Tornike Kadeishvili

WEEK 6
Reading [SB], 5.1-5.6, pp. 82-102

1 Exponents and Logarithms

1.1 Types of Growth

1.1.1 Linear or Arithmetic Growth

Arithmetic sequence

$$a_1,\ a_2,\ a_3,\ \dots ,a_n,\ \dots ,$$

recursive description

$$a_n = a_{n-1} + d.$$

The n-th term formula

$$a_n = a_1 + (n-1)d.$$

If $d > 0$ the sequence increases, if $d < 0$ the sequence decreases, if $d = 0$ the sequence is constant.

Arithmetic sum formula: let

$$S_n = a_1 + a_2 + a_3 + \ \dots\ + a_n,$$

then

$$S_n = \frac{a_1 + a_n}{2}n = \frac{2a_1 + (n-1)d}{2}n.$$

Why arithmetic? Because each term is arithmetic mean of two neighbor terms

$$a_n = \frac{a_{n-1} + a_{n+1}}{2}.$$

1.1.2 Exponential or Geometric Growth

Geometric sequence

$$a_1,\ a_2,\ a_3,\ \dots ,a_n,\ \dots ,$$

recursive description

$$a_n = a_{n-1} \cdot q.$$

The n-th term formula

$$a_n = a_1 \cdot q^{n-1}.$$

If $d > 1$ the sequence increases, if $1 > q > 0$ the sequence decreases, if $q < 0$ the sequence alternates in sign.

Geometric sum formula: let

$$S_n = a_1 + a_2 + a_3 + \ \dots \ + a_n,$$

then

$$S_n = \frac{a_1(q^n - 1)}{q - 1} = \frac{a_1 - a_n q}{1 - q}.$$

If $|q| < 1$ then the infinite sum

$$S_\infty = a_1 + a_2 + a_3 + \ \dots \ + a_n + \ \dots$$

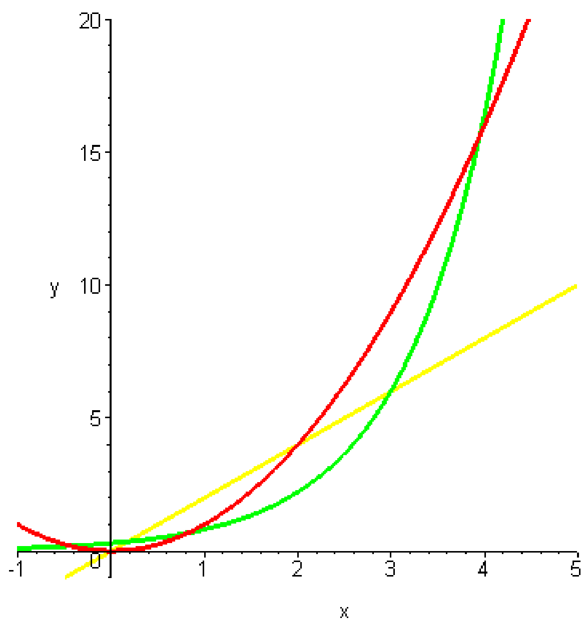
equals to

$$S_\infty = \frac{a_1}{1 - q}.$$

Why geometric? Because each term is geometric mean of two neighbor terms

$$a_n = \sqrt{a_{n-1} \cdot a_{n+1}}.$$

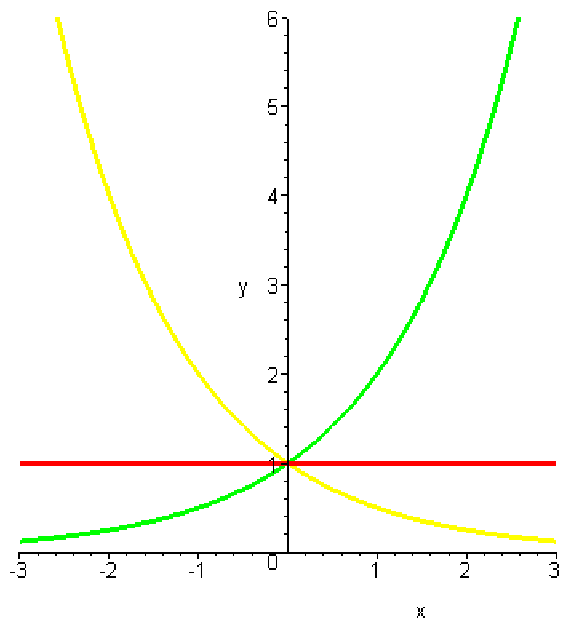
1.1.3 Functions with Linear, Polynomial and Exponential Growth



$f(x) = 2x, \quad g(x) = x^2, \quad h(x) = 0.3exp(x)$

1.2 Exponential Function

Exponential function $y = a^x$ is increasing for $a > 1$, is decreasing for $0 < a < 1$, and is constant for $a = 1$.



