Math for Economists, Calculus 1

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WEEK 2

1 Derivatives

1.1 The Slope of Nonlinear Function

If we want approximate a nonlinear function $y = f(x)$ by a linear one around some point x_0, the best approximation is the line *tangent* to the graph of the function $y = f(x)$ at the point $(x_0, f(x_0))$. The slope of this tangent line is the **derivative** of $y = f(x)$ at x_0 and is denoted as

$$f'(x_0) \qquad or \qquad \frac{df}{dx}(x_0).$$

More precisely:

The tangent line of the function $y = f(x)$ at a point x_0 is the limit of secant which passes trough two points $(x_0, f(x_0))$ and $(x, f(x))$, when $x \to x_0$.

What is the slope of this secant? This is

$$\frac{f(x) - f(x_0)}{x - x_0}.$$

Thus the slope of the tangent line is

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

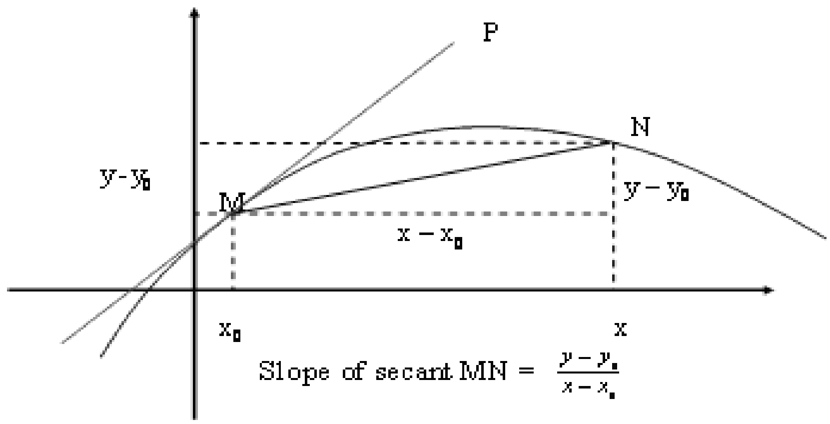
Definition 1 *The derivative of a function $y = f(x)$ at x_0 is*

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

equivalently

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

Derivative



Slope of tangent MP = $\lim_{N \to M}$ (slope of secant MN) =

$$f'(x) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Example. Let us calculate using the definition the derivative of quadratic function $f(x) = x^2$ at a point x_0:

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} =$$

$$\lim_{h\to 0} \frac{(x_0+h)^2-x_0^2}{h} = \lim_{h\to 0} \frac{x_0^2+2x_0h+h^2-x_0^2}{h} =$$

$$\lim_{h\to 0} \frac{2x_0h+h^2}{h} = \lim_{h\to 0}(2x_0+h) = 2x_0.$$

Example. In previous proof we have used the formula

$$(a+b)^2 = a^2 + 2ab + b^2.$$

This is a particular case of general **Newton Binom** formula

$$(a+b)^k =$$
$$C_k^0 a^k + C_k^1 a^{k-1} b + C_k^2 a^{k-2} b^2 + \ \dots \ + C_k^{k-1} a b^{k-1} + C_k^k b^k$$

where $C_k^i = \frac{k!}{i!(k-i)!}$ are *binomial coefficients* given by

$$C_k^i = \frac{k!}{i!(k-i)!},$$

that is

$$C_k^0 = 1,\ C_k^1 = k,\ C_k^2 = \frac{(k-1)\cdot k}{2},\ \dots ,C_k^{k-1} = k,\ C_k^k = 1.$$

In particular

$$C_1^0 = 1,\ C_1^1 = 1,$$

thus $(a+b)^1 = a+b$ (wow!)

Furthermore

$$C_2^0 = 1,\ C_2^1 = 2,\ C_2^2 = 1,$$

thus $(a+b)^2 = a^2 + 2ab + b^2$.

And furthermore

$$C_3^0 = 1,\ C_3^1 = 3,\ C_3^2 = 3, C_3^3 = 1,$$

thus $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

The binomial coefficients C_k^j form Pascal's triangle

$$\begin{array}{ccccccccccc} & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \\ \dots & \dots & & \dots & & \dots & & \dots & \dots \end{array}$$

where each number is the sum of the two directly above it.

