Reading [SB] Ch. 10, pp. 199-236.

1 Vector Algebra

1.1 Euclidian Space R^n

R^1 is the real line.
$R^2 = \{(x_1, x_2),\ x_1, x_2 \in R\}$ is the Euclidian 2-space.
$R^n = \{(x_1, ..., x_n),\ x_i \in R\}$ is the Euclidian n-space which consists of n-tuples of real numbers.

1.1.1 Vectors

A vector is an object which has a *magnitude* (or length) and *direction*. Graphically a vector is represented as an arrow, connecting an *initial point* P with a *terminal point* Q, notation $\overrightarrow{PQ}$.

Two arrows represent the same vector if they have the same magnitude and direction.

Any two points $P = (p_1, ..., p_n),\ Q = (q_1, ..., q_n) \in R^n$ determine the vector $\overrightarrow{PQ}$. This vector has *coordinates* and $\overrightarrow{PQ}$ can be written as *row vector*

$$(q_1 - p_1, ..., q_n - p_n)$$

or *column vector*

$$\begin{pmatrix} q_1 - p_1 \\ ... \\ q_n - p_n \end{pmatrix}.$$

Any vector is equivalent to the vector of the same magnitude and direction whose initial point is the origin.

Any vector can be identified with its terminal point when as initial point is assumed the origin.

1.1.2 The Algebra of Vectors

Vector addition: For $x = (x_1,\ ...\ , x_n),\quad y = (y_1,\ ...\ , y_n) \in R^n$ the sum is defined by

$$x + y = (x_1 + y_1,\ ...\ , x_n + y_n).$$

Scalar multiplication: for $c \in R$ and $x = (x_1, \ ... \ , x_n) \in R^n$ let

$$c \cdot x = (c \cdot x_1, \ ... \ , c \cdot x_n).$$

These operations satisfy the following conditions:
(1) $u + (v + w) = (u + v) + w$,
(2) The zero vector $O = (0, \ ... \ , 0) \in R^n$ is neutral with respect to summation $O + v = v + O = v$,
(3) for each $v = (x_1, \ ... \ , x_n) \in R^n$ there exists *opposite vector* $(-v) = (-x_1, .., -x_n) \in R^n$ s.t. $v + (-v) = O$,
(4) $v + w = w + v$.
(5) $r \cdot (s \cdot v) = (r \cdot s) \cdot v, \ 1 \cdot v = v$ for each $r, s \in R$.
(6) $(r + s) \cdot v = r \cdot v + s \cdot v$, $r \cdot (v + w) = r \cdot v + r \cdot w$.

1.2 Inner Product

The *inner product* of two vectors $x = (x_1, \ ... \ , x_n), \ y = (y_1, \ ... \ , y_n) \in R^n$ is defined as the number

$$x \cdot y = x_1 \cdot y_1 + ... + x_n \cdot y_n = \sum_{k=1}^{n} x_k \cdot y_k.$$

Properties of inner product:
(1) $u \cdot v = v \cdot u$,
(2) $u \cdot (v + w) = u \cdot v + u \cdot w$,
(3) $u \cdot rv = r(u \cdot v) = ru \cdot v$,
(4) $u \cdot u \geq 0$,
(5) $u \cdot u = 0 \ \Rightarrow u = 0$,

Some important concepts concerning vectors such as length, angle, distance can be expressed in terms of inner product.

1.2.1 Norm of a Vector

The **Euclidian norm** (length) of a vector $x = (x_1, \ ... \ , x_n) \in R^n$ is given by $||x|| = \sqrt{x_1^2 + ... + x_n^2}$. It fact the norm can be expressed in terms of inner product

$$||x|| = \sqrt{x \cdot x}.$$

Generally, a norm on R^n is a function $||...|| : R^n \to R$ which satisfies the conditions

1. $||v|| > 0$ if $v \neq 0$ and $||v|| = 0$ if $v = 0$.

2. $||\alpha \cdot v|| = |\alpha| \cdot ||v||$.

3. $||v + w|| \leq ||v|| + ||v||$.

The above **Euclidian norm**

$$||x||_E = \sqrt{x_1^2 + ... + x_n^2}$$

is a norm: it satisfies the above three conditions.