$f(x) = 2^x, \quad g(x) = 1^x, \quad h(x) = (\frac{1}{2})^x$

1.2.1 Properties of Exponent

$a^m \cdot a^n = a^{m+n};$
$a^{-n} = \frac{1}{a^n};$
$\frac{a^m}{a^n} = a^{m-n};$
$(a^m)^n = a^{m \cdot n};$
$a^0 = 1.$

1.3 The Napier Number e

The Napier number $e \approx 2.718281693...$ is an important irrational number (as $\pi \approx 3.14.14.16...$). It is defined as the limit

$$e = \lim_{n \to \infty} (1 + \frac{1}{n})^n.$$

n	(1 + 1/n) ^n
1	2
2	2.25
3	2.3703704
4	2.4414063
5	2.48832
10	2.5937425
15	2.6328787
20	2.6532977
25	2.6658363
30	2.6743188
35	2.6804393
40	2.6850638
45	2.6886812
50	2.691588

Consequently

$$\lim_{n\to\infty}(1+\frac{k}{n})^n = e^k.$$

Indeed

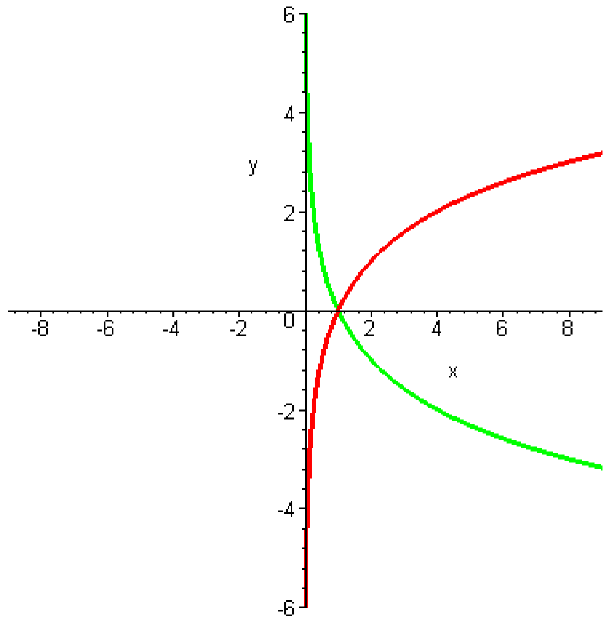
$$\lim_{n\to\infty}(1+\frac{k}{n})^n = \lim_{n\to\infty}(1+\frac{1}{\frac{n}{k}})^n = \lim_{n\to\infty}[(1+\frac{1}{\frac{n}{k}})^{\frac{n}{k}}]^k =$$

$$[\lim_{n\to\infty}(1+\frac{1}{\frac{n}{k}})^{\frac{n}{k}}]^k = [\lim_{\frac{n}{k}\to\infty}(1+\frac{1}{\frac{n}{k}})^{\frac{n}{k}}]^k = e^k.$$

1.4 Logarithmic Function

Exponential function $f(x) = a^x$, $f : (-\infty, \infty) \to (0, \infty)$ for $a > 0$, $a \neq 1$ is monotonic, thus it is invertible. The inverse function $(0, \infty) \to (-\infty, \infty)$ is called logarithmic function and is denoted by $g(x) = \log_a x$.

For $0 < a < 1$ the function $\log_a x$ is decreasing, for $a > 1$ it is increasing.



$$f(x) = log_2 x, \quad g(x) = log_{\frac{1}{2}} x$$

1.4.1 Properties of Logarithm

$\log_a(r \cdot s) = \log_a r + \log_a s;$
$\log_a \frac{1}{r} = -\log_a r;$
$\log_a \frac{r}{s} = \log_a r - \log_a s;$
$\log_a r^s = s \cdot \log_a r;$
$\log_a 1 = 0;$
$\log_r s = \frac{1}{\log_s r};$
$\log_r s = \frac{\log_a s}{\log_a r}.$

