

# Systems of Linear Equations

## Introduction

### 1 Linear Equation

$$a \cdot x = b.$$

Solution:

**Case 1.**  $a \neq 0$ , then  $x = \frac{b}{a}$  (one solution).

**Case 2.**  $a = 0$ ,  $b \neq 0$ , then  $x \in \emptyset$  (no solutions).

**Case 3.**  $a = 0$ ,  $b = 0$ , then  $x \in R$  (infinitely many solutions, moreover, any  $x \in R$  is a solution).

#### 1.1 Geometrical Interpretation

Solution is the  $x$ -intercept of the graph of the function  $y = a \cdot x - b$ .

**Case 1.** Slope =  $a \neq 0$  - one intersection.

**Case 2.** Slope =  $a = 0$ ,  $b \neq 0$  - the graph is parallel to  $x$  axes - no intersection.

**Case 3.** Slope =  $a = 0$ ,  $b = 0$  - the graph coincides with  $x$  axes - infinitely many intersections.

### 2 System of Linear Equations

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

#### 2.1 Substitution Method

Suppose  $b_1 \neq 0$ , then from the first equation  $y = \frac{c_1 - a_1x}{b_1}$ .

Substituting to the second we obtain one variable equation

$$a_2x + b_2 \cdot \frac{c_1 - a_1x}{b_1} = c_2.$$

## 2.2 Elimination Method

Multiply the first equation by  $-\frac{a_2}{a_1}$  and add to the second

$$\begin{cases} a_1x + b_1y = c_1 & | & a_1x & + & b_1y & = & c_1 \\ a_2x + b_2y = c_2 & | & (a_2 - \frac{a_2}{a_1} \cdot a_1) \cdot x & + & (b_2 - \frac{a_2}{a_1} \cdot b_1)y & = & c_2 - \frac{a_2}{a_1} \cdot c_1, \end{cases}$$
$$\begin{cases} a_1x + & b_1y & = & c_1 \\ & (b_2 - \frac{a_2}{a_1} \cdot b_1)y & = & c_2 - \frac{a_2}{a_1} \cdot c_1. \end{cases}$$

Thus

$$y = \frac{c_2 - \frac{a_2c_1}{a_1}}{b_2 - \frac{a_2b_1}{a_1}} = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

and the back substitution gives

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}.$$

## 2.3 Determinant Method

Assign to a system

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

three DETERMINANTS

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1,$$

$$\Delta_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = c_1b_2 - c_2b_1, \quad \Delta_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = a_1c_2 - a_2c_1.$$

### 2.3.1 Cramer's Rule

**Case 1.** If  $\Delta \neq 0$  then

$$x = \frac{\Delta_x}{\Delta}, \quad y = \frac{\Delta_y}{\Delta}.$$

**Case 2.** If  $\Delta = 0$  and either  $\Delta_x \neq 0$  or  $\Delta_y \neq 0$  then the system has NO SOLUTIONS.

**Case 3.** If  $\Delta = 0$  and  $\Delta_x = 0$ ,  $\Delta_y = 0$  then the system has INFINITELY MANY SOLUTIONS.

**Example.** Try to solve the system

$$\begin{cases} x + 0 \cdot y = c_1 \\ x + 0 \cdot y = c_2 \end{cases}$$

for various  $c_1$  and  $c_2$ .

## 2.4 Geometrical Interpretation

Each equation of the system defines linear function

$$y = -\frac{a_1}{b_1}x + \frac{c_1}{b_1}, \quad y = -\frac{a_2}{b_2}x + \frac{c_2}{b_2}.$$

A solution of the system is the intersection point of their graphs.

