1 Determinants

[SB], Chapter 9, p.188-196. [SB], Chapter 26, p.719-739.

Bellow w'll study the central question: *which additional conditions must satisfy a quadratic matrix A to be invertible, that is to have A^{-1}?* This question is DETERMINED by DETERMINANT.

1.1 Determinant

There is a function which assigns to an $n \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

the real number denoted as

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

or *det* A, called **determinant** of A which has the properties described bellow.

1.1.1 Properties of Determinant

1.

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{i1}+b_{i1} & a_{i2}+b_{i2} & \dots & a_{in}+b_{in} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} =$$

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ b_{i1} & b_{i2} & \dots & b_{in} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

2. If B is obtained from A by multiplying of each entry of row i by a scalar r then $|B| = r \cdot |A|$.

3. If a matrix B is obtained by interchanging two rows of A then $|B| = -|A|$.

4. $|I| = 1$;

5. If two rows of A equal then $|A| = 0$ (prove it using 3).

6. If a matrix A has an all-zero row then $|A| = 0$ (prove it using 2).

7. Transform matrix A to matrix B by performing the *elementary row operation* of adding r times row i to row j of A to form row j of B (the other rows remain the same), then $|B| = |A|$ (prove it using 1,2,5).

8. $|A \cdot B| = |A| \cdot |B|$;

9. $|A^{-1}| = |A|^{-1}$ (prove it using 4,8).

10. $|A^T| = |A|$.

Remark 1. Since of the property 10 all the properties remain correct if we replace *row* by *column*.

Remark 2. The properties 1,2,3,4 are very essential. They define determinant uniquely: using these properties, and their consequences 5,6,7, we can transform a matrix to reduced row echelon form and trace the evolution of the determinant during this transformation, the final reduced row echelon form is either identity matrix with determinant 1, or a matrix with zero row, with determinant 0.

The *inductive* definition of determinant will be given bellow.

1.2 Minors and Cofactors

For an $n \times n$ matrix A let A_{ij} be the $(n-1) \times (n-1)$ submatrix obtained by deleting the i-th row and j-th column. The determinant of this matrix $M_{ij} = |a_{ij}|$ is called (i, j)-th *minor* of A.

$C_{ij} = (-1)^{i+j} M_{ij}$ is called (i, j)-th *cofactor* of A.

For example for $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ we have

$$A_{21} = \begin{pmatrix} 2 & 3 \\ 8 & 9 \end{pmatrix}, \qquad M_{21} = 2 \cdot 9 - 8 \cdot 3 = -6,$$

$$C_{21} = (-1)^{2+1}(-6) = (-1)^3(-6) = -(-6) = 6.$$

1.3 Laplas Expansion - Inductive Definition of Determinant

For a matrix $A = \begin{pmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{pmatrix}$ the determinant $|A|$ can be calculated by i-th row expansion

$$|A| = a_{i1} \cdot C_{i1} + a_{i2} \cdot C_{i2} + \dots + a_{in} \cdot C_{in} = \sum_{k=1}^{n} a_{ik} \cdot C_{ik},$$

or by j-th column expansion

$$|A| = a_{1j} \cdot C_{1j} + a_{2j} \cdot C_{2j} + \dots + a_{nj} \cdot C_{nj} = \sum_{k=1}^{n} a_{kj} \cdot C_{kj}.$$

All row expansions as well as all column expansions give the *same result*, so Laplas expansion can be used as an *inductive* definition of determinant.

