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WEEK 5
Reading [SB], 4.1-4.2, pp. 70-81

# 1 Chain Rule

## 1.1 Composition of Functions

Suppose $f : X \to Y$ and $g : Y \to Z$. The the *composition* $g \cdot f : X \to Z$ is defined by $g \cdot f(x) = g(f(x))$. In this composition $g \cdot f$ the function $f$ is the **inside function**, and the function $g$ is the **outside function**.

**Examples**

**1.** Let $f(x) = x^2$ and $g(x) = 2x + 3$, then $f \cdot g(x) = (2x+3)^2$ and $g \cdot f(x) = 2x^2 + 3$.

**2.** Let $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$, then $f \cdot g(x) = (\sqrt[3]{x})^3 = x$ and $g \cdot f(x) = \sqrt[3]{x^3} = x$, so the compositions both are identity functions $f \cdot g = id, \;\; g \cdot f = id$.

**3.** Let $f(x) = e^x$ and $g(x) = \ln x$, then $f \cdot g(x) = e^{\ln x} = x$ and $g \cdot f(x) = \ln e^x = x$, so the compositions both are identity functions $f \cdot g = id, \;\; g \cdot f = id$.

**Exercise**

For the composite function $f \cdot g(x) = 5e^{2x} + 3e^x + 1$, what are the inside and outside functions?
**Solution.** $5e^{2x} + 3e^x + 1 = 5(e^x)^2 + 3e^x + 1$, so the inside function is $g(x) = e^x$ and the outside function is $f(x) = 5x^2 + 3x + 1$.

## 1.2 Differentiating of Composite Functions: the Chain Rule

**Theorem.** The derivative of composite function $(h \circ g)(x)$ can be calculated as

$$(h \circ g)'(x) = h'(g(x)) \cdot g'(x)$$

(the chain rule).

**Proof*.**

$$(h\circ g)'(x_0) = lim_{x_1\to x_0}\frac{(h\circ g)(x_1)-(h\circ g)(x_0)}{x_1-x_0} = lim_{x_1\to x_0}\frac{h(g(x_1))-h(g(x_0))}{x_1-x_0} =$$

$$= lim_{x_1\to x_0}\frac{h(g(x_1))-h(g(x_0))}{g(x_1)-g(x_0)}\cdot\frac{g(x_1)-g(x_0)}{x_1-x_0} =$$

$$lim_{x_1\to x_0}\frac{h(g(x_1))-h(g(x_0))}{g(x_1)-g(x_0)}\cdot lim_{x_1\to x_0}\frac{g(x_1)-g(x_0)}{x_1-x_0} =$$

$$lim_{g(x_1)\to g(x_0)}\frac{h(g(x_1))-h(g(x_0))}{g(x_1)-g(x_0)}\cdot lim_{x_1\to x_0}\frac{g(x_1)-g(x_0)}{x_1-x_0} =$$

$$h'(g(x_0)\cdot g'(x_0).$$

Well, this proof has small gap, but forget it!

In particular

$$\frac{d}{dx}(g(x))^k = k(g(x))^{k-1}\cdot g'(x).$$

**Examples**

1. Find the derivative of $f(x) = (2x+3)^7$.

**Solution.** The function $f(x)$ is a composition $f(x) = h(g(x))$ with $g(x) = 2x+3$ and $h(z) = z^7$. Thus, by chain rule

$$f'(x) = h'(g(x)\cdot g'(x) = 7(2x+3)^6\cdot(2x+3)' = 7(2x+3)\cdot 2 = 14(2x+3)^6.$$

2. A firm computes that at the present moment its output is increasing at the rate of 2 units per hour and that its marginal cost is 12. At what rate is its cost increasing per hour?

**Solution.** Let $x(t)$ be the production function (output $x$ depends on time $t$) and in this moment $t = t_0$ we have $x'(t_0) = 2$. Let $C(x)$ be the cost function, so we have $C'(x_0) = 12$, where $x_0 = x(t_0)$. Then

$$\frac{dC}{dt}(t_0) = \frac{dC}{dx}(x(t_0))\cdot\frac{dx}{dt}(t_0) = 12\cdot 2 = 24.$$

**Exercises 4.1-4.6**

# 2 Again About Functions

A function (map, transformation) from the set $X$ (domain, or source) to the set $Y$ (codomain, or target)

$$f : X\to Y$$

is a rule that assigns to each element $x \in X$ one element $f(x) \in Y$.