1.2.2 Some Exotic Norms*

There are other norms too:

Manhattan norm (Taxicab norm)

$$||x||_1 = |x_1| + \ ... \ + |x_n|.$$

***p*-norm**

$$||x||_p = (|x_1|^p + \ ... \ + |x_n|^p)^{\frac{1}{p}}.$$

Maximum norm

$$||x||_\infty = max(|x_1|, \ ... \ , |x_n|).$$

Note that $||x||_p$ for $p = 1$ coincides with Manhattan norm $||x||_M$ and for $p = 2$ coincides with Euclidian norm $||x||_E$. Besides the limit $lim_{p\to\infty}||x||_p$ coincides with Maximum norm $||x||_\infty$.

1.2.3 Metric in R^n

Metric (distance) is a function of two arguments $d(x, y)$ which satisfies the following axioms

1. $d(a, b) \geq 0$, $d(a, b) = 0 \ \Leftrightarrow \ a = b$;
2. $d(a, b) = d(b, a)$;
3. $d(a, c) + d(c, b) \geq d(a, b)$.

Any norm $||x||$ determines a metric

$$d(x, y) = ||x - y||.$$

The **Euclidian metric** in R^n is given by

$$d(x, y) = ||(x_1, \ ... \ , x_n) - (y_1, \ ... \ , y_n)|| =$$

$$\sqrt{(x_1 - y_1)^2 + ... + (x_n - y_n)^2}.$$

1.2.4 Some Exotic Metrics*

The Manhattan norm induces the **Manhattan metric**

$$d_M(x,y) = |x_1 - y_1| + \ ... \ + |x_n - y_n|.$$

The Maximum norm induces the **Chessboard metric**: the minimal number of moves of chess king to travel from x to y.

$$d_{Max}(x,y) = max(|x_1 - y_1|, \ ... \ , |x_n - y_n|.$$

Any norm $||x||$ induces the **British rail metric**

$$d_{BR}(x,y) = ||x|| + ||y|| \ \ and \ \ d_{BR}(x,x) = 0.$$

1.2.5 Angle Between Two Vectors

Any two vectors $x, y \in R^n$ (with initial points at the origin) determine a plane. In that plane we can measure the angle α between these two vectors. In fact the inner product can be expressed in terms of the length and the angle between them: if the angle between vectors $x, y \in R^n$ is α, then

$$x \cdot y = ||x|| \cdot ||y|| \cdot cos\ \alpha.$$

Why? Because

$$||x - y||^2 = ||x||^2 + ||y||^2 - 2||x|| \cdot ||y|| \cdot cos\ \alpha$$

and

$$\begin{gathered} ||x - y||^2 = (x - y) \cdot (x - y) = x \cdot x + 2x \cdot y + y \cdot y = \\ ||x||^2 + ||y||^2 - 2x \cdot y, \end{gathered}$$

that is it!

This formula can be used to find the angle between two vectors:

$$cos\ \alpha = \frac{x \cdot y}{||x|| \cdot ||y||} = \frac{x_1 \cdot y_1 + ... + x_n \cdot y_n}{\sqrt{x_1^2 + ... + x_n^2} \cdot \sqrt{y_1^2 + ... + y_n^2}}.$$

The denominator of this expression is positive thus the sign of $cos\ \alpha$ coincides with the sign of $x \cdot y$. Consequently

α is acute if $x \cdot y > 0$;

α is obtuse if $x \cdot y < 0$;

α is right if $x \cdot y = 0$.

The last condition means that x and y are *orthogonal* if and only if

$$x \cdot y = 0.$$

Example. Consider the rectangle determined by vectors $(\sqrt{3}, 0)$ and $(0, 1)$. Find the angle between the diagonals of this rectangle.
Solution. The diagonals are the vectors $d_1 = (\sqrt{3}, 1)$ and $d_2 = (\sqrt{3} - 0, 0 - 1) = (\sqrt{3}, -1)$, thus

$$cos\ \alpha = \frac{d_1 \cdot d_2}{||d_1|| \cdot ||d_1||} = \frac{3-1}{2 \cdot 2} = \frac{1}{2}.$$

1.3 Convexity

1.3.1 Lines, Half-Lines, Segments

Let x, y be two points in R^n and s, t two real numbers such that $s + t = 1$. The *weighted average* of x and y with weights s and t is defined as the point

$$s \cdot x + t \cdot y.$$

Note that if $s = t = \frac{1}{2}$ this is the usual average, or the *midpoint*.
If $s = 1,\ t = 0$ then this is x.
If $s = 0,\ t = 1$ then this is y.