1.3.1 Expansion by Alien Cofactors

For a matrix

$$A = \begin{pmatrix} a_{11} & ... & ... & a_{1n} \\ ... & ... & ... & \\ a_{i1} & ... & ... & a_{in} \\ ... & ... & ... & ... \\ a_{k1} & ... & ... & a_{kn} \\ ... & ... & ... & ... \\ a_{n1} & ... & ... & a_{nn} \end{pmatrix}$$

the i-th row expansion gives the determinant of A:

$$|A| = a_{i1} \cdot C_{i1} + a_{i2} \cdot C_{i2} + ... + a_{in} \cdot C_{in} = \sum_{k=1}^{n} a_{ik} \cdot C_{ik}.$$

Here the entries of the i-th row

$$a_{i1},\ a_{i2},\ ... ,a_{in}$$

are multiplied by cofactors of the *same* i-th row

$$C_{i1},\ C_{i2},\ ... ,C_{in}.$$

What happens if we multiply these cofactors by the entries of an *alien*, say k-th, row

$$a_{k1},\ a_{k2},\ ... ,a_{kn}?$$

Theorem. The expansion of a determinant by *alien* cofactors gives zero.
Proof. Consider the alien expansion which uses the entries of k-th row $a_{k1}, ..., a_{kn}$ and cofactors of i-th row $C_{i1}, ..., C_{in}$

$$a_{k1} \cdot C_{i1} + a_{k2} \cdot C_{i2} + ... + a_{in} \cdot C_{in}.$$

This is Laplas expansion of the matrix

$$A' = \begin{pmatrix} a_{11} & ... & ... & a_{1n} \\ ... & ... & ... & \\ a_{k1} & ... & ... & a_{kn} \\ ... & ... & ... & ... \\ a_{k1} & ... & ... & a_{kn} \\ ... & ... & ... & ... \\ a_{n1} & ... & ... & a_{nn} \end{pmatrix}$$

with two equal rows, thus $|A'| = 0$.

1.3.2 Determinant of a 3×3 matrix

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$a_{11} \cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \cdot \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \cdot \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{33} \end{vmatrix} =$$

$$\begin{aligned} &(a_{11} \cdot a_{22} \cdot a_{33} - a_{11} \cdot a_{23} \cdot a_{32}) - \\ &(a_{12} \cdot a_{21} \cdot a_{33} - a_{12} \cdot a_{23} \cdot a_{31}) + \\ &(a_{13} \cdot a_{21} \cdot a_{33} - a_{13} \cdot a_{22} \cdot a_{31}) = \\ &(a_{11} \cdot a_{22} \cdot a_{33} + a_{12} \cdot a_{23} \cdot a_{31} + a_{13} \cdot a_{21} \cdot a_{32}) - \\ &(a_{13} \cdot a_{22} \cdot a_{31} + a_{11} \cdot a_{23} \cdot a_{32} + a_{12} \cdot a_{21} \cdot a_{33}). \end{aligned}$$

1.4 Inverse Matrix

1.4.1 Adjoint Matrix

For a matrix

$$A = \begin{pmatrix} a_{11} & ... & ... & a_{1n} \\ ... & ... & ... & ... \\ a_{i1} & ... & ... & a_{in} \\ ... & ... & ... & ... \\ a_{n1} & ... & ... & a_{nn} \end{pmatrix}$$

The **adjoint matrix** $adj\ A$ is defined is the matrix

$$adj\ A = \begin{pmatrix} C_{11} & ... & ... & C_{n1} \\ ... & ... & ... & ... \\ C_{1i} & ... & ... & C_{ni} \\ ... & ... & ... & ... \\ C_{1n} & ... & ... & C_{nn} \end{pmatrix}$$

This is the transpose of the matrix which consists of cofactors C_{ij} of the elements a_{ij} of A.

Theorem.

$$A \cdot adj\ A = \begin{pmatrix} |A| & 0 & ... & 0 \\ 0 & |A| & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & |A| \end{pmatrix}$$

Proof. Suppose $A \cdot adj\ A = B = (b_{ij})$. Let us calculate these b_{ij}-s. First calculate the diagonal elements b_{ii}:

$$b_{ii} = \sum_{k+1}^{n} a_{ik} \cdot C_{ik},$$

but this is the Laplas expansion by the i-th row, so $b_{ii} = |A|$.

Now calculate b_{ij} for $i \neq j$:

$$b_{ij} = \sum_{k+1}^{n} a_{ik} \cdot C_{jk},$$

and this is the expansion by alien row, so $b_{ij} = 0$. This completes the proof.