The *image* of $f$ is the set of all elements $y \in Y$ that correspond to some $x$:

$$Im\ f = \{y \in Y, y = f(x)\}.$$

For an element $y \in Y$ its *preimage* $f^{-1}(y)$ is the set of all elements $x \in X$ such that $f(x) = y$:

$$f^{-1}(y) = \{x \in X, f(x) = y\}.$$

## 2.1 Again About Surjections, Injections, Bijections

A function $f : X \to Y$ is called *surjective* (**onto**) if

$$\forall\ y \in Y \quad \exists\ x \in X \quad s.t. \quad f(x) = y.$$

A function $f : X \to Y$ is called *injective* (**one-to-one**) if

$$f(x_1) = f(x_2) \quad \Rightarrow \quad x_1 = x_2.$$

A function is called *bijection* if it is a surjection and injection simultaneously.

In other words:
$f$ is a surjection if the equation $f(x) = y$ has *at least* one solution;
$f$ is an injection if the equation $f(x) = y$ has *at most* one solution.
$f$ is bijection if the equation $f(x) = y$ has *exactly* one solution.

## 2.2 Inverse Function

When $f : X \to Y$ is *bijective*, there is an *inverse* function $g : Y \to X$ which assigns to $y \in Y$ the unique element $g(y) = x$ such that $f(x) = y$.
**Definition** Function $g$ is the inverse of $f$ if $g(f(x)) = x$ and $f(g(y)) = y$ for arbitrary $x \in X$ and $y \in Y$. In other words

$$f \cdot g = id, \quad g \cdot f = id.$$

If $f$ is invertible, then its inverse function often is denoted as $f^{-1}$.

**Theorem 1** *If $f : X \to Y$ is invertible then it is a bijection.*

**Proof.**
**(i) Surjectivity.** For any $y \in Y$ we must find $x \in X$ s.t. $f(x) = y$. Let us take $x := g(y)$. Then

$$f(x) = f(g(y)) = y \quad since \quad f \circ g = id_Y, \quad QED.$$

**(i) Injectivity.** Suppose $f(x_1) = f(x_2)$, we must show that $x_1 = x_2$. Indeed,

$$f(x_1) = f(x_2) \ \Rightarrow\ g(f(x_1)) = g(f(x_2)) \ \Rightarrow\ x_1 = x_2 \ since\ g \circ f = id_X,\ QED.$$

**Theorem 2** *If $f : X \to Y$ is invertible then its inverse is uniquely determined.*

**Proof.** Suppose $g, h : Y \to X$ are two inverses of $f$:

$$f \circ g = id_Y, \quad g \circ f = id_X \quad and \quad f \circ h = id_Y, \quad h \circ f = id_X.$$

Then $g = h$, i.e. $g(y) = h(y)$ for arbitrary $y \in Y$, indeed, since of bijectivity (in fact by surjectivity) of $f$

$$\exists \; x \in X \quad s.t. \quad f(x) = y.$$

Then

$$g(y) = g(f(x)) = x \quad and \quad h(y) = h(f)x)) = x \quad since \quad g \circ f = h \circ f = id_x,$$

thus $g(y) = h(y)$, QED.

**Theorem 3** *A continuous function $f$ defined on an interval $I \subset R$ is invertible if and only if it is monotonically increasing or or monotonically decreasing.*

**Examples**

**1.** The function $f : R \to R$ given by $f(x) = x^2$ is not invertible (why?), but the function $f : [0, \infty) \to [0, \infty)$ is: The inverse function $g = f^{-1} : [0, \infty) \to [0, \infty)$ is $g(y) = \sqrt{y} = y^{1/2}$. Indeed,

$$f(g(y)) = (\sqrt{y})^2 = y, \quad g(f(x)) = \sqrt{x^2} = x.$$

**Remark.** This example shows that in the definition of inverse function both conditions

$$f \cdot g = id, \quad g \cdot f = id.$$

are essential: here we have $f(g(x)) = (\sqrt{x})^2 = x$, i.e. the first condition $f \cdot g = id$ is satisfied, but $g(f(-3)) = \sqrt{(-3)^2} = \sqrt{9} = 3 \neq -3$ , that is the second condition $g \cdot f = id$ is *not* satisfied for $f : R \to [0, \infty)$.

$$R \xrightarrow{f} R$$
*neither inj. nor surj.*

$$[0, +\infty) \xrightarrow{f} R$$
*inj. but not surj.*

$$R \xrightarrow{f} [0, +\infty)$$
*not inj. but surj.*

$$[0, +\infty) \xrightarrow{f} [0, +\infty)$$
*inj. and surj.*

**2.** The function $f : R \to R_+$ given by $f(x) = e^x$ is invertible, and its inverse is $g : R_+ \to R$ given by $f(y) = \ln y$ (why?).