We use this formula to find the derivative of the function $f(x) = x^k$:

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0} \frac{(x_0+h)^k - x_0^k}{h} =$$
$$\lim_{h\to 0} \frac{x_0^k + kx_0^{k-1}h + C_k^2 x_0^{k-2}h^2 + \ \dots\ + kx_0h^{k-1} + h^k - x_0^k}{h} =$$
$$\lim_{h\to 0} \frac{kx_0^{k-1}h + C_k^2 x_0^{k-2}h^2 + \ \dots\ + kx_0h^{k-1} + h^k}{h} =$$
$$\lim_{h\to 0}(kx_0^{k-1} + C_k^2 x_0^{k-2}h + \ \dots\ + kx_0h^{k-2} + h^{k-1}) = kx_0^{k-1}.$$

1.1.1 Rules for Computing Derivatives

(a) $(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$,

(b) $(kf)'(x_0) = kf'(x_0)$,

(c) $(f \cdot g)'(x_0) = f'(x_0)\cdot g(x_0) + f(x_0)\cdot g'(x_0)$,

(d) $(\frac{f}{g})'(x_0) = \frac{f'(x_0)\cdot g(x_0) - f(x_0)\cdot g'(x_0)}{g(x_0)^2}$,

(e) $(x^k)' = kx^{k-1}$,

(f) $((f(x)^n)' = n(f(x))^{n-1}\cdot f'(x)$,

1.2 Tangent line

There are infinitely many lines which pass trogh given ONE point. But two different points determine a line uniquely.
Example. Write the equation of the line which passes trough points $A = (1, 3)$ and $B = (5, 11)$.
Solution. This is $y = ax + b,\ a =?,\ b =?$. Since A when $x = 1 => y = 3$ and since B when $x = 5 => y = 11$, so we have the system

$$\{ \begin{array}{l} 3 = a \cdot 1 + b \\ 11 = a \cdot 5 + b \end{array},$$

solution gives $a = 2,\ b = 1$, so this line is $y = 2x + 1$.
Example. Write the equation of the tangent line to the graph of the function $y = x^2$ at the point with $x = 2$.
Solution. This is $y = ax + b,\ a =?,\ b =?$. But $a = f'(2) = 2 \cdot 2 = 4$, so we need only b. Substitution in $y = 4x + b$ of $x = 2,\ y = 2^2 = 4$ gives $4 = 4 \cdot 2 + b,\quad b = -4$, so the tangent line is $y = 4x - 4$.

If you prefare generally the equation of the tangent line to $f(x)$ at x_0 is $y = f(x_0) + f'(x_0) \cdot (x - x_0)$ (try to prove!).

1.3 Continuous Functions

A function is continuous if its graph has no brakes.

Precise definition: a function $y = f(x)$ is continuous at x if for any sequence

$$\{x_1, x_2,\ \dots\ , x_n,\ \dots\}$$

which converges to x the sequence

$$\{f(x_1), f(x_2),\ \dots\ , f(x_n),\ \dots\}$$

converges to $f(x)$, that is

$$\lim_{n\to\infty} x_n = x \quad \Rightarrow \quad \lim_{n\to\infty} f(x_n) = f(x).$$

Example. The function

$$f(x) = \begin{cases} -x, & x \le 0 \\ x + 1, & x > 0 \end{cases}$$

is discontinuous at $x = 0$: for a sequence

$$\{x_n = -\frac{1}{n}\} = \{-1, -\frac{1}{2}, -\frac{1}{3}, \dots, -\frac{1}{n}, \dots\},$$

which converges to $x = 0$ from the left, the sequence

$$\{f(x_n) = \frac{1}{n}\} = \{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\}$$

converges to $0 = f(0)$, but for the sequence

$$\{x_n = \frac{1}{n}\} = \{1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ...\}$$

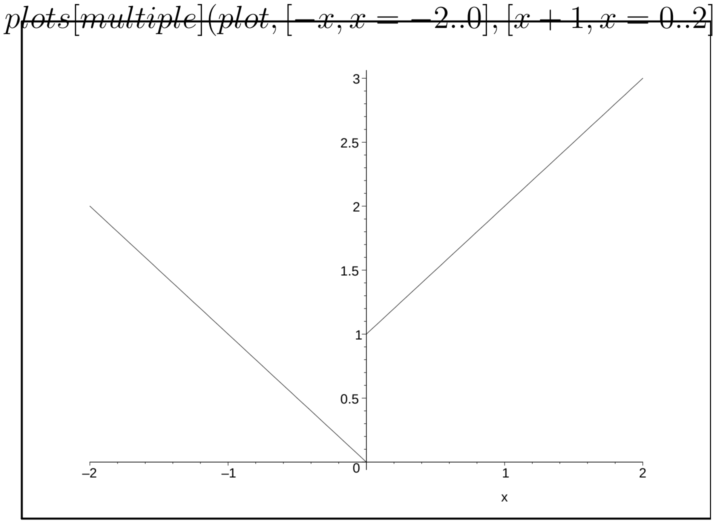
which converges to $x = 0$ from the right, the sequence

