ISET MATH II Term Final

Answers without work or justification will not receive credit.

1. Diagonalize $A = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$, i.e. show that it is similar to a diagonal matrix, that is find a matrix S such that $S^{-1}AS$ is a diagonal matrix.

$S = \begin{pmatrix} & \\ & \end{pmatrix}, \quad S^{-1}AS = \begin{pmatrix} & \\ & \end{pmatrix}$

2. Let $U = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $V = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
(2a) Are the U and W similar?
(2b) Are U and V similar?
(write "yes" or "no" and justify your answer).

(2a)

(2b)

3. Let $A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \\ 2 & 5 \end{pmatrix}$. Find or show the nonexistence of a matrix B such that $B \cdot A$ is unit 2×2 matrix.

4. Let V and W be vectors in the plane R^2 with lengths $||V|| = 3$ and $||W|| = 5$. (4a) What are the maxima and minima of $||V + W||$? (4b) When do these occur?

$(4a)$ $max(\|\|V + W\|\|) =$ $min(\|\|V + W\|\| =$
$(4b)$

5. Find a vector which (5a) does belong, and (5b) does not belong to $L((1, 3, 4), (4, 0, 1), (3, 1, 2))$.

$(5a)$ $(5b)$

6. In the plane (through the origin) spanned by $V = (1, 1, -2)$ and $W = (-1, 1, 1)$, find all vectors that are perpendicular to the vector $Z = (2, 1, 2)$.

7. Let $S \subset R^3$ be the subspace spanned by the two vectors $u = (1, -1, 0)$ and $v = (1, -1, 1)$. Write the equation of a line orthogonal to S which passes trough the origin.

8. Find a basis of the subspace of solutions of the equation

$$x_1 + x_2 + x_3 + x_4 = 0.$$

9. Give a proof or counterexample to the following.

a) Suppose that $u,\ v$ and w are vectors in R^n and $T : R^n \to R^m$ is a linear map. If $u,\ v$ and w are linearly dependent, is it true that $T(u),\ T(v)$ and $T(w)$ are linearly dependent? Why?

b) If $T : R^6 \to R^4$ is a linear map, is it possible that the nullspace of T is one dimensional?

(9*a*)

(9*b*)

10. **Remainder.** Let $T : R^n \to R^k$ be a linear transformation determined by $T(X) = A \cdot X$.

Equation $AX = Y$ has a solution iff $Y \in Im(T) = Col(A)$.

X is a solution of $AX = 0$ iff $X \in Ker(T) = Null(A)$.

Particularly, if $rank(A) = r < k$, then $dimIm(T) = dimCol(A) = r < k$ thus $Im(T)$ does not fulfill R^k, i.e. T is not surjective. Hence there exists $Y \in R^k$ which is not in $Im(T)$, that is $AX = Y$ does not have a solution.

Now the problem:

Say you have k linear algebraic equations in n variables; in matrix form $AX = Y$. For each of the following write "yes" and justify or write "no" and give a counterexample.

(a) If $n = k$ then for each Y the system $AX = Y$ has at most one solution.

(b) If $n > k$ you can solve $AX = Y$ for any Y.

(c) If $n > k$ then $AX = 0$ has nonzero solutions.

(d) If $n < k$ then for some Y there is no solution of $AX = Y$.

(e) If $n < k$ the only solution of $AX = 0$ is $X = 0$.

(10a)	
(10b)	
(10c)	
(10d)	
(10e)	

11. Each of three elementary row operations may be performed on a matrix A by multiplication from the left by certain elementary matrices. For example the elementary row operation

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \longrightarrow \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} + r \cdot a_{11} & a_{22} + r \cdot a_{12} & a_{23} + r \cdot a_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

in fact is the matrix product $\begin{pmatrix} 1 & 0 & 0 \\ r & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.

(a) Write 3×3 elementary matrices for the following row operations

a1. Multiplication of each element of the third row by r.

a2. Interchanging of second and third rows.

a3. Adding to the third row the second row multiplied by k.