1.5 Derivatives of Exp and Log

Theorem 1

$$(\ln x)' = \frac{1}{x}.$$

Proof.

$$\frac{\ln(x+h) - \ln x}{h} = \frac{1}{h} \ln(\frac{x+h}{x}) = \ln(1 + \frac{h}{x})^{\frac{1}{h}} = \ln(1 + \frac{\frac{1}{x}}{\frac{1}{h}})^{\frac{1}{h}}.$$

Now denote $m = \frac{1}{h}$, then

$$(\ln x)' = \lim_{h \to 0} \frac{\ln(x+h) - \ln x}{h} = \lim_{h \to 0} \ln(1 + \frac{\frac{1}{x}}{\frac{1}{h}})^{\frac{1}{h}} =$$
$$\ln \lim_{h \to 0} (1 + \frac{\frac{1}{x}}{\frac{1}{h}})^{\frac{1}{h}} = \ln \lim_{m \to \infty} (1 + \frac{\frac{1}{x}}{m})^m = \ln e^{\frac{1}{x}} = \frac{1}{x}.$$

Corollary 1

$$(e^x)' = e^x.$$

Proof. $g(x) = e^x$ is the inverse function of $f(x) = \ln x$, then

$$(e^x)' = (g(x))' = \frac{1}{f'(g(x))} = \frac{1}{\ln' e^x} = \frac{1}{\frac{1}{e^x}} = e^x.$$

Corollary 2

$$(a^x)' = a^x \cdot \ln a.$$

Proof. $a = e^{\ln a}$, thus

$$(a^x)' = ((e^{\ln a})^x)' = (e^{(\ln a)\cdot x})' = e^{(\ln a)\cdot x} \cdot \ln a = (e^{\ln a})^x \cdot \ln a = a^x \cdot \ln a.$$

Corollary 3

$$(\log_a x)' = \frac{1}{x \cdot \ln a}.$$

Proof. $\log_a x = \frac{\ln x}{\ln a} = \frac{1}{\ln a} \cdot \ln x$, thus

$$(\log_a x)' = (\frac{1}{\ln a} \cdot \ln x)' = \frac{1}{\ln a} \cdot \frac{1}{x} = \frac{1}{x \cdot \ln a}.$$

Exercises 5.1-5.10

1.6 Applications

1.6.1 Interest

1. Simple interest: A-starting investment, r-rate, t-time in years,

$$B = A(1 + rt).$$

2. Interest compounded yearly: after t years the starting amount A increases to

$$B = A(1 + r)^t.$$

3. If the bank compounds interest n times a year then

$$B = A(1 + \frac{r}{n})^{nt}.$$

4. If the bank compounds interest continuously then

$$B = \lim_{n\to\infty} A(1+\frac{r}{n})^{nt} = \lim_{n\to\infty} A[(1+\frac{r}{n})^n]^t = [\lim_{n\to\infty} A(1+\frac{r}{n})^n]^t = [Ae^r]^t = Ae^{r\cdot t}.$$

1.6.2 Doubling Time

In what time t the amount A deposited for the annual rate r compounded continuously will be doubled? This time is a solution of the equation

$$2A = Ae^{r\cdot t},\ 2 = e^{r\cdot t},\ r\cdot t = \ln 2, t = \frac{\ln 2}{r}.$$

From this formula follows so called **Rule of 70**: To estimate how long it will take to double your money, just divide 70 by the rate of return *in percents*:

$$t = \frac{\ln 2}{r} = \frac{0.69}{r} = \frac{69}{100r} \approx \frac{70}{100r}.$$

Example. What is the doubling time if the return is 10%?

Solution. $t = \frac{70}{10} = 7$.

Example. The population of the word was 6.04 billion in 2000. Assuming that population groves exponentially at rate of 0.016 (that is 1,6%),
(a) estimate the population in 2010,
(b) after what period of time the population be double that in 2000?

Solution.
(a) $P_{2010} = P_{2000}\cdot e^{r\cdot 10} = 6.04\cdot e^{0.016\cdot 10} = 6.04\cdot e^{0.16} = 6.04\cdot 1.173 = 7.09.$
(b) $t = \frac{70}{1.6} = 43.75.$

1.6.3 Present value

The **present value** PV **of** B **dollars** t **years from now at interest rate** r is the amount A which must be deposited now in order to get B after t years.