$$\{f(x_n) = \frac{1}{n} + 1\} = \{1 + 1, \frac{1}{2} + 1, \frac{1}{3} + 1, ..., \frac{1}{n} + 1, ...\}$$

converges to $1 \neq f(0)$. We write in this case

$$\lim_{x \to 0^-} f(x) = 0 = f(0) \neq 1 = \lim_{x \to 0^+} f(x).$$

> $plots[multiple](plot, [-x, x = -2..0], [x + 1, x = 0..2]);$



Example. The function

$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is discontinuous at $x = 0$:

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty, \ f(0) = 0, \ \lim_{x \to 0^+} \frac{1}{x} = \infty.$$

The function $\frac{1}{x}$ is continuous at each point of its domain $(-\infty, 0) \bigcup (0, \infty)$ but not at $x = 0$.

1.4 Differentiability

A function $y = f(x)$ is called *differentiable* if it has the derivative at every point of its domain. The graph of such function has tangent everywhere, that is its graph is a *smooth* curve.

A function $y = f(x)$ is called *continually differentiable* function (a C^1 **function** in short) if
(a) $f(x)$ is continuous, (b) $f(x)$ is differentiable, (c) $f'(x)$ is continuous.

Example. The function $y = |x|$ has *no tangent* at $x = 0$, so it has *no derivative* at this point, it *is not differentiable*, it *is not smooth.*
Example. The function

$$f(x) = \begin{cases} -x^2, & x \leq 0 \\ x, & x > 0 \end{cases}$$

is continuous at $x = 0$: the left limit at $x = 0$ is

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-x^2) = 0$$

as well as the right limit

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0.$$

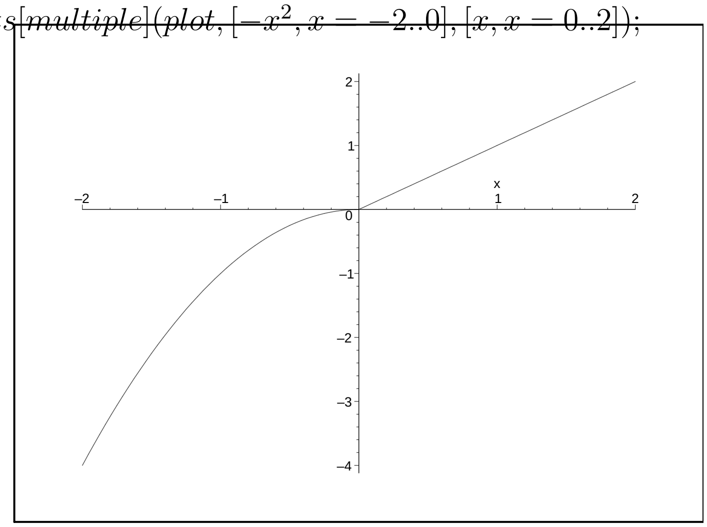
But it is not differentiable at $x = 0$: the left derivative is