Generally, let x, y be two different points in R^n. The straight line x, y is the subset of R^n consisting of points $z = x + t(y - x)$, i.e.

$$\{z = (1 - t) \cdot x + t \cdot y,\ t \in R\}.$$

The *closed* half-line x, y with the origin at x is defined as the subset of R^n consisting of points

$$\{z = (1 - t) \cdot x + t \cdot y,\ t \in R,\ t \geq 0\}.$$

The *open* half-line x, y with the origin at x is defined as the subset of R^n consisting of points

$$\{z = (1 - t) \cdot x + t \cdot y,\ t \in R,\ t > 0\}.$$

The *line segment* x, y is defined as

$$[x, y] = \{(1 - t)x + ty,\ 0 \leq t \leq 1\}.$$

1.3.2 Convex Sets

A subset $X \subset R^n$ is called *convex* if, whenever it contains two points $x, y \in X$, it contains also the line segment

$$[x, y] = \{(1 - t)x + ty,\ 0 \leq t \leq 1\}.$$

1.3.3 Cone

A *cone* with vertex x is a subset $C \subset R^n$ with the following property: if $y \in C$ then the whole closed half line

$$\{(1-t)x + ty,\ t \geq 0\}$$

also belongs to C.

2 Economical Examples

2.1 Budget Sets in Commodity Space

2.1.1 Commodity Bundle

Assume an economy with l commodities. Let x_i denote the amount of commodity i. A *commodity bundle* is defined as a vector with nonnegative coordinates

$$x = (x_1,\ \dots\ , x_l) \in R^l,\ x_i \geq 0.$$

The set of all commodity bundles is in the *nonnegative orthant* of R^l

$$\{(x_l, \dots, x_l), x_1 \geq 0,\ \dots\ , x_l \geq 0\}$$

and is called a *commodity space.*

2.1.2 Price System

Let $p_i > 0$ denote the price of commodity i. The vector $p = (p_1,\ \dots\ , p_l) \in R^l$ is called a *price system.*

2.1.3 The Value of an Action

The cost of purchasing commodity bundle $x = (x_1 \dots, x_l) \in R^l$ by the price system $p = (p_1,\ \dots\ , p_l) \in R^l$ is

$$p_1 x_1 + \dots + p_l x_l = p \cdot x,$$

that is the inner product of vectors $p \cdot x$.

2.1.4 Consumers Budget Set

A consumer with income I can purchase only bundles x such that

$$p \cdot x = p_1 \cdot x_1 + \dots + p_l \cdot x_l \leq I$$

This subset of commodity space is called the *consumer's budget set.*

It is interesting to observe the budget set for $l = 2$.

In this case the budget set is the triangle with vertices

$$(0,0),\ (\frac{I}{p_1},0),\ (0,\frac{I}{p_2}).$$

Note that the price vector p is *orthogonal* (normal) to the line $(\frac{I}{p_1},0),\ (0,\frac{I}{p_2})$: indeed, the direction vector of this line is

$$v = (0,\frac{I}{p_2}) - (\frac{I}{p_1},0) = (0-\frac{I}{p_1},\frac{I}{p_2}-0) = (-\frac{I}{p_1},\frac{I}{p_2}),$$

then the inner product is

$$p \cdot v = (p_1,p_2) \cdot (-\frac{I}{p_1},\frac{I}{p_2}) = -I + I = 0.$$

2.2 Production Sets

2.2.1 Production Set for One Producer

Assume again that we have an economy with l commodities.

A *production plan*, or briefly a *production*, is a vector

$$y = (y_1,\ \dots\ ,y_l) \in R^l$$

where y_i is the quantity of the commodity i which is produced by our producer, in this case y_i is positive; or the quantity of the commodity i which is used in the production, in this case y_i is negative. If a commodity i is neither in the input, nor in the output of our production, then $y_i = 0$.

The set of all *possible* productions (production plans) for our producer is called *production set* and is denoted by Y. This is a subset of R^l.