**Exercise**

Calculate an expression for the inverse of the function $y = \frac{1}{x+1}$ specifying the domain.

**Solution.** Solve $x$ from the equation $y = \frac{1}{x+1}$:

$$y \cdot (x+1) = 1, \quad x+1 = \frac{1}{y}, \quad x = \frac{1}{y} - 1.$$

So the inverse function for $f(x) = \frac{1}{x+1}$ is $g(y) = \frac{1}{y} - 1$, indeed

$$f(g(y)) = \frac{1}{(\frac{1}{y} - 1) + 1} = \frac{1}{\frac{1}{y}} = y$$

and

$$g(f(x)) = \frac{1}{\frac{1}{x+1}} - 1 = (x+1) - 1 = x.$$

The domain of the inverse function is $(-\infty, 0) \bigcup (0, \infty)$.

Notice that just the condition $f \cdot g = id$ guarantees the surjectivity of $f$; just the condition $g \cdot f = id$ guarantees the injectivity of $f$; and only both conditions $f \cdot g = id, \quad g \cdot f = id$ guarantee the bijectivity of $f$, consequently its invertibility.

### 2.2.1 Graph of Inverse Function

Suppose $f$ is invertible and $g$ is its inverse. This means that if $f(a) = b$ then $g(b) = a$.

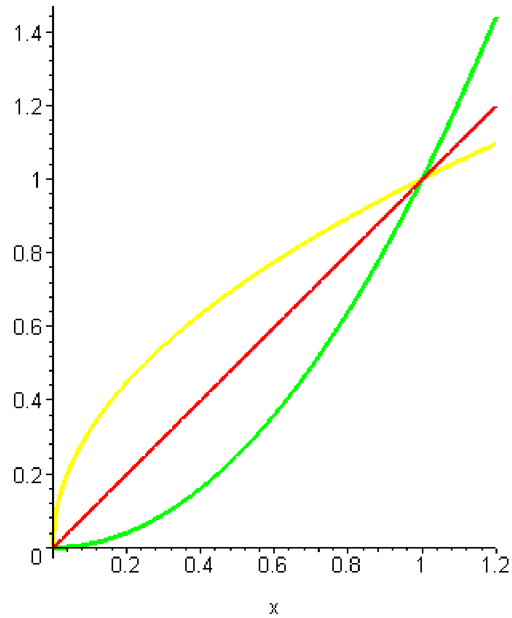
Suppose a point $(a, b)$ belongs to the graph of $f$ (notation $(a, b) \in \Gamma(f)$), i.e. $f(a) = b$. Then we have $g(b) = a$, thus the point $(b, a)$ belongs to the graph of $g$. Shortly

$$(a, b) \in \Gamma(f) \;\Rightarrow\; f(a) = b \;\Rightarrow g(b) = a \;\Rightarrow\; (b, a) \in \Gamma(g).$$

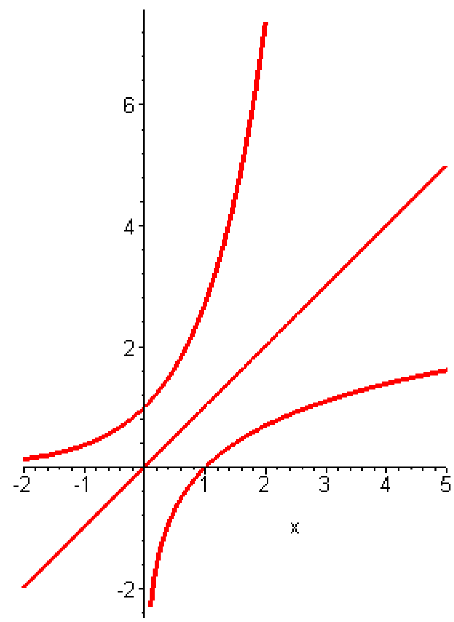
Similarly,

$$(b, a) \in \Gamma(g) \;\Rightarrow\; g(b) = a \;\Rightarrow f(a) = b \;\Rightarrow\; (a, b) \in \Gamma(f).$$

This means that the graphs of $f$ and $g$ are symmetric with respect to the bisectrix $y = x$.