$$lim_{x \to 0^-} f'(x) = (-x^2)'|_{x=0} = -2x|_{x=0} = 0$$

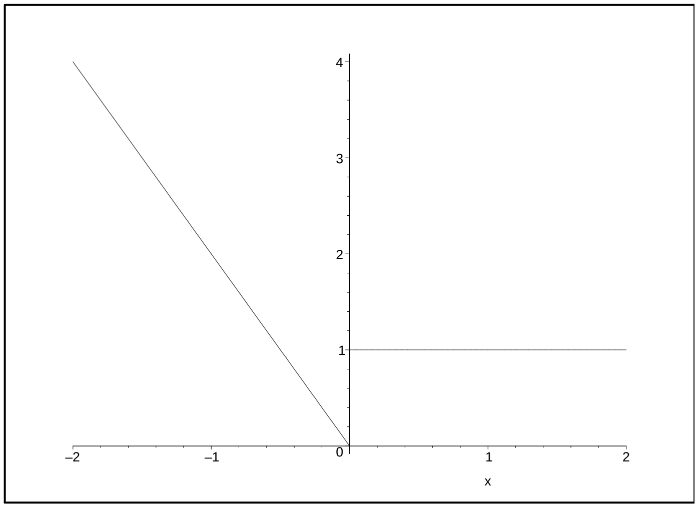
and the right derivative is

$$lim_{x \to 0^+} f'(x) = (x)'|_{x=0} = 1|_{x=0} = 1.$$

$> plots[multiple](plot, [-x^2, x = -2..0], [x, x = 0..2]);$



f(x)



f'(x)

Example. The function

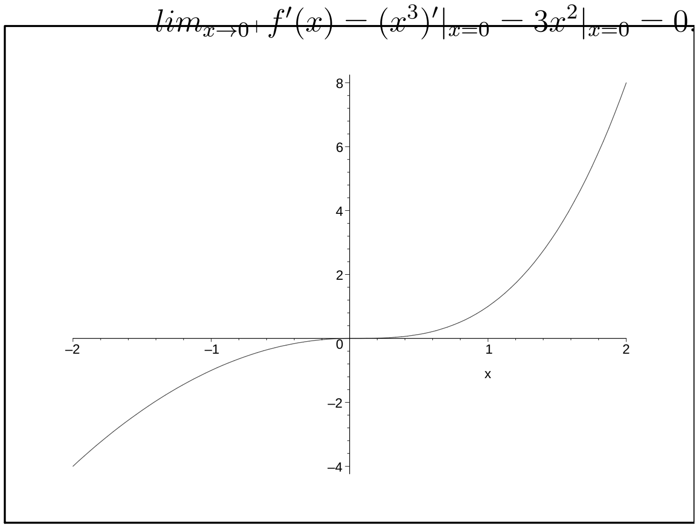
$$f(x) = \begin{cases} -x^2, & x \leq 0 \\ x^3, & x > 0 \end{cases}$$

is differentiable at $x = 0$: the left derivative is

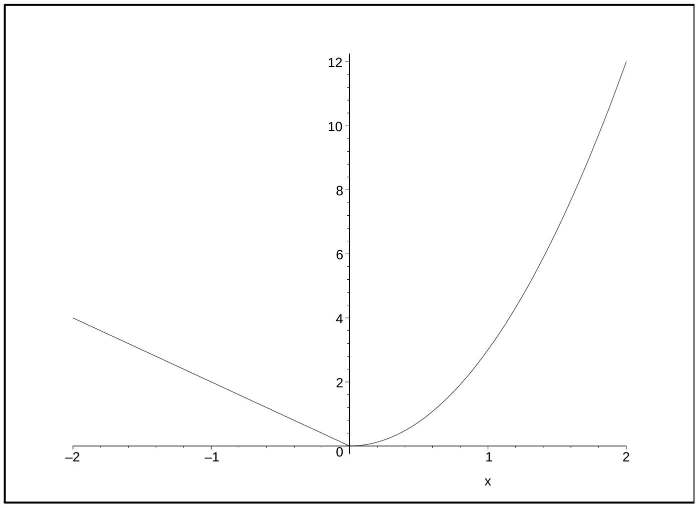
$$lim_{x \to 0^-} f'(x) = (-x^2)'|_{x=0} = -2x|_{x=0} = 0$$

and the right derivative is

$$lim_{x \to 0^+} f'(x) = (x^3)'|_{x=0} = 3x^2|_{x=0} = 0.$$



f(x)



f'(x)

Remark. Here is an example of differentiable but not C^1 function:

$$f(x) = \begin{cases} x^2 sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

Exercises

1. Check the continuity and the differentiability of

$$f(x) = \begin{cases} x^3 & x < 1 \\ x & x \geq 1. \end{cases}$$

Solution. Check the continuity at $x = 1$:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^3 = 1^3 = 1, \quad \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x = 1,$$

so the function is continuous.

The derivative of our function is

$$f'(x) = \begin{cases} 3x^2 & x < 1 \\ 1 & x \geq 1. \end{cases},$$

thus $\lim_{x \to 1^-} f'(x) = \lim_{x \to 1^-} 3x^2 = 3$, and $\lim_{x \to 1^+} f'(x) = \lim_{x \to 1^+} f'x = 1$, so $f'(x)$ does not exist at $x = 1$, the function is not C^1.