2.2.2 Producer's Total Profit

Given a price system $p = (p_1,\ \dots\ ,p_l) \in R^l$ and a production (production plan) $y = (y_1,\ \dots\ ,y_l) \in R^l$. Then the total profit of our producer is

$$p_1 \cdot y_1 + \dots + p_l \cdot y_l = p \cdot y.$$

2.2.3 Profit Maximization

Assuming a price system p is given (an action does not affect prices) a producer chooses in his production set Y so as to maximize his profit. The resulting action is called an equilibrium of the producer relative to p.

Depending on p and Y such an equilibrium y either exists and is unique, or there are many equilibrium actions, or does not exists at all.

Mathematically this means the following problem: given a vector $p \in R^l$ and a subset $Y \subset R^l$. Find $y \in Y$ which maximizes the inner product $p \cdot y$.

2.2.4 Assumptions on a Production Set*

A production set Y *can* satisfy some economically inspired assumptions.

1. $0 \in Y$ **Possibility of inaction.** This property means that it is possible for a producer not to do anything: no input, no output.

2. $Y \bigcap \Omega \subset \{0\}$ **Impossibility of free production.** This property means that a production with no negative coordinates (no input!) is only zero production (here $\Omega = \{y = (y_1, \ ... \ , y_l), \ each \ y_i \geq 0\}$ is the set of all vectors with non-negative coordinates).

3. $Y \bigcap -Y \subset \{0\}$ **Irreversibility.** This property means that if a nonzero production y is possible, then $-y$ is not possible.

4. $Y + Y \subset Y$ **Additivity.** This property means that if production plans y and y' both are possible, then their sum $y+y'$ is possible too, that is $y, y' \in Y$ implies $y + y' \in Y$.

5. Y **is convex set.** This property means that if production plans y and y' both are possible, then their weighted average $(1-t)y + ty'$ is also possible, that is $y, y' \in Y$ implies $(1-t)y + ty' \in Y$.

6. Y **is a cone with vertex** 0 **(constant return to scale).** This property means that if a production plan y is possible, then $t{\cdot}y$ for $t \geq 0$ is also possible, that is $y \in Y$ implies $ty \in Y, \ t \geq 0$.

3 Lines and Planes

3.1 Lines in R^n

In R^n a line is completely determined by two things: a *point* $p = (p_1, \ ... \ , p_n)$ on the line and a *direction vector* $v = (v_1, \ ... \ , v_n)$ from the point p. The parametric representation of this line is

$$x(t) = p + tv, \quad t \in R.$$

In coordinates

$$(x_1(t), \ ... \ , x_n(t)) = (p_1 + tv_1, \ ... \ , p_n + tv_n),$$

or

$$\begin{aligned} x_1(t) &= p_1 + tv_1, \\ x_2(t) &= p_2 + tv_2, \\ ... \ & ... \ ... \\ x_n(t) &= p_n + tv_n. \end{aligned}$$

Another way to determine a line is to identify two points on the line. Suppose a line goes trough the points p and q. Then this is a line which passes trough p in the direction of the vector $v = q - p$, thus

$$x(t) = p + tv = p + t(q - p) = (1 - t)p + tq.$$

Equivalently, the equation of this line is

$$x(t_1, t_2) = t_1 p + t_2 q, \quad t_1 + t_2 = 0.$$

Line segment joining p to q:

$$\{(1 - t)p + tq, \quad 0 \leq t \leq 1\},$$

indeed, for $t = 0$ we have $x(0) = p$ and for $t = 1$ we have $x(t) = q$.

The *midpoint* between p and q corresponds to $t = \frac{1}{2}$ thus this is the point $\frac{1}{2}p + \frac{1}{2}q$.

Example. Is the point $x = (0.5, 1, 1)$ on the line which goes trough $p = (1, 0, 0)$ and $q = (0, 2, 2)$?

Solution. The parameterized equation of this line is $x(t) = (1 - t)p + tq = (1 - t)(1, 0, 0) + t(0, 2, 2) = (1 - t, 2t, 2t)$. The coordinates $(0.5, 1, 1)$ of the point x satisfy this equation for $t = 0.5$, so x is on this line. Moreover, since $t = 0.5$, then x is the midpoint of the $[p, q]$ segment.