**Case 1.** These two graphs have different slopes thus they have one intersection point:

$$\Delta = a_1b_2 - a_2b_1 \neq 0 \Rightarrow a_1b_2 \neq a_2b_1 \Rightarrow -\frac{a_1}{b_1} \neq -\frac{a_2}{b_2}.$$

**Case 2.** These two graphs have equal slopes but different y-intercepts thus they are parallel:

$$\Delta = a_1b_2 - a_2b_1 = 0 \Rightarrow a_1b_2 = a_2b_1 \Rightarrow -\frac{a_1}{b_1} = -\frac{a_2}{b_2},$$

but

$$\Delta_x = c_1b_2 - c_2b_1 \neq 0 \Rightarrow c_1b_2 \neq c_2b_1 \Rightarrow -\frac{c_1}{b_1} \neq -\frac{c_2}{b_2}.$$

**Case 3.** These two graphs have equal slopes and the same y-intercepts thus they coincide:

$$\Delta = a_1b_2 - a_2b_1 = 0 \Rightarrow a_1b_2 = a_2b_1 \Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2},$$

but

$$\Delta_x = c_1b_2 - c_2b_1 = 0 \Rightarrow c_1b_2 = c_2b_1 \Rightarrow \frac{c_1}{b_1} = \frac{c_2}{b_2}.$$

## 2.5 More About the Case 3

In this case the system has infinitely many solution but not all the couples  $(x, y)$  form a solution!

$\Delta = a_1b_2 - a_2b_1 = 0$ , means that  $a_1b_2 = a_2b_1$  thus  $\frac{a_1}{a_2} = \frac{b_1}{b_2}$ .

Similarly  $\Delta_x = c_1b_2 - c_2b_1 = 0 \Rightarrow c_1b_2 = c_2b_1 \Rightarrow \frac{c_1}{c_2} = \frac{b_1}{b_2}$ .

So in this case  $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$  i.e. the equations are proportional

$$\begin{cases} a_1x + b_1y = c_1 \\ ka_1x + kb_1y = kc_1 \end{cases}$$

and any solution of the first equation satisfies the second one.

The couples  $(x, y = \frac{c_1 - a_1x}{b_1})$  form all the solutions.

### 3 General form of $m$ linear equations with $n$ variables

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = c_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = c_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = c_m. \end{cases}$$

Three ingredients of the system

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} \quad c = \begin{pmatrix} c_1 \\ c_2 \\ \dots \\ c_m \end{pmatrix}$$

A coefficient matrix of the system of order  $m \times n$ ,  $x$  vector of variables of order  $n \times 1$ ,  $c$  vector of constants of order  $m \times 1$ .

$A$  and  $C$  together form so called **augmented matrix**

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & c_1 \\ a_{21} & a_{22} & \dots & a_{2n} & c_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & c_m \end{pmatrix}.$$

The method of elimination of variables described bellow uses the following **elementary equation operations**:

1. Adding a multiple of one equation to another.
2. Multiplying both sides of an equation by nonzero scalar.
3. Interchanging of two equations.

If a system is obtained from another one by these operations, they are **equivalent**, that is they have same solutions.

To these three **elementary equation operations** correspond three **elementary row operations** on the augmented matrix:

1. Adding a multiple of one row to another.
2. Multiplying each element of a row by the same nonzero scalar.
3. Interchanging of two rows.

**Definition.** A row of a matrix is said to have  $k$  **leading zeros** if first  $k$  elements of the row are all zeros.

**Definition.** A matrix is in **row echelon form** if each row has more leading zeros than the row preceding it.

The first non-zero entry in each row is called a **pivot**.

**Definition.** A row echelon matrix is in **reduced row echelon form** if each pivot is 1 and each column containing a pivot has no other nonzero entries.

**Example.** The matrix  $A$  is in row echelon form, and  $B$  is in reduced row echelon form

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Every non-zero matrix can be reduced to an infinite number of row echelon forms (they can all be multiples of each other, for example) via elementary matrix transformations.

### 3.1 Gaussian Elimination

Step 1. Forward elimination:

Choose a pivot element on the main diagonal (start at top left corner)

Eliminate (set to zero) all elements below the main diagonal

Move to the next column

Repeat until matrix is in row echelon form.

Step 2. Solve the unknowns using backward substitution. If necessary introduce **free** variables and solve **basic** variables.

