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WEEK 7

1 Integral Calculus

Reading: [SB] appendix A4

1.1 Antiderivative = Indefinite integral

Definition. Indefinite integral $\int f(x)dx$ is a function $F(x)$ such that $F'(x) = f(x)$.

Examples

$$(x^2)' = 2x \;\Rightarrow\; \int 2xdx = x^2 + C, \qquad (x^3)' = 3x^2 \;\Rightarrow\; \int 3x^2dx = x^3 + C,$$
$$(e^x)' = e^x \;\Rightarrow\; \int e^xdx = e^x + C, \qquad (\ln x)' = \frac{1}{x} \;\Rightarrow\; \int \frac{1}{x}dx = \ln x + C.$$

1.1.1 Indefinite Integral Formulas

1. $\int kdx = kx + C;$
2. $\int x^ndx = \frac{x^{n+1}}{n+1} + C \quad for \quad n \neq -1$
3. $\int x^{-1}dx = \ln x + C;$
4. $\int kf(x)dx = k\int f(x)dx;$
5. $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx;$
6. $\int [u(x)]^n u'(x)dx = \frac{[u(x)]^{n+1}}{n+1} + C \quad for \quad n \neq -1$
7. $\int [u(x)]^{-1}u'(x)dx = \ln u(x) + C;$
8. $\int e^{u(x)}u'(x)dx = e^{u(x)} + C.$

Examples

1. $\int 5dx = 5x + C;$
2. $\int x^3dx = \frac{x^4}{4} + C;$
3. $\int \frac{4}{x}dx = 4\int x^{-1}dx = 4\ln x + C;$
4. $\int 3e^xdx = 3e^x + C;$
5. $\int (6x^2 - 4x + 2)dx = 2x^3 - 2x^2 + 2x + C;$
6. $\int 4x(x^2 + 1)dx = 2\int (x^2 + 1)d(x^2 + 1) = (x^2 + 1)^2 + C;$
7. $\int 2x(x^2 + 1)^{-1}dx = \int (x^2 + 1)^{-1}d(x^2 + 1) = \ln(x^2 + 1) + C;$
8. $\int e^{4x+3}dx = \frac{1}{4}\int e^{4x+3}4dx = \frac{1}{4}\int e^{4x+3}d(4x + 3) = \frac{1}{4}e^{4x+3} + C$.

Exercises

1. $\int(2x+3)^{10}dx.$
2. $\int(2x+3)^{-1}dx.$
3. $\int(2x+3)^{-3}dx.$
4. $\int x(2x^2+3)^{-1}dx.$
5. $\int x^2(2x^3+3)^5dx.$
6. $\int e^{3x}dx.$
7. $\int e^x(3e^x+1)^7dx.$
8. $\int \frac{x}{3x^2+1}dx.$
9. $\int \frac{x^2}{1-3x^3}dx.$

Find $f(x)$ if

10. $f'(x) = 2x-3, \quad and \quad f(0) = 5.$
11. $f'(x) = 2x^{-2}+3x^{-1} \quad and \quad f(1) = 0.$

12. Marginal cost function is given by $C'(x) = 3x^2 - 24x + 53$, and the fixed cost at 0 output are \$ 30,000. What is the cost for manufacturing of 4,000 items?
13. The weekly marginal revenue from the sale of x pairs of shoes is given by $R'(x) = 40 - 0.02x + \frac{200}{x+1}$. Besides $R(0) = 0$. Find the revenue function. Find the revenue from the sale of 1,000 shoes.
14. The weekly marginal cost of producing x pairs of shoes is $C'(x) = 12 + \frac{500}{x+1}$, and the weekly fixed costs are \$ 2,000. Find the cost function. What is the average cost per one pair of shoes if 1,000 pair of shoes are produced?