2. Check the continuity and the differentiability of

$$f(x) = \begin{cases} x^3 & x < 1 \\ 3x - 2 & x \geq 1. \end{cases}$$

Solution. Check the continuity at $x = 1$:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^3 = 1^3 = 1, \quad \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(3x-2) = 3\cdot 1 - 2 = 1,$$

so the function is continuous.

The derivative of our function is

$$f'(x) = \begin{cases} 3x^2 & x < 1 \\ 3 & x \geq 1. \end{cases},$$

thus $\lim_{x\to 1^-} f'(x) = \lim_{x\to 1^-} 3x^2 = 3$, and $\lim_{x\to 1^+} f'(x) = \lim_{x\to 1^+} 3 = 3$, so $f'(x)$ is continuous, the function is C^1.

3. Check the continuity and the differentiability of $f(x) = x^{\frac{1}{3}}$.

1.5 Higher order derivatives

The **second derivative** of a function $y = f(x)$ is the derivative of the derivative $f'(x)$. Notation

$$f''(x) \quad \textit{or} \quad \frac{d}{dx}\left(\frac{df}{dx}(x)\right) = \frac{d^2 f}{dx^2}(x).$$

For example $(x^3)'' = ((x^3)')' = (2x^2)' = 4x$.

A C^2 function is a **twice continuously differentiable function**.

The k-th derivative of f is denoted by

$$f^{[k]} = \frac{d^k f}{dx^k}(x).$$

If this k-th derivative is continuous, then we say f is C^k. If f has continuous $f^{[k]}$-s for all k, then we say f is C^∞. All polynomials are C^∞.

Exercises

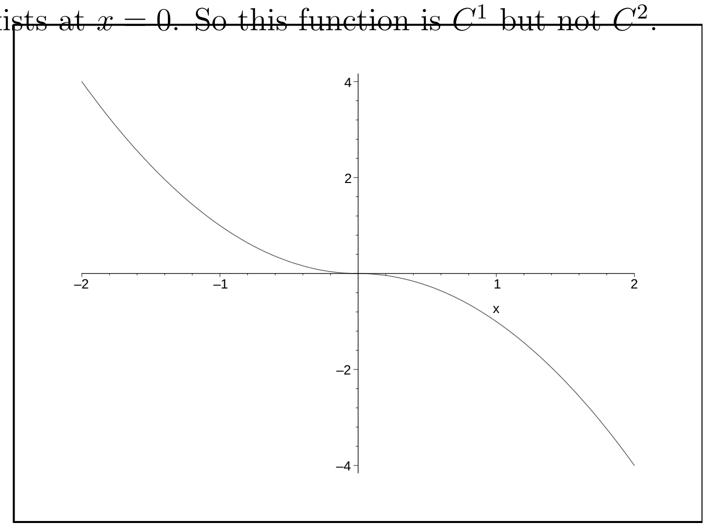
4. Check the continuity and the differentiability of

$$f(x) = \begin{cases} x^2 & x \leq 0 \\ -x^2 & x > 0. \end{cases}$$

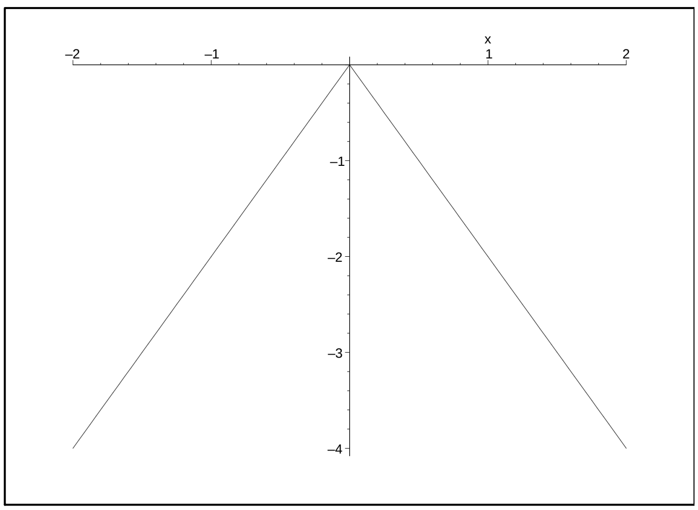
Solution. The derivative of our function is

