1 Matrix Algebra

Reading [SB] 8.1-8.5, pp. 153-180.

1.1 Matrix Operations

1. Addition

$$\begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & ... & b_{1n} \\ b_{21} & b_{22} & ... & b_{2n} \\ ... & ... & ... & ... \\ b_{m1} & b_{m2} & ... & b_{mn} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & ... & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & ... & a_{2n}+b_{2n} \\ ... & ... & ... & ... \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & ... & a_{mn}+b_{mn} \end{pmatrix}.$$

This operation is

- associative $A+(B+C)=(A+B)+C$;
- commutative $A+B=B+A$;
- has a neutral element $O+A=A$, here O is the **null matrix**

$$O = \begin{pmatrix} 0 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 0 \end{pmatrix}.$$

2. Scalar multiplication

$$k \cdot \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix} = \begin{pmatrix} ka_{11} & ka_{12} & ... & ka_{1n} \\ ka_{21} & ka_{22} & ... & ka_{2n} \\ ... & ... & ... & ... \\ ka_{m1} & ka_{m2} & ... & ka_{mn} \end{pmatrix}.$$

3. Multiplication of matrixes

$$\begin{pmatrix} a_{11} & ... & a_{1j} & ... & a_{1n} \\ ... & ... & ... & ... & \\ a_{i1} & ... & a_{ij} & ... & a_{in} \\ ... & ... & ... & ... & ... \\ a_{m1} & ... & a_{mj} & ... & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & ... & b_{1j} & ... & b_{1k} \\ ... & ... & ... & ... & ... \\ b_{i1} & ... & b_{ij} & ... & b_{ik} \\ ... & ... & ... & ... & ... \\ b_{n1} & ... & b_{nj} & ... & b_{nk} \end{pmatrix} =$$

$$\begin{pmatrix} c_{11} & ... & c_{12} & ... & c_{1k} \\ ... & ... & ... & ... & ... \\ c_{i1} & ... & c_{ij} & ... & c_{ik} \\ ... & ... & ... & ... & ... \\ c_{m1} & ... & c_{mj} & ... & c_{mk} \end{pmatrix}$$

where

$$c_{ij} = a_{i1}b_{1j} + ... + a_{ik}b_{kj} + ... + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

1.2 Transpose of a matrix

Transpose of a $m \times n$ matrix A is the $n \times m$ matrix A^T whose i-th column is is the i-th row of A.

For example the transpose for $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ is $A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$.

Properties of transposes
1. $(A^T)^T = A$;
2. $(A + B)^T = A^T + B^T$;
3. $(A \cdot B)^T = B^T \cdot A^T$.

1.3 Multiplication matrix $\times$ column vector

$$\begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \\ .... \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n \end{pmatrix}$$

thus a system of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = c_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = c_2 \\ .... \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = c_m. \end{cases}$$

can be written in matrix form as

$$\begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ ... \\ c_n \end{pmatrix}$$

or

$$A \cdot x = c.$$

1.4 Special Kinds of Matrices

Bellow k denotes the number of rows and n denotes the number of columns.

Square matrix. $k = n$.

Column matrix. $n = 1$.

Row matrix. $k = 1$.

Diagonal matrix. $k = n$ and $a_{ij} = 0$ for $i \neq j$.
What can you say about the product of two diagonal matrices?

Upper-triangular matrix. $k = n$ and $a_{ij} = 0$ for $i > j$.
What can you say about the product of two upper-triangular matrices?

Lower-triangular matrix. $k = n$ and $a_{ij} = 0$ for $i < j$.
What can you say about the product of two lower-triangular matrices?

Symmetric matrix. $a_{ij} = a_{ji}$, equivalently $A^T = A$.
Is a symmetric matrix necessarily square matrix?
What can you say about the product of two symmetric matrices?

Idempotent matrix. $A \cdot A = A$.
Is an idempotent matrix necessarily square matrix?
What can you say about the determinant of an idempotent matrix?
What can you say about a nonsingular idempotent matrix?

Orthogonal matrix. $A^{-1} = A^T$.
What can you say about the determinant of an orthogonal matrix?
What can you say about the inverse of an orthogonal matrix?
What can you say about the transpose of an orthogonal matrix?
What can you say about the product of two orthogonal matrices?
What can you say about the sum of two orthogonal matrices?