If the annual interest r is compounded continuously, then the present value A can be solved from the equation

$$B = Ae^{rt},\quad PV = A = \frac{B}{e^{rt}} = Be^{-rt}.$$

Example. Find the present value of \$200,000 due 25 yr from now at 8.7%, compounded continuously.

Solution.

$$PV = 200,000\cdot e^{-8.7\cdot 25} = 200,000\cdot e^{-0.1136} = 22,721.63.$$

The PV for **flow of payments** B_1 dollars after t_1 years, B_2 dollars after t_2 years, ... , B_n dollars after t_n years is

$$PV = B_1e^{-rt_1} + \ ...\ + B_ne^{-rt_n}.$$

1.6.4 Annuities

An annuity is a sequence of equal payments at a regular intervals over specified period of time.

Case 1.Continuously Compounded Interest

The annuity that pays B dollars anally during N years needs the present value

$$PV = PV = Be^{-r\cdot 1} + Be^{-r\cdot 2} + \ ... \ + Be^{-r\cdot N} = \\ B(e^{-r\cdot 1} + e^{-r\cdot 2} + \ ... \ + e^{-r\cdot N}),$$

the sum of geometrical progression. Thus

$$PV_N = B \cdot \frac{e^{-r}(1-e^{-rN})}{1-e^{-r}} = B \cdot \frac{1-e^{-rN}}{e^r - 1}.$$

To calculate the present value of annuity which pays B dollars **forever**, let $N \to \infty$, then

$$PV_\infty = \frac{B}{e^r - 1}.$$

Case 2. Yearly Compounded Interest

In this case

$$PV_N = \frac{B}{1+r} + \ ... \ + \frac{B}{(1+r)^N}.$$

Again this is the sum of geometrical progression, so

$$PV_N = \frac{B}{r} \cdot (1 - (\frac{1}{1+r})^N).$$

Then the present value for **forever** annuity is

$$PV_\infty = \frac{B}{r}.$$

Example. I want \$ 5000 annuity. How much I must invest now if

(a) the rate is 10% compounded continuously and I want this annuity during next 20 years?

(b) the rate is 10% compounded continuously and I want this annuity forever?

Solution. Recall, for a geometric progression

$$b_1, \ b_2, \ ... \ , \ b_n, \ ...$$

$b_{k+1} = b_k \cdot q$, the sum of first n terms is

$$\sum_{k=1}^{n} b_k = \frac{b_1 - b_n \cdot q}{1-q} = \frac{b_1(1-q^{n+1})}{1-q},$$

and the sum of *all* terms (in case $|q| < 1$) is

$$\sum_{k=1}^{\infty} b_k = \frac{b_1}{1-q}.$$

(a) $PV_a = 5000 \cdot e^{-0.1 \cdot 1} + 5000 \cdot e^{-0.1 \cdot 2} \ ... \ + 5000 \cdot e^{-0.1 \cdot 20} =$
$5000 \cdot (e^{-0.1 \cdot 1} + e^{-0.1 \cdot 2} + \ ... \ + e^{-0.1 \cdot 20}) =$
$5000 \cdot \frac{e^{-0.1} \cdot (1-e^{-0.1 \cdot 21})}{1-e^{-0.1}} = 41720.$

(b) $PV_a = 5000 \cdot e^{-0.1 \cdot 1} + 5000 \cdot e^{-0.1 \cdot 2} \ ... \ + 5000 \cdot e^{-0.1 \cdot 20} + \ ... =$
$5000 \cdot (e^{-0.1 \cdot 1} + e^{-0.1 \cdot 2} + \ ... \ + e^{-0.1 \cdot 20} + \ ...) =$
$5000 \cdot \frac{e^{-0.1}}{1-e^{-0.1}} = 47542.$

1.6.5 Optimal Holding Time

Suppose the value of a real estate after t years from now will be $V(t)$. And suppose the bank rate in this period of time is r.

What is the best strategy for getting maximal income after, say, T years?

There are the following possibilities:

1. To sell the property now, at the moment $t = 0$, and deposit the money in bank.
2. To keep the property till to the end, i.e. to sell it at the moment $t = T$.
3. To sell it at some *optimal* time t_0 and deposit the received money in a bank till to the moment T.

Of course the answer depends on the function $V(t)$ and r.