3.2 Lines in R^2

In R^2 a line can be determined also by *non-parameterized* equation

$$ax_1 + bx_2 + c = 0.$$

For example $x_2 = 2x_1 + 3$ and $x_1 = 5$ are non-parameterized equations of lines.

3.2.1 From Non-Parametric to Parametric

If a line is given by an non-parameterized equation

$$ax_1 + bx_2 + c = 0,$$

then it is very easy to turn it to parameterized equation

$$(x_1(t) = p_1 + v_1 t, \quad x_2(t) = p_2 + v_2 t) :$$

just take $x_1(t) = t$ and solve x_2 from the given non-parameterized equation $x_2 = -\frac{a}{b}x_1 - \frac{c}{b}$, thus $x_2(t) = -\frac{a}{b}t - \frac{c}{b}$. So the corresponding parametric equation is

$$(x_1(t) = t, \quad x_2(t) = -\frac{a}{b}t - \frac{c}{b}).$$

3.2.2 From Parametric to Non-Parametric

Conversely, if a line is given by a parameterized equation

$$(x_1(t) = p_1 + v_1 t, \quad x_2(t) = p_2 + v_2 t)$$

then to turn it to non-parameterized one just solve these two equations for t:

$$t = \frac{x_1 - p_1}{v_1}, \quad t = \frac{x_2 - p_2}{v_2}$$

and set the equation

$$\frac{x_1 - p_1}{v_1} = \frac{x_2 - p_2}{v_2}.$$

Try to prove the following

Theorem. Suppose a line l is given by a parametric equation

$$(x_1(t) = p_1 + v_1 t, \quad x_2(t) = p_2 + v_2 t),$$

and by a non-parametric equation

$$ax_1 + bx_2 + c = 0.$$

Then the vector (a, b) is a normal (orthogonal) to the line l, that is

$$(a, b) \cdot (v_1, v_2) = 0.$$

Proof. The parametric form of $ax_1 + bx_2 + c = 0$ is $(x_1(t) = t,\ (x_1(t) = \frac{-c}{b} - \frac{a}{b}t)$, so the direction vector is $(v_1, v_2) = (1, -\frac{a}{b})$. Then $(a, b) \cdot (v_1, v_2) = a - b \cdot \frac{a}{b} = a - a = 0$.

Example. Find the parameterized and non-parameterized equations of the line which passes the points $p = (0, 2),\ q = (4, 0) \in R^2$.

Solution. The parameterized equation of this line is

$$x(t) = (1 - t)p + tq = (1 - t)(0, 2) + t(4, 0) = (4t, 2 - 2t),$$

or

$$\begin{aligned} x_1(t) &= 4t, \\ x_2(t) &= 2 - 2t. \end{aligned}$$

It is easy to turn this parameterized equation to non-parameterized one: solve t from both equations $t = \frac{x_1}{4},\ t = \frac{2 - x_2}{2}$, then the non-parameterized equation is

$$\frac{x_1}{4} = \frac{2 - x_2}{2},\ x_1 = 4 - 2x_2,\ x_1 + 2x_2 - 4 = 0.$$

Example. Let $p_1 = (0, 2),\ q_1 = (4, 0),\ p_2 = (0, 4),\ q_2 = (2, 0)$. Find the intersection of lines $p_1 q_1$ and $p_2 q_2$.

Solution. The equation of the p_1q_1 line is

$$x(t) = (1-t)p_1 + tq_1 = (1-t)(0,2) + t(4,0) = (4t, 2-2t).$$

The equation of the p_2q_2 line is

$$y(s) = (1-s)p_2 + sq_2 = (1-4)(0,2) + s(2,0) = (2s, 4-4s).$$

The intersection point is a point p such that

$$x(t) = p = y(s)$$

for some t and s. So in order to find the intersection of these two lines, we solve the system

$$\left\{ \begin{array}{c|c} 4t = 2s & 4t - 2s = 0 \\ 2 - 2t = 4 - 4s & -2t + 4s = 2 \end{array} \right. ,$$

whose solution is $t = \frac{1}{3},\ s = \frac{2}{3}$. Thus the intersection point is $x(\frac{1}{3}) = (\frac{4}{3}, \frac{4}{3})$, or $y(\frac{2}{3}) = (\frac{4}{3}, \frac{4}{3})$ which of course is the same point.