1.1.2 Integration by Parts

By product rule

$$[u(x) \cdot v(x)]' = u'(x) \cdot v(x) + u(x) \cdot v'(x).$$

Integrating both sides we obtain

$$\int[u(x) \cdot v(x)]'dx = \int u'(x) \cdot v(x)dt + \int u(x) \cdot v'(x)dx,$$

$$u(x) \cdot v(x) = \int u'(x) \cdot v(x)dt + \int u(x) \cdot v'(x)dx,$$

let us rewrite this as

$$\int u(x) \cdot v'(x)dx = u(x) \cdot v(x) - \int u'(x) \cdot v(x)dx.$$

Equivalently

$$\int u(x) \cdot dv(x) = u(x) \cdot v(x) - \int v(x)du(x).$$

Examples

1. Calculate $\int xe^x dx$. Let $u(x) = x,\ dv(x) = e^x dx$, then $du(x) = dx$ and $v(x) = e^x$ thus

$$\int xe^x dx = \int u(x)dv(x) = u(x)v(x) - \int v(x)du(x) =$$
$$xe^x - \int e^x dx = xe^x - e^x + C.$$

2. Calculate $\int \ln x dx$. Let $u(x) = \ln x,\ dv(x) = dx$, then $du(x) = \frac{1}{x}dx$ and $v(x) = x$ thus

$$\int \ln x dx = \int u(x)dv(x) = u(x)v(x) - \int v(x)du(x) =$$
$$(\ln x) \cdot x - \int xd(\ln x) =$$
$$x \ln x - \int x \cdot \tfrac{1}{x} dx = x \ln x - \int dx = x \ln x - x + C.$$

3. Calculate $\int x\sqrt{x+1}dx$. Let $u(x) = x,\ dv(x) = (x+1)^{\frac{1}{2}}dx$, then $du(x) = dx$ and $v(x) = \int dv(x) = \int (x+1)^{\frac{1}{2}}dx = \frac{2}{3}(x+1)^{\frac{3}{2}}$ thus

$$\int x\sqrt{x+1}dx = \int u(x)dv(x) = u(x)v(x) - \int v(x)du(x) =$$
$$x \cdot \tfrac{2}{3}(x+1)^{\frac{3}{2}} - \int \tfrac{2}{3}(x+1)^{\frac{3}{2}}dx =$$
$$\tfrac{2}{3}x(x+1)^{\frac{3}{2}} - \tfrac{2}{3} \cdot \frac{(x+1)^{\frac{3}{2}+1}}{\frac{3}{2}+1} = \tfrac{2}{3}x(x+1)^{\frac{3}{2}} - \tfrac{4}{15}(x+1)^{\frac{5}{2}} + C.$$

Exercises

1. Calculate $\int x \ln x dx$. Try 4 ways to solve this problem, which way works?

Way 1. Let $u(x) = 1,\quad dv(x) = x \ln x dx.$

Way 2. Let $u(x) = x \ln x,\quad dv(x) = dx.$

Way 3. Let $u(x) = \ln x,\quad dv(x) = xdx.$

Way 4. Let $u(x) = x,\quad dv(x) = \ln x dx.$

2. Calculate $\int x^3 \cdot e^x dx$.

1.2 Definite Integral

The definite integral $\int_a^b f(x)dx$ is defined as follows. Take any natural number N and divide the interval $[a, b]$ into N equal subintervals, for this let $\Delta = \frac{b-a}{N}$ and let's consider the points

$$x_0 = a,\ x_1 = x_0 + \Delta,\ x_2 = x_1 + \Delta,\ \ldots\ ,\ x_N = x_{N-1} + \Delta = b,$$