$f(x) = x^2, \quad g(x) = \sqrt{(x)}$




$f(x) = e^x, \quad g(x) = ln\ x$

### 2.2.2 The Derivative of the Inverse Function

**Theorem 4** *Let $f$ be a $C^1$ function on an interval $I \subset R$ and $f'(x) \neq 0$ for all $x \in I$. Then $f$ is invertible on $I$, its inverse $g$ is $C^1$ on the interval $f(I)$ and*

$$g'(y) = \frac{1}{f'(g(y))}.$$

**Proof.** Invertibility of $f$ on $I$ follows from its monotonicity. Suppose $g = f^{-1}$, then $f(g(y)) = y$ for each $y \in f(I)$. Differentiating this equality using the chain rule we obtain

$$f'(g(y)) \cdot g'(y) = y' = 1,$$

thus $g'(y) = \frac{1}{f'(g(y))}$.

### 2.2.3 Application*

The formula

$$(x^k)' = kx^{k-1},$$

was proven only for **natural** $k$-s. The above theorem allows to generalize this formula for arbitrary **rational** $k$:

1. The function $g(y) = y^{\frac{1}{n}}$ is the inverse of $f(x) = x^n$ (why?). This allows to calculate the derivative of $g(y) = y^{\frac{1}{n}}$:

$$(y^{\frac{1}{n}})' = g'(y) = \frac{1}{f'(g(y))} = \frac{1}{((g(y))^n)'} =$$

$$\frac{1}{n \cdot g(y))^{n-1}} = \frac{1}{n \cdot (y^{1/n})^{n-1}} = \frac{1}{n} \cdot y^{\frac{1-n}{n}} = \frac{1}{n} \cdot y^{\frac{1}{n}-1}.$$

2. Now take any arbitrary rational number $\frac{m}{n} \in Q$. Let us proof that

$$(x^{\frac{m}{n}})' = \frac{m}{n} x^{\frac{m}{n}-1}.$$

Indeed, first let us assume that $m, n \in N$, i.e. $q = \frac{m}{n}$ is a positive rational number. Since $x^{\frac{m}{n}} = (x^{\frac{1}{n}})^m$ by the Chain Rule we have

$$(x^{\frac{m}{n}})' = ((x^{\frac{1}{n}})^m)' = m(x^{\frac{1}{n}})^{m-1} \cdot (x^{\frac{1}{n}})' = mx^{\frac{m-1}{n}} \cdot \frac{1}{n} x^{\frac{1}{n}-1} =$$

$$mx^{\frac{m-1}{n}} \cdot \frac{1}{n} x^{\frac{1-n}{n}} = \frac{m}{n} x^{\frac{m-1}{n}+\frac{1-n}{n}} = \frac{m}{n} x^{\frac{m-1+1-n}{n}} = \frac{m}{n} x^{\frac{m}{n}-1}.$$

So we already have proved $(x^q)' = qx^{q-1}$ for any positive rational $q \in Q$. It remains to generalize this formula for negative rational numbers $(x^{-q})' = -qx^{-q-1}$, indeed,

$$(x^{-q})' = (\frac{1}{x^q})' = \frac{1' \cdot x^q - 1 \cdot (x^q)'}{x^{2q}} = \frac{-qx^{q-1)}}{x^{2q}} = -qx^{-q-1}.$$

The further generalization of the formula $(x^r)' = rx^{r-1}$ for a **real** $r \in R$ uses approximation of a real number by a sequence of rational numbers.

#### Exercise

Calculate the derivative of the inverse of the function $f(x) = \frac{1}{x+1}$ at the point $f(1) = \frac{1}{2}$.

**Solution.**

$$g'(\frac{1}{2}) = g'(f(1)) = \frac{1}{f'(g(f(1)))} = \frac{1}{f'(1)} = \frac{1}{-\frac{1}{(x+1)^2}}|_{x=1} = -(x+1)^2|_{x=1} = -4.$$

By the way, as we know the inverse for $f(x) = \frac{1}{x+1}$ is $g(y) = \frac{1}{y} - 1$. The direct calculation of $g'(\frac{1}{2})$ gives the same result.

**Exercises 4.7-4.10**

**Homework 4**

Exercises 4.3 (c), 4.5 (e,g), 4.6, 4.8 (c), 4.9 (c)