3.3 Planes in R^n

A plane in R^n is completely determined by three things: a *point* $p = (p_1, \ \dots, p_n)$ on the plane and *two linearly independent direction vectors* $v = (v_1, \ \dots, v_n)$ and $w = (w_1, \ \dots, w_n)$ from the point p. The parametric representation of this plane is

$$x(s,t) = p + sv + tw, \quad s, t \in R.$$

In coordinates

$$(x_1(s,t), \ \dots, x_n(s,t)) = (p_1 + sv_1 + tw_1, \ \dots, p_n + sv_n + tw_n),$$

or

$$\begin{aligned} & x_1(s,t) = p_1 + sv_1 + tw_1, \\ & x_2(s,t) = p_2 + sv_2 + tw_2, \\ & \dots \ \dots \ \dots \\ & x_n(s,t) = p_n + sv_n + tw_n. \end{aligned}$$

Another way to determine a plane is to identify three points on it. Suppose a plane goes trough the points p, q and r. Then this is a plane which passes trough p in the direction of the vectors $v = q - p$ and $w = r - p$, thus

$$x(s,t) = p + sv + tw = p + s(q-p) + t(r-p) = (1-s-t)p + sq + tr.$$

Equivalently, the equation of this plane is

$$x(t_1, t_2, t_3) = t_1p + t_2q + t_3r, \quad t_1 + t_2 + t_3 = 0.$$

If we assume $t_i \geq 0$, then this describes a point from the triangle on the plane with vertices $p,\ q,\ r$. The numbers $(t_1,\ t_2,\ t_3)$ are called barycentric coordinates of this point.

What are brycentric coordinates of vertices p, q, r? What are the barycentric coordinates of the barycenter of triangle?

3.4 Planes in R^3

A plane in R^3 is determined by the following two things: by a point $p = (x_0, y_0, z_0)$ on it and by a **normal** - a vector $n = (a, b, c)$ orthogonal to the plane. So a point $\mathbf{x} = (x, y, z)$ belongs to plane iff $x - p$ is orthogonal to n:

$$0 = n \cdot (\mathbf{x} - p) = (a, b, c) \cdot (x - x_0, y - y_0, z - z_0) =$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0).$$

So the nonparametric equation of a plane in R^3 is

$$ax + by + cz = d,$$

here $d = ax_0 + by_0 + cz_0$.

3.4.1 From Non-Parametric to Parametric

How to go from non-parametric equation

$$ax + by + cz = d,$$

to parametric one?

Just find three point on the plane, say

$$p = (\frac{d}{a}, 0, 0),\ \ q = (0, \frac{d}{b}, 0),\ \ r = (0, 0, \frac{d}{c})$$

and then write the parametric equation determined by these three points.

3.4.2 From Parametric to Non-Parametric

How to go from parametric equation

$$(x_0 + sv_1 + tw_1,\ \ y_0 + sv_2 + tw_2\ ,z_0 + sv_3 + tw_3)$$

to a nonparametric one $ax + by + cz = d$?

Just find a normal - a vector $n = (a, b, c)$ orthogonal to v and w, that is solve the system

$$\begin{cases} n \cdot v = 0 \\ n \cdot w = 0 \end{cases}$$

and then find d substituting (x_0, y_0, z_0).

Example. Suppose a plane is given by

$$x(s, t) = s, \quad y(s, t) = t, \quad z(s, t) = s.$$

Write for this plane $p,\ v,\ w$, a normal vector n and the nonparametric equation.

Solution. $p = (0,0,0)$, $v = (1,0,1)$, $w = (0,1,0)$. Let $n = (a,b,c)$, then $n \cdot v = 0$ and $n \cdot w = 0$ give the system

$$\begin{cases} a + c = 0 \\ b = 0 \end{cases}$$

whose solution is, for example, $a = 1$, $b = 0$, $c = -1$, so the nonparametric equation looks as

$$x - z = d.$$

To find d let us substitute $p = (0,0,0)$:

$$0 + 0 = d \quad \Rightarrow \quad d = 0.$$

Finally the nonparametric equation is $x - z = 0$. Try to draw this plane.