these points divide $[a, b]$ into N intervals

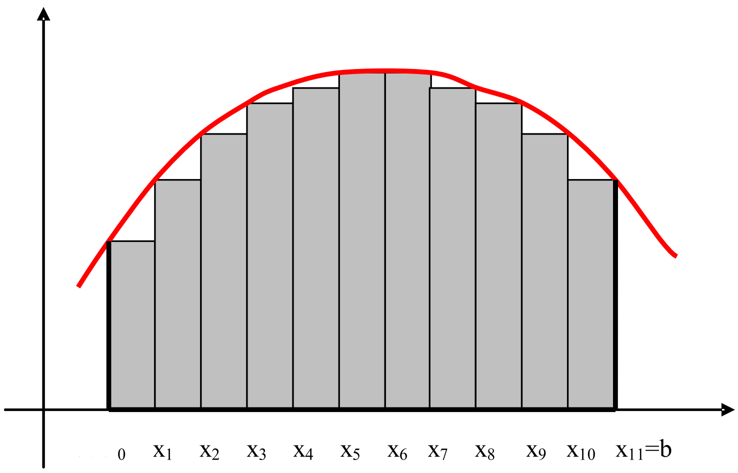
$$[a = x_0, x_1],\ [x_1, x_2],\ \ldots\ ,\ [x_k, x_{k+1}],\ \ldots\ ,\ [x_{N-1}, x_N = b]$$

each of length Δ.

Now consider the **Riemann sum**

$$f(x_0)\cdot(x_1-x_0)+f(x_1)\cdot(x_2-x_1)+\ \ldots\ +f(x_{N-1})\cdot(x_N-x_{N-1}) = \sum_{i=1}^{N} f(x_{i-1})\cdot\Delta.$$

This sum approximates the area under the graph of $f(x)$ from a to b:



Definition. The definite integral $\int_a^b f(x)dx$ is defined as the limit

$$\lim_{N\to\infty} \sum_{i=1}^{N} f(x_{i-1}) \cdot \Delta.$$

Remark. Actually in the general definition of definite integral partitions of $[a, b]$ *not necessarily into equal intervals* $[x_{k-1}, x_k]$ are used, and in the limit process it is assumed that the length of *largest* subinterval goes to zero. Besides, as $f(x_k)$ can be taken not necessarily the left end but any point of the k-th subinterval $[x_{k-1}, x_k]$.

1.2.1 The Fundamental Theorem of Calculus

As we see the definitions of indefinite and definite integrals are totally different. This theorem connects these two notions: suppose $F'(x) = f(x)$, i.e. $\int f(x)dx = F(x) + C$, then

$$\int_a^b f(x)dx = F(b) - F(a) = F(x)|_a^b.$$

Example. Calculate $\int_1^e \ln x dx$.

Solution. We have already calculated the indefinite integral

$$\int \ln x dx = x \ln x - x + C.$$

Then

$$\begin{gathered}\int_1^e \ln x dx = (x\ln x - x + C)|_1^e = \\ (e \cdot \ln e - e + C) - (1 \cdot \ln 1 - 1 + C) = \\ (e - e + C) - (1 \cdot 0 - 1 + C) = 1.\end{gathered}$$

1.3 Applications

1.3.1 Average Value

Let f be a continuous function over a closed interval $[a, b]$. Its *average value* over $[a, b]$ is

$$y_{av} = \frac{1}{b-a}\int_a^b f(x)dx.$$

Geometrical meaning: multiply both sides by $b - a$

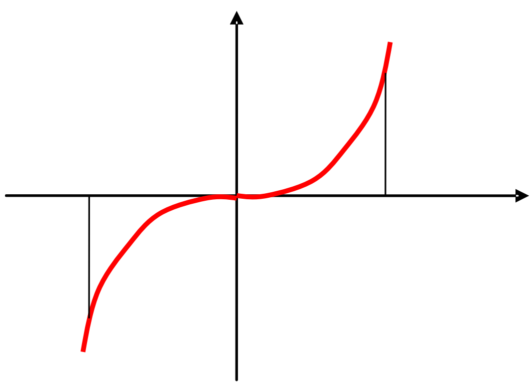
$$(b-a) \cdot y_{av} = \int_a^b f(x)dx.$$

So the average value y_{av} is the height of the rectangle of length $b - a$ whose area is the same as the area under the graph of $f(x)$ over the interval $[a, b]$, which equals to $\int_a^b f(x)dx$.