How to find the optimal time? If we sell the property at the time t_0 we get $V(t_0)$. If we deposit at this moment this money at a bank we obtain at the end (in the moment T), i.e. after $T - t_0$ years the amount

$$W(t_0) = V(t_0)e^{r(T-t_0)}.$$

Let us find a critical point of $W(t)$:

$$W'(t) = V'(t)e^{r(T-t_0)} + V(t_0)e^{r(T-t_0)}(r(T-t))' = V'(t)e^{r(T-t_0)} - V(t_0)e^{r(T-t_0)}r,$$

so a critical point t_0 is a solution of the equation $V'(t)e^{r(T-t_0} = V(t_0)e^{r(T-t_0)}r$ which is equivalent to

$$\frac{V'(t)}{V(t)} = r.$$

Example

Suppose you have a book than now costs 2 lari and each year its price increases by 1 lari! Suppose the bank rate is 10% compounded continuously.

(a) How much you get after 10 years if you sell your book immediately and deposit your money in bank?

(b) How much you get if you sell your book after 10 years?

(c) How much you get after 10 years if you sell your book in optimal time and deposit your money in bank?

Solution
$(a) \quad B = 2 \cdot e^{0.1 \cdot 10} = 5.44!$

$(b) \quad V(t) = t + 2, \quad V(10) = 10 + 2 = 12!!$

$(c) \quad V(t) = t + 2, \quad V'(t) = 1, \quad \frac{V'(t)}{V(t)} = r, \quad \frac{1}{t+2} = 0.1, \quad t + 2 = 10, \quad t = 8,$ so the optimal holding time is 8 years, and if we sell in this moment we get $V(8) = 8 + 2 = 10$ lari. Now deposit this 10 lari in bank for 2 more years: $B = 10 \cdot e^{0.1 \cdot 2} = 12.2!!!$

1.7 Logarithmic Derivative

Sometimes the logarithm $\ln f(x)$ of a function $f(x)$ is easier to calculate and handle than the function $f(x)$ itself.

By the chain rule

$$(\ln f(x))' = \frac{f'(x)}{f(x)},$$

thus

$$f'(x) = (\ln f(x))' \cdot f(x).$$

This trick can be used to calculate $f'(x)$ in some cases.
Example. Let $f(x) = \sqrt[4]{x^2 - 1}$. Then

$$\ln f(x) = \frac{1}{4} \ln(x^2 - 1),$$

and

$$(\ln f(x))' = \frac{1}{4} \cdot (\ln(x^2 - 1))' = \frac{1}{4} \cdot \frac{2x}{x^2 - 1} = \frac{x}{2(x^2 - 1)}.$$

Finally

$$f'(x) = (\ln f(x))' \cdot f(x) = \frac{x}{2(x^2 - 1)} \cdot \sqrt[4]{x^2 - 1}.$$

Example. Let $f(x) = x^x$. Then

$$\ln f(x) = \ln x^x = x \ln x$$

and

$$(\ln f(x))' = \ln x + 1.$$

Finally

$$f'(x) = (\ln f(x))' \cdot f(x) = (\ln x + 1) \cdot x^x.$$

Elasticity and Logarithm

The differential of a function f is defined as $df(x) = f'(x)dx$. In this terms the elasticity $\epsilon(x) = \frac{f'(x) \cdot x}{f(x)}$ can be expressed as $\epsilon(x) = \frac{d \ln f(x)}{d \ln x}$. Indeed

$$\frac{d \ln f(x)}{d \ln x} = \frac{\frac{f'(x) \cdot dx}{f(x)}}{\frac{dx}{x}} = \frac{f'(x) \cdot x}{f(x)} = \epsilon(x).$$

Exercises 5.11-5.17

Homework

1. How much one would invest for \$ 5000 annuity now if

(a) the rate 10% is compounded anally and the duration is 20 years.

(b) the rate 10% is compounded annually and one wants this annuity forever?

2. Suppose you have a book than now costs 36 lari and after t years its price will be $V(t) = t^2 + 36$. Suppose the bank rate is 10% compounded continuously.

(a) How much you get after 20 years if you sell your book immediately and deposit your money in bank?

(b) How much you get if you sell your book after 20 years?

(c) How much you get after 20 years if you sell your book in optimal time and deposit your money in bank?

3. What annual interest rate compounded semiannually gives an annual yield of 21%.

4. Consider a population that grows according to a linear growth model. The initial population is $p_1 = 7$, and the population in the first generation is $P_2 = 11$. Find $P_{10} + P_{11} + \ ... \ + P_{20}$.

5. Give an example of geometric sequence in which $a_1, \ a_2, \ a_3, \ a_4$, are integers, and all the terms from a_5 on are fractions.