Involutory matrix. $A \cdot A = I$.
Is an involutory matrix necessarily square matrix?
What can you say about the determinant of an involutory matrix?
What can you say about the inverse of an involutory matrix?
What can you say about a singular involutory matrix?
What can you say about the transpose of an involutory matrix?
What can you say about the symmetric involutory matrix?

Nilpotent matrix. $k = n$ and $A^m = 0$ for some positive integer m.

Permutation matrix. $k = n$ and each row and each column contains exactly one 1 and all other entries are 0.

Non singular matrix. $k = n$ is an invertible = nonzero determinant = maximal rank (see bellow).

2 Algebra of Square Matrices

The sum, difference, product of square $n \times n$ matrices is $n \times n$ again, besides the $n \times n$ identity matrix I is true multiplicative identity

$$I \cdot A = A \cdot I = A.$$

So the set of all $n \times n$ matrices M_n carries algebraic structure similar to that of real numbers R. But there are some differences:

1. Multiplication in M_n is not commutative: generally $A \cdot B \neq B \cdot A$.

2. M_n has zero divisors: there exist nonzero matrices $A,\ B \in M_n$ such that $A \cdot B = O$.

3. Not all nonzero matrices have inverse.

2.1 Inverse Matrix

Definition. A matrix $A \in M_n$ is called invertible there exists its inverse, a matrix A^{-1} such that

$$A \cdot A^{-1} = A^{-1} \cdot A = I.$$

Theorem. An $n \times n$ matrix A can have at most one inverse.
Proof. Suppose A^{-1} and $\bar{A}^{-1}$ are two inverses for A. Then from one hand side

$$A^{-1} \cdot A \cdot \bar{A}^{-1} = A^{-1} \cdot (A \cdot \bar{A}^{-1}) = A^{-1} \cdot I = A^{-1},$$

and from another hand side

$$A^{-1} \cdot A \cdot \bar{A}^{-1} = (A^{-1} \cdot A) \cdot \bar{A}^{-1} = I \cdot \bar{A}^{-1} = \bar{A}^{-1}$$

thus $A^{-1} = A^{-1} \cdot A \cdot \bar{A}^{-1} = \bar{A}^{-1}$.

2.1.1 Some Properties of Inverse

1. $(A^{-1})^{-1} = A$.
2. $(A^T)^{-1} = (A^{-1})^T$.
3. $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$.

2.2 Solving Systems Using Inverse

As we know each system of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = c_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = c_2 \\ .... \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = c_m. \end{cases}$$

can be written in matrix form $A \cdot x = c$.
Theorem. If A is invertible then the system of linear equations $A \cdot x = c$ has the unique solution given by $x = A^{-1} \cdot c$.
Proof.

$$A \cdot x = c \;\Rightarrow\; A^{-1} \cdot (A \cdot x) = A^{-1} \cdot c \;\Rightarrow\; (A^{-1} \cdot A) \cdot x = A^{-1} \cdot c \;\Rightarrow\; I \cdot x = A^{-1} \cdot c \;\Rightarrow\; x = A^{-1} \cdot c.$$

2.3 How to Find the Inverse matrix if it Exists

If a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{pmatrix}$$

is invertible, then A^{-1} can be found using Gauss-Jordan elimination for the matrix $(A|I)$ which is augmentation of A by the unit matrix I

$$(A|E) = \left(\begin{array}{cccc|cccc} a_{11} & a_{12} & ... & a_{1n} & 1 & 0 & ... & 0 \\ a_{21} & a_{22} & ... & a_{2n} & 0 & 1 & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... & \\ a_{n1} & a_{n2} & ... & a_{nn} & 0 & 0 & ... & 1. \end{array}\right)$$

Performing Gauss-Jordan elimination for $(A|I)$ we obtain $(I|A^{-1})$.

2.4 Inverse of an 2×2 Matrix

The Gauss-Jordan elimination for an 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

gives the following inverse matrix

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

The condition

$$ad - bc \neq 0$$

is necessary and sufficient for the existence of inverse matrix.

2.5 Input-output analysis for two goods production

To produce 1 unit of good I_1 a_{11} *units of good* I_1 *and* a_{21} *units of good* I_2 are needed.
To produce 1 unit of good I_2 a_{12} *units of good* I_1 *and* a_{22} *units of good* I_2 are needed.
The input-output matrix (or technology matrix) is $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.
Problem. How many units of good I_1 and good I_2 must be produced to satisfy the demand d_1 units of good I_1 and d_2 of good I_2?
Solution. Suppose x_1 units of good I_1 and x_2 units of good I_2 are produced. One part will be used for *production* and other for *consumption*.