(b) For $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 4 \\ 3 & 2 & 2 \end{pmatrix}$, find a matrix B such that $B \cdot A$ will be in Gauss row echelon form;

$(a1)$	$\begin{pmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{pmatrix}$
$(a2)$	$\begin{pmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{pmatrix}$
$(a3)$	$\begin{pmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{pmatrix}$
(b)	$B = \begin{pmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{pmatrix}, \qquad BA = \begin{pmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{pmatrix}$

12. (a) Find a 3×3 matrix that acts on R^3 as follows: it keeps the x_1 axis fixed but rotates the x_2 x_3 plane by 90 degrees (counterclockwise when you look from $(1, 0, 0)$).

b) Find a 3×3 matrix A mapping $R^3 \to R^3$ that rotates the x_1 x_3 plane by 180 degrees and leaves the x_2 axis fixed.

(a)	$\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$
(b)	$\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$

ADDITIONAL PAPER

ADDITIONAL PAPER

Solutions

1. $\begin{pmatrix} -3 & 0 \\ 0 & -5 \end{pmatrix} = \begin{pmatrix} -0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$.

2. (a) No: $det(U) \neq det(W)$.
(b) Yes: $U = S^{-1}VS$ with $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

3. $B = \begin{pmatrix} 2+z & -3-4z & z \\ -1+t & 2-4t & t \end{pmatrix}$. Particularly
$B = \begin{pmatrix} 2 & -3 & 0 \\ -1 & 2 & 0 \end{pmatrix}$.

4. $max(||V+W|| = 8$ when V and W are colinear and of same direction; $min(||V+W|| = 2$ when V and W are colinear and of opposite direction

5. (a) Say $(1,3,4)$. (b) Say $(0,0,1)$.

6. Solve $(\alpha \cdot V + \beta \cdot U) \cdot Z = 0$. Answer $(0, 2\alpha, -\alpha)$.

7. $(x = t, y = t, z = 0)$.

8. General solution $\begin{pmatrix} -x_1 - x_2 - x_3 - x_4 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$.

Basis $\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

9. (a) Yes: if $\alpha u + \beta v + \gamma w = 0,\ \ (\alpha, \beta, \gamma) \neq 0$ then $0 = T(\alpha u + \beta v + \gamma w) = T(\alpha u) + T(\beta v) + T(\gamma w) = \alpha T(u) + \beta T(v) + \gamma T(w)$.

(b)No: It is clear that $r = rank(T) \leq min(6,4) = 4$, thus $dim\ Null(T) = 6 - r \geq 6 - 4 = 2$.

10. a) No, if $det(A) = 0$ and $Y = 0$ there are infinitely many solutions.
Example: take $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

b) No: Suppose $rank(A) = r < k$. Then for $T : R^n \to R^k$ given by $T(X) = AX$ we have $dimIm(T) = dimCol(A) = r < k$ thus $Im(T)$ does not fulfill R^k, i.e. T is not surjective. Hence there exists $Y \in R^k$ which is not in $Im(T)$, that is $AX = Y$ does not have a solution.

Example: take $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

c) Yes: $dimNull(A) = n - r > n - k > 0$ thus $Null(A)$ contains nonzero vectors which are nonzero solutions of $AX = 0$.

d) Yes: $T : R^n \to R^k$ can not be surjective since $dim\, Im(A) = dim\, Col(A) = r \leq n < k$, thus there exist Y which is not in $Im(T)$ that is $AX = Y$ does not have a solution.

e) No: If $r < n$ then $dim\ Null(A) = n - r > 0$ so there exist nonzero vectors in $Null(A)$, they are nonzero solutions of $AX = 0$. Example: take $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{pmatrix}$

11. a1. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r \end{pmatrix}$

a2. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

a3. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{pmatrix}$

b.

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix}$$

$$B \cdot A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

12.

(a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$. (b) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.