Here is the MAPLE command to plot this plane given by parametric equation

> plot3d([s,t,s],s=-3..3,t=-3..3);

and here when it is given by nonparametric one

> plot3d(x,x=-3..3,y=-3..3);

Example. Suppose a plane is given by

$$x + y - z = 0.$$

Write the parametric equation for this plane. Plot this plane.

Solution. Let us find first three (noncolinear) points on this plane.

First substitute $x = 0$, $y = 0$, then $z = 0$, so one point is $p = (0,0,0)$, the origin.

Now substitute $x = 1$, $y = 0$, then $z = 1$, so the second point is $q = (1,0,1)$.

Finally, take $x = 0$, $y = 1$, then $z = 1$, so the third point is $r = (0,1,1)$.

Thus the parametric equation of this plane is

$$p + s(q - p) + t(r - p) = (0,0,0) + s(1,0,1) + t(0,1,1) = (s, t, s + t).$$

Here is the MAPLE command to plot this plane given by parametric equation

> plot3d([s,t,s+t],s=-3..3,t=-3..3);

and here when it is given by nonparametric one

> plot3d(x+y,x=-3..3,y=-3..3);

Example. Let l be the line of intersection of the planes

$$x + y - z = 4, \quad and \quad x + 2y + z = 3.$$

Write the parametric equation of this line. At which point intersects this line the plane $x0y$?

Solution. First find the equation of l: solve the system

$$\begin{cases} x + y - z = 4 \\ x + 2y + z = 3 \end{cases}$$

$> solve(\{x + y - z = 4, x + 2 * y + z = 3\});$

y = -1 - 2 z, x = 5 + 3 z, z = z

so the equation of this intersection line is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 + 3t \\ -1 - 2t \\ t \end{pmatrix}.$$

At which point intersect this line the plane $x0y$? Just substitute $z = t = 0$, then $x = 5,\ y = -1$.

Just recheck the solution using maple
$> solve(\{x + y - z = 4, x + 2 * y + z = 3, z = 0\});$

z = 0, y = -1, x = 5

$> plot3d(\{4 - x - y, 3 - x - 2 * y\}, x = -6..6, y = -6..6);$

Exercises

1. Find the lengths of the vectors (i) $(3,4)$, (ii) $(1,2,3)$.

2. Find the distances (i) $d((0,0),(3,4))$, (ii) $d((5,2),(1,2))$.

3. Find the lengths of the vectors (i) $(3,0,0,0)$, (ii) $(1,1,1,1)$.

4. Find the distances
(i) $d((1,2,3,4),(1,0,-1,0))$, (ii) $d((1,2,1,2),(2,1,2,1))$.

5. Find the angle between the vectors u and v if (i) $u=(1,0),\ v=(-1,1)$; (ii) $u=(1,0,0),\ v=(0,0,1)$.

6. Find the angle between the vectors u and v if (i) $u=(1,0),\ v=(2,2)$; (ii) $u=(\sqrt{3},0),\ v=(0,1)$.

7. Find a vector of length 1 which points in the same direction as (i) (3,4); (ii) (6,0); (iii) (1,1,1); (iv) (-1,2,-3).

8. Find a vector of length 2 which points in the opposite direction to (i) (3,4); (ii) (6,0); (iii) (1,1,1); (iv) (-1,2,-3).

9. Consider the parallelogram determined by vectors $(1,0)$ and $(1,1)$. Find the angle between the diagonals of this parallelogram.

12. Derive parametric and nonparametric equations for the lines which pass through each of the following pairs of points in R^2:
a) (1,2) and (3, 6); b) (1,1) and (4,10); c) (3, 0) and (0, 4).

13. Write the parametric equations for each of the following lines:

$$\begin{aligned} &a)\ \ x_2 = 3x_1 - 7; \\ &b)\ \ 3x_1 + 4x_2 = 12. \end{aligned}$$

12. Write nonparametric equations for each of the following lines:

$$\begin{aligned} &a)\ \ x_1(t) = 3 - 4t, \quad x_2(t) = 1 + 2t; \\ &b)\ \ x_1(t) = 2t, \quad x_2(t) = 1 + t. \end{aligned}$$

13. For which value of k the lines given by nonparameteric equations $x_2 = 3x_1 - 7$ and $3x_1 + kx_2 = 12$ does not intersect each other.