$a_{11} \cdot x_1$ units of good I_1 will be used for production of good I_1;
$a_{12} \cdot x_2$ units of good I_1 will be used for production of good I_2;
d_1 units of good I_1 will be used for consumption.
Thus $x_1 = a_{11} \cdot x_1 + a_{12} \cdot x_2 + d_1$.
Similarly $x_2 = a_{21} \cdot x_1 + a_{22} \cdot x_2 + d_2$.

To find x_1 and x_2 for the consumptions d_1 and d_2 the following *Leontieff* system must be solved

$$\left\{ \begin{array}{l} x_1 = a_{11} \cdot x_1 + a_{12} \cdot x_2 + d_1 \\ x_2 = a_{21} \cdot x_1 + a_{22} \cdot x_2 + d_2 \end{array} \right| \Rightarrow$$

$$\left\{ \begin{array}{l} (1 - a_{11}) \cdot x_1 - a_{12} \cdot x_2 = d_1 \\ -a_{21} \cdot x_1 + (1 - a_{22}) \cdot x_2 = d_2 \end{array} \right| .$$

The matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is called input-output, or technology matrix.

The matrix of this system

$$I - A = \begin{pmatrix} 1 - a_{11} & -a_{12} \\ -a_{21} & 1 - a_{22} \end{pmatrix}$$

is called *Leontieff* matrix.

2.5.1 Example

A company produces two products CORN (C) and FERTILIZER (F).
To produce 1 ton of C are needed: 0.1 tons of C and 0.8 tons of F.
To produce 1 ton of F are needed: 0.5 tons of C and 0 tons of F.
Problem. How many tons of C and F must be produced for consumption

of 4 tons of C and 2 tons of F?
Solution. Suppose for this x_C tons of C and x_F tons of F. Then

$$\left\{\begin{array}{l} x_C = 0.1x_C + 0.5x_F + 4 \\ x_F = 0.8x_C + 2 \end{array}\right. \left|\begin{array}{l} 0.9x_C - 0.5x_F = 4 \\ -0.8x_C + x_F = 2 \end{array}\right. .$$

2.6 Productivity of Input-output Matrix

The economical nature of the problem dictates the following restrictions:
(i) All input-output coefficients a_{ij} are nonnegative.
(ii) All the demands d_i are nonnegative.
(iii) The solutions x_i also must be nonnegative.
So we need not all but just nonnegative solutions of the Leontief system.

Definition 1 *An input-output matrix A is called productive if the Leontief system has unique nonnegative solution for each nonnegative demand vector d.*

What conditions must satisfy the input-output matrix A for this?

Definition 2 *An industry I_k is called profitable if the sum of the k-th column of the input-output matrix is less than 1.*

Theorem 1 *If an input-output matrix A is nonnegative, that is $a_{ij} \geq 0$ and the sum of entries of each column is less than 1, then the Leontief matrix $I - A$ has inverse $(I - A)^{-1}$ with all nonnegative entries.*

Proof. Let us proof the theorem for

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

By the assumptions of theorem, the input-output matrix has a form

$$A = \begin{pmatrix} 1 - c - \epsilon & b \\ c & 1 - b - \delta \end{pmatrix}$$

with $0 \leq b < 1,\ 0 \leq c < 1,\ \epsilon > 0,\ \delta > 0$. Then its Leontieff matrix looks as

$$I - A = \begin{pmatrix} c + \epsilon & -b \\ -c & b + \delta \end{pmatrix},$$

thus its determinant is

$$\Delta = bc + c\delta + b\epsilon - bc = c\delta + b\epsilon + \epsilon \cdot \delta$$

which, by assumption, is positive, thus $I - A$ is invertible. Furthermore, the inverse $(I - A)^{-1}$ looks as

$$\frac{1}{\Delta} \cdot \begin{pmatrix} b+\delta & b \\ c & c+\epsilon \end{pmatrix}$$

where all entries are nonnegative. Q.E.D.

Corollary 1 *If each industry is profitable, then the economy is productive.*

Proof. For each demand vector d the solution of the Leontief system is given by $x = (I - A)^{-1} \cdot d$ and since all entries of $(I - A)^{-1}$ and d are nonnegative, each x_i is nonnegative too. Q.E.D.