Example. Find the average value of $f(x) = x^3$ over the interval $[-1, 1]$.

$$y_{av} = \frac{1}{1-(-1)}\int_{-1}^1 x^3 dx = \frac{1}{2} \cdot \frac{x^4}{4}|_{-1}^1 = \frac{1}{2}(\frac{1}{4} - \frac{1}{4}) = 0.$$

Is it strange?



1.3.2 Consumer's Surplus

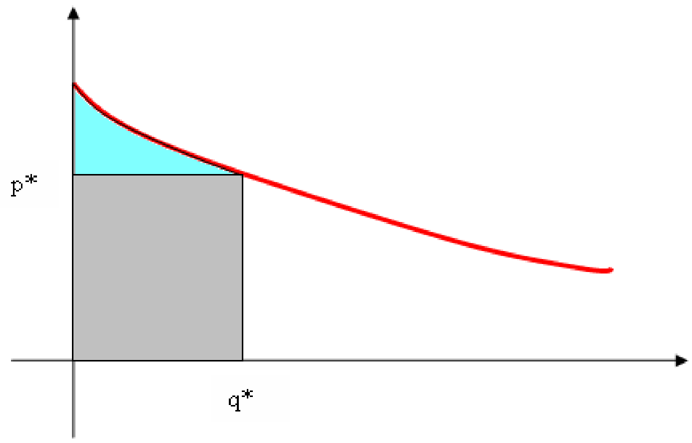
Consider the *inverse* demand function $p = D(q)$ (p is the price at the demand q). We know that this is a decreasing function.

Take any demand q^*, the corresponding price is $p^* = D(q^*)$. If we purchase q^* units immediately for the price $p^* = D(q^*)$, then our expenditure will be $q^* \cdot D(q^*)$.

On the other hand the total willingness to pay for q^* units of commodity is the area under the graph of inverse demand function $p = D(q)$ on the interval $[0, q^*]$, so it is $\int_0^{q^*} f(q)dq$.

The *consumer's surplus* is defined as

$$\int_0^{q^*} D(q)dq - q^* \cdot D(q^*).$$



Example. Let the price of a book depending on number of books purchased is given by $p = 10 - q$, this means that the price of the first book is \$9; of the second is \$8; of the third book is \$7, etc. Let us analyze what does it mean.

Consider three strategies of selling books.

1. Trivial: each book costs 9, so if you buy two books, you pay 9+9=18; if you buy three books you pay 9+9+9=27, etc.

2. Weak discount: if you buy one book you pay 9; if you buy two books you pay 9 for the first book and 8 for the second, so 9+8=17 for both; if you buy three books you pay 9 for the first, 8 for the second, 7 for the third, so you pay 9+8+7=25 for all three, etc.

3. Strong discount: if you buy one book you pay 9; if you buy two books you pay 8 for each, so you pay 8+8=16 for both; if you buy three books you pay 7 for each, so you pay 7+7+7=21 for all three, etc.

The consumer's surplus is the difference between the second and the third methods, that is:

If $q=1$, the surplus is 9-9=0.
If $q=2$, the surplus is 17-16=1.
If $q=3$, the surplus is 25-21=4.
Generally, if $q=q_0$, the surplus is

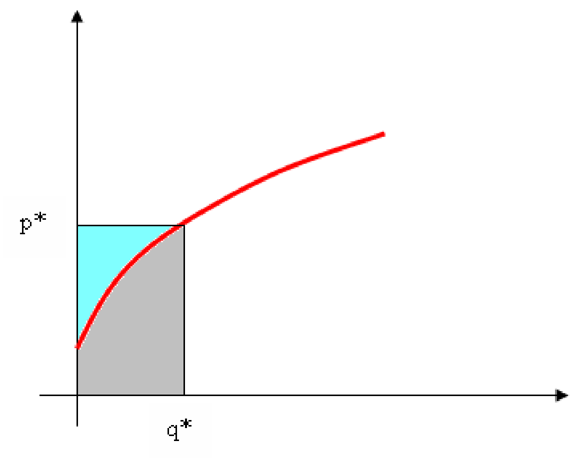
$$\int_0^{q_0}(10-q)dq - q_0\cdot(10-q_0) = (10q-\tfrac{q^2}{2})|_0^{q_0} - 10q_0 + q_0^2 = 10q_0 - \tfrac{q_0^2}{2} - 10q_0 + q_0^2 = \tfrac{q_0^2}{2}.$$

Why this result differs from above for $q_0 = 1, 2, 3$?