$$f'(x) = \begin{cases} 2x & x \leq 0 \\ -2x & x > 0. \end{cases} = -2|x|,$$

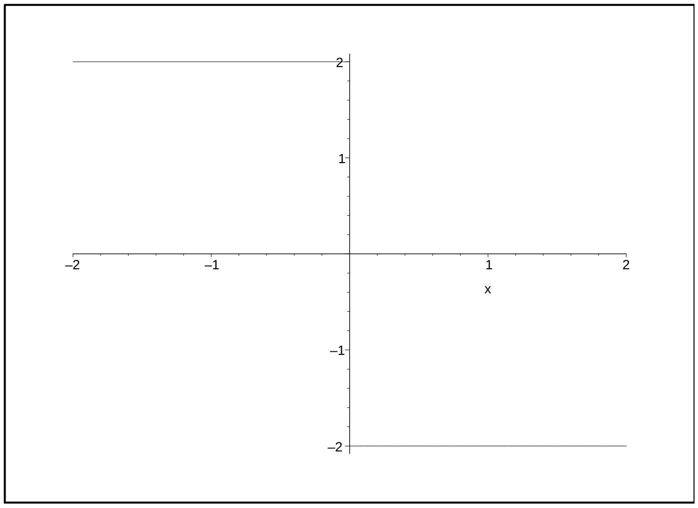
thus the function is continuous, differentiable, but the second derivative does not exists at $x = 0$. So this function is C^1 but not C^2.



f(x)



f'(x)



$f''(x)$

5. We have already checked that the function

$$f(x) = \begin{cases} x^3 & x < 1 \\ 3x - 2 & x \geq 1. \end{cases}$$

is C^1. But is it C^2? The second derivative of our function is

$$f''(x) = \begin{cases} 6x & x < 1 \\ 0 & x \geq 1. \end{cases},$$

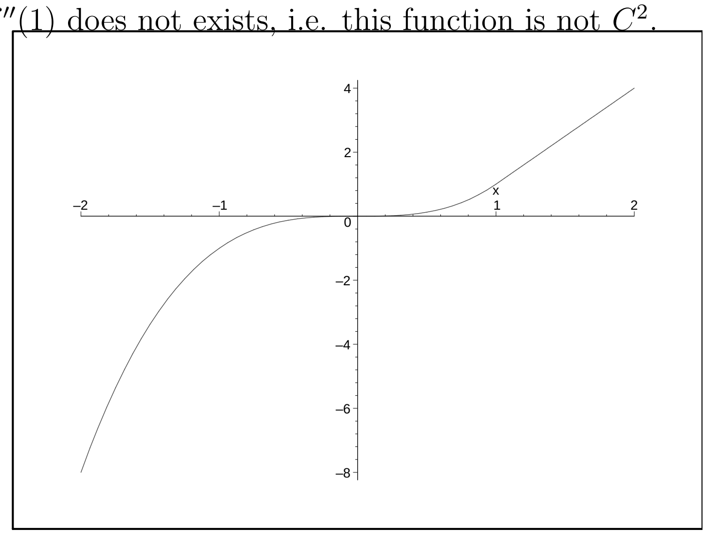
so at $x = 1$ the left second derivative is