So the profitability of each industry is sufficient for productivity. But not necessary: take

$$A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

where the second industry is definitely nonprofitable, nevertheless its Leontieff matrix looks as

$$I - A = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

and its inverse

$$(I - A)^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

is a nonnegative matrix.

3 Markov Chains

Reading: [Chang], section 4.7, p. 78-81, [Simon], section 23.6, p. 615-619.

The Markov process is a process when the state of the system depends on the previous state and not on the whole history.

Suppose there are n populations. A Markov process is described by a *transition matrix (Stochastic matrix, Markov matrix)*

$$M = \begin{pmatrix} P_{11} & P_{12} & \ldots & P_{1n} \\ \ldots & \ldots & \ldots & \ldots \\ P_{n1} & P_{n2} & \ldots & P_{nn} \end{pmatrix}$$

where P_{ij} denotes the probability of moving from population j to population i.

The sum of elements of each column of Markov matrix is 1: $\sum_{k=1}^{n} a_{kj} = 1$.

Suppose $x[t] = \begin{pmatrix} x_1[t] \\ ... \\ x_n[t] \end{pmatrix}$ is the state of the system at the moment t. Then after one transition the state of the system will be $x[t+1] = M \cdot x[t]$, and after k transition the state of the system will be $x[t+k] = M^k \cdot x[t]$.

A **steady state** is defined as $x[t]$ such that $x[t+1] = M \cdot x[t] = x[t]$, that is it remains unchanged and the process remains stabile starting from this state.

If $x[t]$ is a steady state, then $k \cdot x[t]$ is a steady state too:

$$M \cdot (k \cdot x[t]) = k \cdot (M \cdot x[t]) = k \cdot x[t].$$

If you remember the definitions of eigenvectors and eigenvalue (if not - do not mind, we come back to this question later):

Theorem. A Markov matrix has an eigenvalue equal to 1.

An eigenvector x corresponding to this eigenvalue $\lambda = 1$ is a *stady-state* that is it does not change by further transitions: $M \cdot x = x$.

How to find a steady state for a Markov matrix $\begin{pmatrix} p & q \\ 1-p & 1-q \end{pmatrix}$? Just solve $\begin{pmatrix} p & q \\ 1-p & 1-q \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$:

$$\begin{cases} px + qy = x \\ (1-p)x + (1-q)y = y \end{cases} \left| \begin{array}{l} px + qy = x \\ px + qy = x \end{array} \right| (p-1)x + qy = 0.$$

3.1 Examples

Example 1

Two populations A and B.
P_{AA} probability that A remains in A
P_{AB} probability that A moves to B
P_{BB} probability that B remains in B
P_{BA} probability that B moves to A

$$P_{AA}, P_{AB}, P_{AA}, P_{BB} \geq 0,$$
$$P_{AA} + P_{AB} = 1, \quad P_{BA} + P_{BB} = 1.$$

Transition matrix (Stochastic matrix, Markov matrix)

$$M = \begin{pmatrix} P_{AA} & P_{BA} \\ P_{AB} & P_{BB} \end{pmatrix}.$$

Let $x_A(0)$ be the initial population of A and $x_B(0)$ be the initial population of B. After one transition the distribution of population will be

$$\begin{pmatrix} x_A(1) \\ x_B(1) \end{pmatrix} = \begin{pmatrix} P_{AA} & P_{BA} \\ P_{AB} & P_{BB} \end{pmatrix} \cdot \begin{pmatrix} x_A(0) \\ x_B(0) \end{pmatrix} =$$

$$\begin{pmatrix} P_{AA} \cdot x_A(0) + P_{BA} \cdot x_B(0) \\ P_{AB} \cdot x_A(0) + P_{BB} \cdot x_B(0) \end{pmatrix}$$

After n transition the distribution of population will be

$$\begin{pmatrix} x_A(n) \\ x_B(n) \end{pmatrix} = \begin{pmatrix} P_{AA} & P_{BA} \\ P_{AB} & P_{BB} \end{pmatrix}^n \cdot \begin{pmatrix} x_A(0) \\ x_B(0) \end{pmatrix}$$

Example 2

Let for the population L the probability to stay in L during one year be 0.8 and the probability to pass to D be 0.2.

Let for the representatives of D the probability to stay in D be 1 and the probability to pass to L be 0.

Let the initial population of L and D be $x_L(0) = 1000,\ x_D(0) = 0$. What will be the populations of L and D after 1 year? After 2 years?