1.3.3 Producer's Surplus

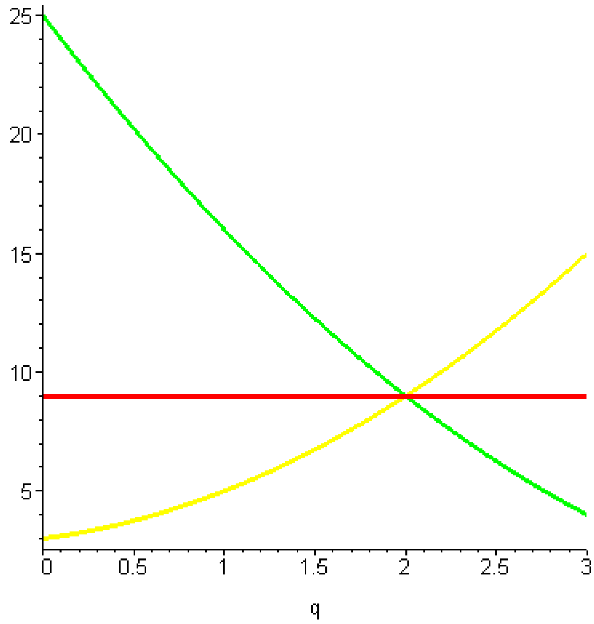
Let $p = S(q)$ be the supply function for a commodity, it is an increasing function. The producer's surplus at $q = q^*$ is defined as

$$q^*\cdot S(q^*) - \int_0^{q^*} S(q)dq.$$



Example. Let $D(q) = (q-5)^2$ and $S(q) = q^2 + q + 3$, find
(a) The equilibrium point.
(b) The consumer's surplus at the equilibrium point.
(c) The producer's's surplus at the equilibrium point.

Solution.



(a) $S(q) = D(q), \quad (q-5)^2 = q^2 + q + 3, \quad q^* = 2, \quad p^* = D(2) = S(2) = 9.$

$$(b) \quad \int_0^2 D(q)dq - q^* \cdot D(q^*) = \int_0^2 (q-5)^2 dq - 2 \cdot 9 = \frac{(q-5)^3}{3}|_0^2 - 18 = \frac{44}{3}.$$

$$(c) \quad q^* \cdot S(q^*) - \int_0^2 S(q)dq = 2 \cdot 9 - \int_0^2 (q^2+q+3)dq = 18 - (\frac{q^3}{3} + \frac{q^2}{2} + q)|_0^2 = \frac{22}{3}.$$

```
 d:=(q-5)^2; s:=q^2+q+3;
> solve(d=s,q);
> eval(s,q=2);
Consumer surplus
> CS:=int(d,q=0..2)-2*9;
Producer surplus
> PS:=2*9-int(s,q=0..2);
```

Exercises

A4.1-A4.5 from [SB], pp. 983.

1. Find the consumer's surplus at a price level \$150 if the price-demand equation is $D(x) = 400 - 0.05x$.

2. Find the producer's surplus at a price level \$67 if the price-supply equation is $S(x) = 10 + 0.1x + 0.0003x^2$.

3. Suppose the price-demand function is $D(x) = 50 - 0.1x$ and the price-supply function is $S(x) = 11 + 0.05x$. Find:

(a) The equilibrium price level.

(b) Consumer's surplus at equilibrium.

(c) Producer's surplus at equilibrium.