### 3.2 Gaussian-Jordan Elimination

This method reduces the matrix even further to reduced row echelon form. For this:

Choose a pivot element on the main diagonal (start at top left corner)

Eliminate (set to zero) all elements **above and below** the main diagonal

Move to the next column

**Another way:** suppose we have a matrix already in row echelon form. Then first convert each pivot  $a$  to 1 multiplying its row by  $\frac{1}{a}$ , then use these pivots (starting with the 1 in the last row) to kill all the entries above it.

### 3.2.1 Example of a system with one solution.

Consider the system

$$\begin{cases} 2x + y - z = 8 \\ -3x - y + 2z = -11 \\ -2x + y + 2z = -3 \end{cases}$$

The augmented matrix of this system is

$$\left( \begin{array}{cccc} 2 & 1 & -1 & 8 \\ -3 & -1 & 2 & -11 \\ -2 & 1 & 2 & -3 \end{array} \right).$$

The Gauss elimination algorithm gives

$$\left( \begin{array}{cccc} 2 & 1 & -1 & 8 \\ -3 & -1 & 2 & -11 \\ -2 & 1 & 2 & -3 \end{array} \right)$$

$$\Downarrow \frac{3}{2} \cdot I + II$$

$$\left( \begin{array}{cccc} 2 & 1 & -1 & 8 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ -2 & 1 & 2 & -3 \end{array} \right).$$

$$\Downarrow I + III$$

$$\left( \begin{array}{cccc} 2 & 1 & -1 & 8 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 2 & -1 & 5 \end{array} \right).$$

$$\Downarrow -4 \cdot II + III$$

$$\left( \begin{array}{cccc} 2 & 1 & -1 & 8 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 0 & -1 & 1 \end{array} \right).$$

This is the row echelon form.

So our system is equivalent to the following system

$$\begin{cases} 2x + y - z = 8 \\ \frac{1}{2} \cdot y + \frac{1}{2} \cdot z = 1 \\ -z = 1 \end{cases}$$

and  $x$ ,  $y$ ,  $z$  can be found by back substitution:

$$x = 2, \quad y = 3, \quad z = -1.$$

The Gauss-Jordan elimination algorithm gives the reduced row echelon form

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

So the solution is the last column

$$x = 2, \quad y = 3, \quad z = -1.$$

### 3.2.2 Example of a system with many solutions.

Consider the system

$$\begin{cases} x + 2y + 0 \cdot z + 3t = 5 \\ 2x + 3y + 2z + 5t = 4 \\ 0 \cdot x + 2y + z + 5t = 2 \end{cases}$$

The augmented matrix of this system is

$$\begin{pmatrix} 1 & 2 & 0 & 3 & 5 \\ -2 & 3 & 2 & 5 & 4 \\ 0 & 2 & 1 & 5 & 2 \end{pmatrix}.$$

The Gauss elimination algorithm gives the following row echelon form

$$\begin{pmatrix} 1 & 2 & 0 & 3 & 5 \\ 0 & -1 & 2 & -1 & -6 \\ 0 & 0 & 5 & 3 & -10 \end{pmatrix}.$$

Corresponding system looks as

$$\begin{cases} x + 2y + 0 \cdot z + 3t = 5 \\ -y + 2z - t = -6 \\ 5z + 3t = -10 \end{cases}$$

Take free variable  $t$ . Then the back substitution gives

$$z = -\frac{3}{5}t - 2, \quad y = -\frac{11}{5}t + 2, \quad x = \frac{7}{5}t + 1.$$

### 3.2.3 Example of a system with no solutions.

Consider the system

$$\begin{cases} 2x + 3y + z = 1 \\ x + y + 2z = 2 \\ 3x + 4y + 3z = 4 \end{cases}$$

The augmented matrix of this system is

$$\begin{pmatrix} 2 & 3 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 3 & 4 & 3 & 4 \end{pmatrix}.$$

The Gauss elimination algorithm gives the following row echelon form

$$\begin{pmatrix} 2 & 3 & 1 & 1 \\ 0 & -1 & 3 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Corresponding system looks as

$$\begin{cases} 2x + 3y + z = 1 \\ 0 \cdot x - y + 3z = 3 \\ 0 \cdot x + 0 \cdot y + 0 \cdot z = 4 \end{cases}$$

so the system has no solutions, that is it is non consistent.