Solution.

$$\begin{pmatrix} x_L(1) \\ x_D(1) \end{pmatrix} = \begin{pmatrix} 0.8 & 0 \\ 0.2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1000 \\ 0 \end{pmatrix} = \begin{pmatrix} 800 \\ 200 \end{pmatrix},$$

$$\begin{pmatrix} x_L(2) \\ x_D(2) \end{pmatrix} = \begin{pmatrix} 0.8 & 0 \\ 0.2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 800 \\ 200 \end{pmatrix} = \begin{pmatrix} 640 \\ 360 \end{pmatrix}.$$

Example 3. Unemployment

Assume the population is divided into two parts: E-employed persons and U-unemployed persons.

Suppose further that the probability of an employed person to remain employed during one week is q. Then the probability to loose a job is $1 - q$.

Similarly, suppose the probability of an unemployed person to find a job after one week is p. Then the probability to satay unemployed is $1 - p$.

So the transition matrix looks as

$$\begin{pmatrix} q & p \\ 1-q & 1-p \end{pmatrix}.$$

Let x_0 be the percentage of employed persons and y_0 be the percentage of unemployed persons (**the unemployment rate**), so $x_0 + y_0 = 1$.

How the unemployment rate will change after one week?

The answer is

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} q & p \\ 1-q & 1-p \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} qx_0 + py_0 \\ (1-q)x_0 + (1-p)y_0 \end{pmatrix}.$$

Exercises

1. Let

$$A=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \quad B=\begin{pmatrix} b_{11} & b_{12} & \dots & b_{1q} \\ b_{21} & b_{22} & \dots & b_{2q} \\ \dots & \dots & \dots & \dots \\ b_{p1} & b_{p2} & \dots & b_{pq} \end{pmatrix}, \quad x=\begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_r \end{pmatrix}$$

Find the values of m, n, p, q, r for which exist the following products, and find the dimensions of these products when they exist:
(a) $A \cdot B$, (b) $B \cdot A$, (c) $B^T \cdot A^T$, (d) $A^T \cdot B^T$, (e) $A \cdot B^T$,
(f) $A \cdot x$, (g) $A \cdot x^T$, (h) $x \cdot A$, (i) $x^T \cdot A$ (j) $x \cdot x^T$, (k) $x^T \cdot x$, (l) $x \cdot x$, (m) $x^T \cdot x^T$.

2. Suppose for the population of L the probability to stay in L during one year is 0.8 and the probability to pass to D is 0.2.

Suppose for the representatives of D the probability to stay in D is 0.1 and the probability to pass to L is 0.9.

(a) Let the initial population of L be $x[0] = 1000$ and the initial population of D be $y[0] = 2000$. What will be the populations of L and D after 1 year $x[1], y[1]$? After 2 years $x[2], y[2]$?

(b) Find the steady-state with $y = 1500$.

(c) Find the steady-state with $x = 5000$.

3. Let $M = \begin{pmatrix} q & p \\ r & s \end{pmatrix}$ be a Markov matrix.

(a) Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ be a starting state and

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = M \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Show that $x_0 + y_0 = x_1 + y_1$.

(b) Find the steady-state(s) $\begin{pmatrix} x_s \\ y_s \end{pmatrix}$ with $x_s + y_s = d$.

4. Suppose we consider a simple economy with a lumber industry and a power industry. Suppose further that production of 10 units of power require 4 units of power and 25 units of power require 5 units of lumber. 10 units of lumber require 1 unit of lumber and 25 units of lumber require 5 units of power. If surplus of 30 units of lumber and 70 units of power are desired, find the gross production of each industry.

5. The economy of a developing nation is based on agricultural products, steel, and coal. An input of 1 ton of agricultural products requires an input

of 0.1 ton of agricultural products, 0.02 ton of steel, and 0.05 ton of coal. An output of 1 ton of steel requires an input of 0.01 ton of agricultural products, 0.13 tons of steel, and 0.18 tons of coal. An output of 1 ton of coal requires an input of 0.01 ton of agricultural products, 0.2 tons of steel, and 0.05 ton of coal. Find the necessary gross productions to provide surpluses of 2350 tons of agricultural products, 4552 tons of steel, and 911 tons of coal.

Exercises 6.3, 6.4, 6.5, 6.6, 8.1-8.5, 8.6-8.10, 8.15-8.22, 8.24-8.29

Homework

Exercises 8.6, 8.19, 8.24 from [SB], problem 3, problem 4.