1.4.2 Inverse Matrix

From this theorem follows that the inverse A^{-1} of a matrix A exits if and only if A is *nonsingular*, that is $|A| \neq 0$, and it is defined as

$$A^{-1} = \frac{1}{|A|} \cdot adj\ A = \begin{pmatrix} \frac{C_{11}}{|A|} & ... & ... & \frac{C_{n1}}{|A|} & \\ ... & ... & ... & & \\ \frac{C_{1i}}{|A|} & ... & ... & \frac{C_{ni}}{|A|} & \\ ... & ... & ... & ... & ... \\ \frac{C_{1n}}{|A|} & ... & ... & \frac{C_{nn}}{|A|} & \end{pmatrix}$$

1.5 Cramer's rule

For a system of n linear equations with n variables

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = c_1 \\ .... \\ .... \\ .... \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = c_n \end{cases}$$

we define $n+1$ matrixes $A,\ A_1,\ A_2,\ ...,,\ A_n$:

$$A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{pmatrix}, \quad A_k = \begin{pmatrix} a_{11} & ... & c_1 & ... & a_{1n} \\ ... & ... & ... & ... & ... \\ a_{n1} & ... & c_n & ... & a_{nn} \end{pmatrix}$$

here A_k is obtained by replacing in A the k-th column by the column of constants c.

Bellow we'll use the k-th column expansion of $|A_k|$:

$$|A_k| = c_1 \cdot C_{1k} + ... + c_n \cdot C_{nk}.$$

Theorem. (Cramer's Rule) Let A be a nonsingular matrix i.e. $|A| \neq 0$. Then the system $A \cdot x = c$ has unique solution given by

$$x_k = \frac{|A_k|}{|A|}, \quad k = 1, 2, ..., n \quad .$$

Proof. The solution in vector form is given by $x = A^{-1}c$, that is

$$\begin{pmatrix} x_1 \\ ... \\ x_n \end{pmatrix} = \frac{1}{|A|} \cdot \begin{pmatrix} C_{11} & C_{21} & ... & C_{n1} \\ ... & ... & ... & ... \\ C_{1n} & C_{2n} & ... & C_{nn} \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ ... \\ c_n \end{pmatrix} =$$

$$\frac{1}{|A|} \cdot \begin{pmatrix} c_1 \cdot C_{11} + c_2 \cdot C_{21} + \; ... \; + c_n \cdot C_{n1} \\ ... \\ c_n \cdot C_{1n} + c_2 \cdot C_{2n} + \; ... \; + c_n \cdot C_{nn} \end{pmatrix} = \frac{1}{|A|} \cdot \begin{pmatrix} |A_1| \\ ... \\ |A_n| \end{pmatrix} = \begin{pmatrix} \frac{|A_1|}{|A|} \\ ... \\ \frac{|A_n|}{|A|} \end{pmatrix},$$

this completes the proof.

What happens if the matrix A is *singular* that is if $|A| = 0$? This will be answered latter.

1.5.1 Homogenous System

Homogenous system is a system with all $c_i = 0$:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = 0 \\ .... \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = 0. \end{cases},$$

in matrix form $A \cdot x = 0$. Such a system is always consistent: $x_1 = 0, ..., x_n = 0$ is a solution.

But are there nontrivial solutions too?

Theorem. A homogeneous system has nontrivial solution if and only if $\Delta = 0$.

Now turn to the nonhomogenous system $A \cdot x = c$. If $|A| \neq 0$, then this system has unique solution given by Cramer formula.

If $|A| = 0$, then, as we know, the system has no, or infinitely many solutions.

Theorem. Suppose $\overline{x}$ is one particular solution of $A \cdot x = c$. Then any other solution looks as $\overline{x} + x_0$ where x is a solution of homogenous system $A \cdot x = 0$.

2 Rank of a Matrix

2.1 Definition of the Rank

The rank of a matrix is *maximum order of nonzero determinant that can be constructed from the rows and columns of that matrix.*

2.2 How to Calculate the Rank

2.2.1 Calculating Minors

By definition the rank of a matrix A is r if there exists nonzero minor of degree r but *all minors of higher degrees* are zero.