## 4 Economical Examples

### 4.1 One-commodity market equilibrium

$Q_d$  - demand of commodity,  $Q_s$  - supply of commodity,  $P$  - price per unit.

**Linear model**

$Q_d(P) = -kP + b$  (decreasing function),

$Q_s(P) = lP + c$  (increasing function).

Equilibrium condition  $Q_d(P) = Q_s(P)$ .

Equilibrium price

$$-k \cdot P + b = l \cdot P + c, \quad (l + k) \cdot P = b - c,$$

$$P = \frac{b - c}{l + k}.$$



## 4.2 Two-commodity market equilibrium

$Q_{d_1}$  demand of commodity 1;  $Q_{s_1}$  supply of commodity 1;

$Q_{d_2}$  demand of commodity 2,  $Q_{s_2}$  supply of commodity 2;

$P_1$  price per unit of commodity 1;  $P_2$  price per unit of commodity 2.

Demand function for commodity 1:

$$Q_{d_1}(P_1, P_2) = a_0 + a_1P_1 + a_2P_2.$$

Demand function for commodity 2:

$$Q_{d_2}(P_1, P_2) = \alpha_0 + \alpha_1P_1 + \alpha_2P_2.$$

Supply function for commodity 1:

$$Q_{s_1}(P_1, P_2) = b_0 + b_1P_1 + b_2P_2.$$

Supply function for commodity 2:

$$Q_{s_2}(P_1, P_2) = \beta_0 + \beta_1P_1 + \beta_2P_2.$$

Equilibrium conditions  $Q_{d_1} = Q_{s_1}$ ,  $Q_{d_2} = Q_{s_2}$  :

$$\begin{cases} a_0 + a_1P_1 + a_2P_2 = b_0 + b_1P_1 + b_2P_2 \\ \alpha_0 + \alpha_1P_1 + \alpha_2P_2 = \beta_0 + \beta_1P_1 + \beta_2P_2 \end{cases} \Rightarrow$$
$$\begin{cases} (a_1 - b_1) \cdot P_1 + (a_2 - b_2) \cdot P_2 = b_0 - a_0 \\ (\alpha_1 - \beta_1) \cdot P_1 + (\alpha_2 - \beta_2) \cdot P_2 = \beta_0 - \alpha_0 \end{cases}.$$

Equilibrium prices  $P_1$  and  $P_2$  can be solved from this system.

## 4.3 Equilibrium in National Income

$I$  investment,  $G$  spending by government,  $a$  autonomous consumption,  $b$  marginal propensity to consumption,  $Y$  National Income,  $C$  consumption.

Equilibrium condition

$$\begin{cases} Y = C + I + G \\ C = a + bY \end{cases} \quad \begin{cases} Y - C = I + G \\ bY - C = -a \end{cases} \quad \begin{cases} Y = \frac{I+G+a}{1-b} \\ C = \frac{a+b \cdot (I+G)}{1-b} \end{cases}.$$

## Exercises

Solve these systems using all 3 methods (Gauss, Gauss-Jordan, determinant), and give the suitable graphical interpretation for

$$1. \begin{cases} 2x - y = 5 \\ x + y = 4 \end{cases} \quad 2. \begin{cases} 2x - y = 5 \\ 4x - 2y = 4 \end{cases} \quad 3. \begin{cases} 2x - y = 5 \\ 4x - 2y = 10 \end{cases}$$

2. Give examples of systems  $2 \times 2$  and  $3 \times 3$  with (a) one solution, (b) no solutions, (c) infinitely many solutions.

3. Find the values of  $k$  and  $c$  for which the system  $\begin{cases} 2x + ky = 8 \\ 4x + 8y = c \end{cases}$  is (a) *consistent* (i.e. it has solutions), (b) *inconsistent* (i.e. it has no solutions).

**Exercises** 7.1-7.3, 7.7-7.18 from SB.