alpha_1 \cdot v_1 + ... + \alpha_m \cdot v_m$.

Linearly dependent: $\exists i,\ v_i \in L(v_1,\ ...\ , \hat{v}_i,\ ...\ , v_m)$ or $\exists\ (\alpha_1,\ ...\ , \alpha_m) \neq (0,\ ...\ , 0)\ s.t.\ \alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + ... + \alpha_m \cdot v_m = 0$.

Linearly independent: $\alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + ... + \alpha_m \cdot v_m = 0\ \Rightarrow\ \forall\ \alpha_k = 0$, or $A\alpha = 0$ has only zero solution.

$(v_1,\ ...\ , v_k) \in R^n$ **spans** R^n if $L[v_1, v_2,\ ...\ , v_m] = R^n$ or $A\alpha = b$ has a solution for $\forall\ b = (b_1,\ ...\ , b_n)$.

$(v_1,\ ...\ , v_k) \in R^n$ is a **basis** if it is lin. indep. and spans R^n.

n lin. indep. vectors span R^n, so they form a basis. n vectors spanning R^n are lin. indep., so they form a basis.

Subspace $V \subset R^n$: $v, w \in V,\ c \in R\ \ \Rightarrow\ \ v + w \in V,\ c \cdot v \in V$.

Dimension and basis of $L[v_1, v_2,\ ...\ , v_m]$: dimension is $rank\ A$, basis - the columns intersecting main minor.

Spaces Attached to a Matrix

Let $A = \begin{pmatrix} a_{11} & ... & a_{1n} \\ ... & ... & ... \\ a_{m1} & ... & a_{mn} \end{pmatrix}$.

Column space: $Col(A) = L[\begin{pmatrix} a_{11} \\ ... \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ ... \\ a_{m2} \end{pmatrix},\ ...\ , \begin{pmatrix} a_{n1} \\ ... \\ a_{nm} \end{pmatrix}]$. $b \in Col(A)$ iff $Ax = b$ has a solution.

$dim\ Col(A) = rank\ A$, basis - columns that intersect main minor.

Row Space: $Row(A) = L[(a11,\ ...\ , a_{1n}),\ ...\ , (am1,\ ...\ , a_{mn})]$. $dim\ Row(A) = rank\ A$, basis - rows that intersect main minor.

Null-space: $Null(A) = \{x \in R^n,\ \ A{\cdot}x = 0\}$. $dim\ Null(A) = n{-}rank\ A$. Basis of $Null(A)$ - the following solutions of $Ax = 0$

$$v_1 = \begin{pmatrix} x_1^1 \\ ... \\ x_r^1 \\ 1 \\ 0 \\ ... \\ 0 \\ 0 \end{pmatrix},\ v_2 = \begin{pmatrix} x_1^2 \\ ... \\ x_r^2 \\ 0 \\ 1 \\ ... \\ 0 \\ 0 \end{pmatrix},\ ...\ , v_{n-r-1} = \begin{pmatrix} x_1^{n-r-1} \\ ... \\ x_r^{n-r-1} \\ 0 \\ 0 \\ ... \\ 1 \\ 0 \end{pmatrix},\ v_{n-r} = \begin{pmatrix} x_1^{n-r} \\ ... \\ x_r^{n-r} \\ 0 \\ 0 \\ ... \\ 0 \\ 1 \end{pmatrix}.$$

Orthogonal complement: $V^\perp = \{w \in R^n,\ v \cdot w = 0\ for\ \forall\ v \in V\}$.

$Row(A)^\perp = Null(A)$, $Col^\perp = Null A^T$.