14. For which value of k the lines given by parameteric equations $(x_1(t), x_2(t)) = (3,2) + t(-4,2)$ and $(x_1(s), x_2(s)) = (1,2) + s(k,2)$ does not intersect each other.

15. For which value of k the lines given by parameteric equations

$$(x_1(t), x_2(t), x_3(t)) = (3, 2, 1) + t(-4, 2, 1)$$

and

$$(x_1(s), x_2(s), x_3(s)) = (1, 2, 3) + s(k, 2, 1)$$

does intersect each other.

16. Let $p_1 = (0, 0, 4),\ q_1 = (2\sqrt{2}, 2\sqrt{2}, 0),\ p_2 = (0, 0, 4),\ q_2 = (\sqrt{2}, \sqrt{2}, 0)$. Find the intersection of lines p_1q_1 and p_2q_2.

17. Explain why the probability of intersection of randomly taken lines in R^2 is high and in R^3 is very low.

18. Write equations of two lines whose intersection point is (i) $(1, 2) \in R^2$, (ii) $(1, 2, 1) \in R^3$.

Exercises 10.1-10.41 from [SB].

Homework:
10.19, 10.20, 10.29, 10.36, 10.39 from [SB].

Summary

For $x = (x_1,\ \dots, x_n)$, $\ y = (y_1,\ \dots, y_n) \in R^n$:
Vector addition: $x + y = (x_1 + y_1,\ \dots, x_n + y_n)$.
Scalar multiplication: $c \cdot x = (c \cdot x_1,\ \dots, c \cdot x_n)$.
Inner Product: $x \cdot y = x_1 \cdot y_1 + \dots + x_n \cdot y_n = \sum_{k=1}^{n} x_k \cdot y_k$.

$$x \cdot y = ||x|| \cdot ||y|| \cdot cos\ \alpha.$$

Euclidian norm: $||x|| = \sqrt{x \cdot x}$.
Euclidian metric: $d(x, y) = ||(x_1,\ \dots, x_n) - (y_1,\ \dots, y_n)||$.
Angle Between Two Vectors:

$$cos\ \alpha = \frac{x \cdot y}{||x|| \cdot ||y||} = \frac{x_1 \cdot y_1 + \dots + x_n \cdot y_n}{\sqrt{x_1^2 + \dots + x_n^2} \cdot \sqrt{y_1^2 + \dots + y_n^2}}.$$

α is acute if $x \cdot y > 0$; α is obtuse if $x \cdot y < 0$; α is right if $x \cdot y = 0$.
Convex Set: $x, y \in X$ implies $[x, y] = \{(1 - t)x + ty,\ 0 \le t \le 1\} \subset X$.
Cone: $x, y \in X$ implies $\{(1 - t)x + ty,\ 0 \le t\} \subset X$.

Parametric equation of a line in R^n:

By point and direction vector $x(t) = p + tv, \quad t \in R$. In coordinates $(x_1(t),\ \dots, x_n(t)) = (p_1 + tv_1,\ \dots, p_n + tv_n)$.

By two points $x(t) = (1 - t)p + tq$.

Nonparametric equation of a line in R^2: $ax_1 + bx_2 + c = 0$

From non-parametric to parametric:

$$ax_1 + bx_2 + c = 0 \ \longrightarrow\ (x_1(t) = t,\ x_2(t) = -\frac{a}{b}t - \frac{c}{b}).$$

From parametric to non-parametric:

$$(x_1(t) = p_1 + v_1 t, \quad x_2(t) = p_2 + v_2 t) \ \longrightarrow\ \frac{x_1 - p_1}{v_1} = \frac{x_2 - p_2}{v_2}.$$

If a line l is given by $ax_1 + bx_2 + c = 0$ and $(x_1(t) = p_1 + tv_1,\ x_2(t) = p_2 + tv_2)$ then $(a, b) \cdot (v_1, v_2) = 0$.

Parametric equation of a plane in R^n:

By point and two direction vectors $x(s, t) = p + sv + tw, \quad s, t \in R$. In coordinates $(x_1(s, t),\ \dots, x_n(s, t)) = (p_1 + sv_1 + tw_1,\ \dots, p_n + sv_n + tw_n)$.

By three points $x(s, t) = (1 - s - t)p + sq + tr$.

Nonparametric equation of a plane in R^3: $ax + by + cz = d$