In fact there is no need to check *all higher minors*:

Theorem. If in a matrix A there exists nonzero minor M of degree r and all minors bordering it (that is, minors of order $r+1$ and containing M) are equal to zero then **rank A=r**.

2.2.2 Rank and Row Echelon Form

Theorem. If B is a row echelon form of a matrix A then $rank\ A = rank\ B$.

Theorem. The rank of a matrix in row echelon form coincides with the number of it's nonzero rows.

2.3 Solution of Systems of Linear Equations

2.3.1 Criterion of Consistence

Theorem. A linear system $A \cdot X = c$ is consistent if and only if the *rank of the matrix* A equals to the *rank of augmented matrix* $A|c$:

$$rank \begin{pmatrix} a_{11} & ... & a_{1n} \\ a_{21} & ... & a_{2n} \\ ... & ... & ... \\ a_{m1} & ... & a_{mn} \end{pmatrix} = rank \begin{pmatrix} a_{11} & ... & a_{1n}|c_1 \\ a_{21} & ... & a_{2n}|c_2 \\ ... & ... & ... \\ a_{m1} & ... & a_{mn}|c_m \end{pmatrix}.$$

2.3.2 Solution of Consistent Systems

Suppose $rank(A) = rank(A|c) = r$. We can assume that the nonzero minor of degree r (*the basic minor*) is $M_{(1,2,...,r);(1,2,...,r)}$ (left upper corner).

In this case the $(r+1)$-th, $(r+2)$-th, ... , m-th equations are linear combinations of first r equations, so they can be ignored.

The first r equations we write in the form

$$\begin{cases} a_{11}x_1 + ... + a_{1r}x_r = c_1 - (a_{1r+1}x_{r+1} + ... + a_{1n}x_n) \\ a_{21}x_1 + ... + a_{2r}x_r = c_1 - (a_{2r+1}x_{r+1} + ... + a_{2n}x_n) \\ .... \\ a_{r1}x_1 + ... + a_{rr}x_r = c_1 - (a_{r+1}x_{r+1} + ... + a_nx_n). \end{cases}$$

The determinant of this system $M_{(1,2,...,r);(1,2,...,r)}$ is **nonzero**, thus for each values of *free* (or independent, or exogenous) variables $x_{r+1}, x_{r+2}, ..., x_n$ we can find by Cramer's rule unique *basic* (or dependent, or endogenous) variables $x_1, x_2, ..., x_n$.

Then $x_1, x_2, ..., x_n,\ x_{r+1}, x_{r+2}, ..., x_n$ is a solution of our system.

2.3.3 Example

We want to solve the system

$$\begin{cases} x & + & 4y & + & 17z & + & 4t & = & 38 \\ 2x & + & 12y & + & 46z & + & 10t & = & 98 \\ 3x & + & 18y & + & 69z & + & 17t & = & 153 \end{cases}.$$

Write the augmented matrix $(A|c)$ of this system

$$\begin{pmatrix} 1 & 4 & 17 & 4 & | & 38 \\ 2 & 12 & 46 & 10 & | & 98 \\ 3 & 18 & 69 & 17 & | & 153 \end{pmatrix}.$$

Let us start to calculate the rank of

$$A = \begin{pmatrix} 1 & 4 & 17 & 4 \\ 2 & 12 & 46 & 10 \\ 3 & 18 & 69 & 17 \end{pmatrix}.$$

The minor $|a_{11}| = 1$ is nonzero, so the *rank* A is at last 1. Now take the 2×2 minor

$$\begin{vmatrix} 1 & 4 \\ 2 & 12 \end{vmatrix}$$

bordering the previous nonzero minor. It is equal to $1 \cdot 12 - 2 \cdot 4 = 8 \neq 0$, so *rank* A is at last 2.