$$lim_{x \to 1^-} f''(x) = 6x|_{x=1} = 6 \cdot 1 = 6$$

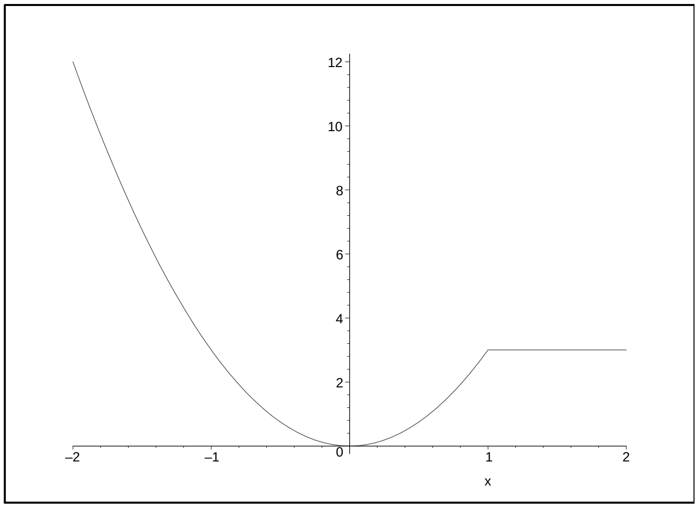
and the right second derivative is

$$lim_{x \to 1^+} f''(x) = 0|_{x=1} = 0$$

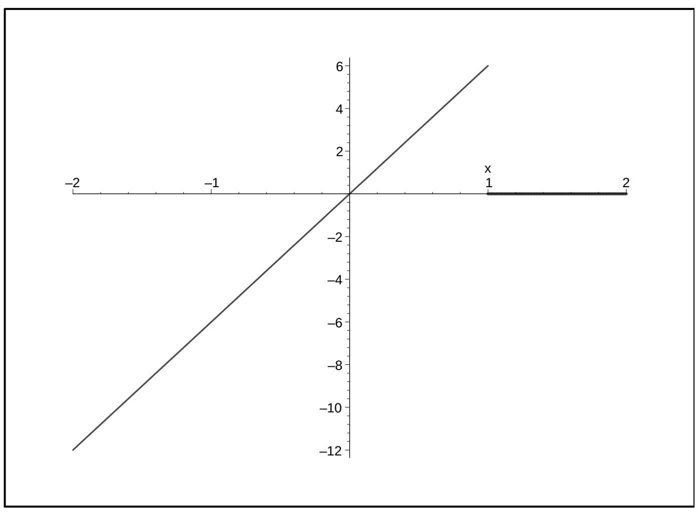
thus $f''(1)$ does not exists, i.e. this function is not C^2.



f(x)



f'(x)



f"(x)

6. We have already checked that the function

$$f(x) = \begin{cases} -x^2, & x \leq 0 \\ x^3, & x > 0 \end{cases}$$

is C^1, but is it C^2?

7. Construct a function which is C^2 but not C^3.

1.6 Approximation by Differential

By definition of the derivative

$$f'(x_0) \approx \frac{f(x_0+h) - f(x_0)}{h},$$

thus

$$f(x_0+h) \approx f'(x_0) \cdot h + f(x_0).$$

Equivalently, taking $x = x_0 + h$ we obtain

$$f(x) \approx f(x_0) + f'(x_0) \cdot (x - x_0).$$

This allows to approximate $f(x)$ by the linear function $f(x_0)+f'(x_0)\cdot(x-x_0)$ around a point x_0, for which $f(x_0)$ and $f'(x_0)$ are easy to calculate.

Denote $f(x) - f(x_0) = \Delta f$ and $x - x_0 = \Delta x$, then the above can be rewritten as

$$\Delta f \approx f'(x_0) \cdot \Delta x.$$

Write df instead of Δf and dx instead of Δx. Then

$$df = f'(x_0) \cdot dx,$$

df is called **differential** of f.

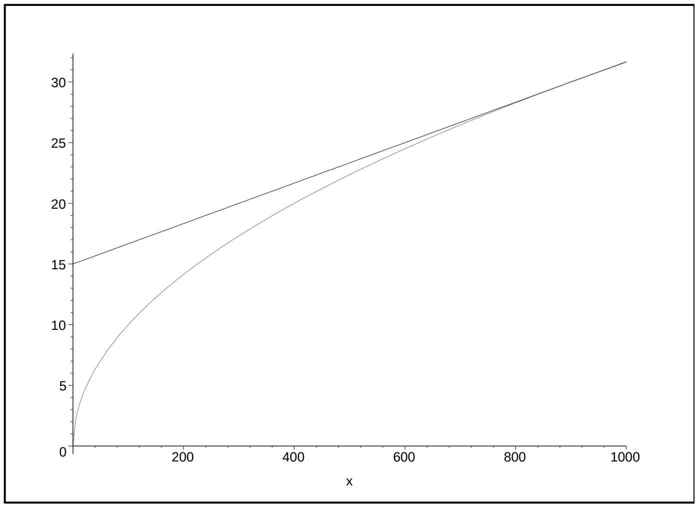
Example. Estimate $\sqrt{920}$.
Solution. Consider the function $f(x) = \sqrt{x}$. The point nearest to 920 for which we can calculate $f(x)$ (and $f'(x)$) is $x = 900$: $f(900) = \sqrt{900} = 30$, furthermore, the derivative of $f(x) = \sqrt{x}$ is $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$, thus $f'(900) = \frac{1}{60}$.