Next we take the 3×3 minor

$$\begin{vmatrix} 1 & 4 & 17 \\ 2 & 12 & 46 \\ 3 & 18 & 69 \end{vmatrix}.$$

bordering the previous one. Calculation shows that it is zero, so this is bad choice. Let us try another 3×3 minor bordering previous nonzero 2×2 minor

$$\begin{vmatrix} 1 & 4 & 4 \\ 2 & 12 & 10 \\ 3 & 18 & 17 \end{vmatrix}.$$

Calculation shows that this minor is equal to 8. There are no larger minors in A, so this is a basic minor and *rank* $A = 3$.

Augmentation of A by c can not increase the rank, so the rank of $(A|c)$ is also 3, thus the system is consistent.

So we have one free variable z and 3 basic variables x, y, t.

Next we rewrite the system so that the *basic minor* becomes the determinant of system

$$\left\{ \begin{array}{rcrcrcrcr} x & + & 4y & + & 4t & = & 38 & - & 17z \\ 2x & + & 12y & + & 10t & = & 98 & - & 46z \\ 3x & + & 18y & + & 17t & = & 153 & - & 69z \end{array} \right.$$

and solve it by Cramer's rule:

$$x = \frac{\Delta_x}{\Delta} = \frac{\begin{vmatrix} 38-17z & 4 & 4 \\ 98-46z & 12 & 410 \\ 153-69z & 18 & 17 \end{vmatrix}}{\begin{vmatrix} 1 & 4 & 4 \\ 2 & 12 & 10 \\ 3 & 18 & 17 \end{vmatrix}} = \frac{80-40z}{8} = 10-5z,$$

$$y = \frac{\Delta_x}{\Delta} = \frac{\begin{vmatrix} 1 & 38-17z & 4 \\ 2 & 98-46z & 410 \\ 3 & 153-69z & 17 \end{vmatrix}}{\begin{vmatrix} 1 & 4 & 4 \\ 2 & 12 & 10 \\ 3 & 18 & 17 \end{vmatrix}} = \frac{32-24z}{8} = 4-3z,$$

$$t = \frac{\Delta_x}{\Delta} = \frac{\begin{vmatrix} 1 & 4 & 38-17z \\ 2 & 12 & 98-46z \\ 3 & 18 & 153-69z \end{vmatrix}}{\begin{vmatrix} 1 & 4 & 4 \\ 2 & 12 & 10 \\ 3 & 18 & 17 \end{vmatrix}} = \frac{24}{8} = 3.$$

So the solution is

$$x = 15-5z, \quad y = 4-3z, \quad z, \quad t = 3.$$

Exercises

1. Evaluate the following determinants

$$(a) \begin{pmatrix} 8 & 1 & 3 \\ 4 & 0 & 1 \\ 6 & 0 & 3 \end{pmatrix}. \quad (b) \begin{pmatrix} 1 & 2 & 3 \\ 4 & 7 & 5 \\ 3 & 6 & 9 \end{pmatrix}. \quad (c) \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}.$$

$$(d) \begin{pmatrix} 1 & 2 & 0 & 9 \\ 2 & 3 & 4 & 6 \\ 1 & 6 & 0 & -1 \\ 0 & -5 & 0 & 8 \end{pmatrix}. \quad (e) \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 3 & 6 & -5 \\ 0 & 4 & 0 & 0 \\ 9 & 6 & -1 & 8 \end{pmatrix}.$$

2. Calculate the determinant of lower-triangular 4×4 matrix

$$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}.$$

3. Calculate the determinant of upper-triangular 4×4 matrix.

4. Check that $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$.

5. Find A^{-1} for (a) $A = \begin{pmatrix} 4 & 5 \\ 4 & 2 \end{pmatrix}$. (b) $A = \begin{pmatrix} 2 & 4 & 0 \\ 4 & 6 & 3 \\ -6 & -10 & 0 \end{pmatrix}$.

6. Invert the coefficient matrix to solve the following systems

$$(a) \begin{cases} 2x_1 + x_2 = 5 \\ x_1 + x_2 = 3 \end{cases} \quad (b) \begin{cases} 2x_1 + 4x_2 = 2 \\ 4x_1 + 6x_2 + 3x_3 = 1 \\ -6x_1 - 10x_2 = 60 \end{cases}$$

7. What is the inverse of the 3×3 diagonal matrix $\begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}$.

8. Show that the inverse of 2×2 upper-triangular matrix is upper-triangular.

9. Show that the inverse of 2×2 lower-triangular matrix is lower-triangular.

10. Show that the inverse of 2×2 symmetric matrix is symmetric.

11. Calculate the rank of each of the following matrixes

$$(a)\ \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}. \quad (b)\ \begin{pmatrix} 2 & -4 & 2 \\ -1 & 2 & 1 \end{pmatrix}. \quad (c)\ \begin{pmatrix} 1 & 6 & -7 & 3 \\ 1 & 9 & -6 & 4 \\ 1 & 3 & -8 & 4 \end{pmatrix}.$$

$$(d)\ \begin{pmatrix} 1 & 6 & -7 & 3 & 5 \\ 1 & 9 & -6 & 4 & 9 \\ 1 & 3 & -8 & 4 & 2 \\ 2 & 15 & -13 & 11 & 16 \end{pmatrix}. \quad (e)\ \begin{pmatrix} 1 & 6 & -7 & 3 & 1 \\ 1 & 9 & -6 & 4 & 2 \\ 1 & 3 & -8 & 4 & 5 \\ & & & & \end{pmatrix}.$$

12. Solve the system whose augmented matrix is $\begin{pmatrix} 2 & -4 & 2 \\ -1 & 2 & 1 \end{pmatrix}$.

13. Solve the system whose augmented matrix is $\begin{pmatrix} 1 & 6 & -7 & 3 & 1 \\ 1 & 9 & -6 & 4 & 2 \\ 1 & 3 & -8 & 4 & 5 \end{pmatrix}$.

14. For the system

$$\begin{cases} x+ & 2y+ & z- & w = 3 & 1 \\ 3x+ & 6y- & z- & 3w = & 2 \end{cases}$$

(a) determine how many variables can be endogenous, (b) determine a successful separation into exogenous and endogenous variables, (c) find an explicit formula for the endogenous variables in terms of exogenous variables.

15. Find numbers a and b that make A the inverse of B when

$$A = \begin{pmatrix} 2 & -1 & -1 \\ a & \frac{1}{4} & b \\ \frac{1}{8} & \frac{1}{8} & \frac{-1}{8} \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 6 \\ 1 & 3 & 2 \end{pmatrix}.$$

Homework

1. For

$$\left(\begin{array}{c} w - x + 3y - z = 0 \\ w + 4x - y + z = 3 \\ 3w + 7x + y + z = 6 \\ 3w + 2x + 5y - z = 3 \end{array} \right.$$

(a) Check the consistence;
(b) Separate free and basic variables;
(c) Solve the system.

2. Solve the system

$$\left(\begin{array}{ccccccc} 2x & + & 3y & + & 3z & = & 2 \\ 2x & + & 2y & + & z & = & 5 \\ x & + & y & + & z & = & 14 \end{array}\right.$$

inverting the coefficient matrix.

3. Compose a system with 3 variables and 4 equations with
(a) No solution;
(b) One solution;
(c) Infinitely many solutions depending on one free variable;
(d) Infinitely many solutions depending on two free variables.

4. (a) Suppose $|A| = a$. Find $|-A|$.

(b) Prove that if all entries of A are all integers and $det\ A = \pm 1$ then the entries of A^{-1} are also integers.

(c) What can you say about the product of two symmetric matrices?

5. (a) There are only two 2×2 permutation matrices and both are symmetric. Is it true that any 3×3 permutation matrix is also symmetric?

(b) What can you say about the determinant of a permutation matrix?

(c) What can you say about the product of two permutation matrices?

(d) Find the inverse of various 2×2 and 3×3 permutation matrices. If you get some idea, prove the general theorem about the inverse of a permutation matrix.