So $f(920)$ can be approximated as

$$f(920) \approx f(900) + f'(900) \cdot 20 =$$

$$\sqrt{920} = 30 + \frac{1}{60} \cdot 20 = 30 + \frac{1}{3} = 30.333... \quad .$$

$> f(x) := sqrt(x); df(x) := diff(f(x), x);$
$> x0 := 900.; k := eval(df(x), x = x0); f(x0) := eval(f(x), x = x0);$
$> g(x) := k * (x - x0) + f(x0);$
$> eval(f(x), x = 920.); eval(g(x), x = 920);$
$> plot(f(x), g(x), x = 0..1000);$



1.7 Taylor Formula

The linear approximation

$$f(x) \approx f(x_0) + f'(x_0) \cdot (x - x_0).$$

is a particular case of more general approximation of a function with **Taylor polynomials** $P_n(x)$

$$f(x) \approx P_n(x) =$$
$$f(x_0) + f'(x_0) \cdot (x - x_0) + \tfrac{f''(x_0)}{2!} \cdot (x - x_0)^2 + \ \dots \ + \tfrac{f^{[n]}(x_0)}{n!} \cdot (x - x_0)^n$$

where $n!$ is the factorial $n! = 1 \cdot 2 \cdot 3 \cdot \ \dots \ \cdot n$. The **Taylor series** of f is "infinite" Taylor polynomial

$$P_\infty(x) = f(x_0) + f'(x_0) \cdot (x - x_0) + \tfrac{f''(x_0)}{2!} \cdot (x - x_0)^2 + \ \dots$$
$$+ \tfrac{f^{[n]}(x_0)}{n!} \cdot (x - x_0)^n + \ \dots \ .$$

Equivalent form

$$P_\infty(x_0 + h) = f(x_0) + f'(x_0) \cdot h + \tfrac{f''(x_0)}{2!} \cdot h^2 + \ \dots$$
$$+ \tfrac{f^{[n]}(x_0)}{n!} \cdot h^n + \ \dots$$

The particular case of this series when $x_0 = 0$

$$P_\infty(x) = f(0) + f'(0) \cdot x + \frac{f''(0)}{2!} \cdot x^2 + \ \dots \ + \frac{f^{[n]}(0)}{n!} \cdot x^n + \ \dots$$

is called **MacLaurin series**.
Example. Estimate $\sqrt{920}$ now using the second order Taylor polynomial.

Solution.

$$f(x) \approx f(900) + f'(900)\cdot(x-900) + \frac{f''(900)}{2!}\cdot(x-900)^2.$$

$$f'(x) = \tfrac{1}{2\sqrt{x}}, \quad f'(900) = \tfrac{1}{2\cdot 30} = \tfrac{1}{60}.$$
$$f''(x) = -\tfrac{1}{4\sqrt{x}^3}, \quad f''(900) = -\tfrac{1}{4\cdot 30^3} = -\tfrac{1}{108000}.$$

Thus

$$f(920) = 30 + \frac{1}{60}\cdot 20 - \frac{1}{2}\cdot\frac{1}{108000}\cdot 20^2 = 30\ +\ 0.33...\ -\ 0.001852 = 30.33148.$$

Compare this by 30.333... obtained by linear approximation and the value $\sqrt{920} = 30.33150178$ given by calculator.

By MAPLE

$> f := sqrt(x);$

$$\sqrt{x}$$

$> T2 := taylor(f, x = 900, 3);$

$$T2 := 30 + \tfrac{1}{60}(x-900) - \tfrac{1}{216000}(x-900)^2 + O((x-900)^3)$$

$> P2 := convert(T2, polynom);$

$$P2 := 15 + \tfrac{x}{60} - \tfrac{(x-900)^2}{216000}$$

$> t := eval(P2, x = 920);$

$$\tfrac{16379}{540}$$

$> evalf(t);$

$$30.33148148$$

Exercises

8. Find the MacLaurin polynomial $P_4(x)$ for the functions $f(x) = \frac{1}{1+x}$ and $f(x) = \frac{1}{1-x}$.

9. Estimate e^x using the MacLaurin polynomials $P_1(x),\ P_2(x),\ P_3(x)$.

10. Estimate $\ln\ x$ using Taylor polynomials $P_1(x),\ P_2(x),\ P_3(x)$ at $x = 1$.

11. Find the MacLaurin polynomials $P_1(x),\ P_2(x),\ P_3(x),\ P_4(x))$ for the a polynomial $f(x) = ax^3 + bx^2 + cx + d$.