Math for Economists, Calculus 1

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WEEK 4

1 Economical Applications

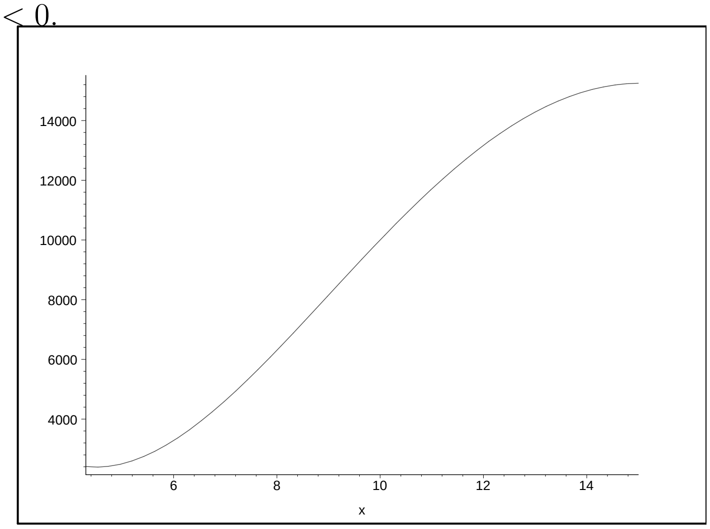
1.1 Production Function

A production function $y = f(q)$ assigns to amount q of input the corresponding output y.

Usually f is

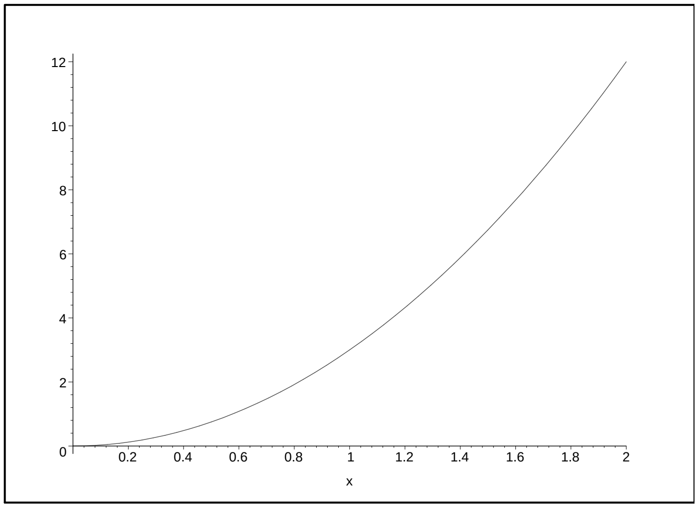
- increasing, that is $f' > 0$;

- there exists a level of input a such that for $0 \leq q < a$ the function f is concave up, that is $f''(q) > 0$, and for $a < q$ is concave down, that is $f''(q) < 0$.



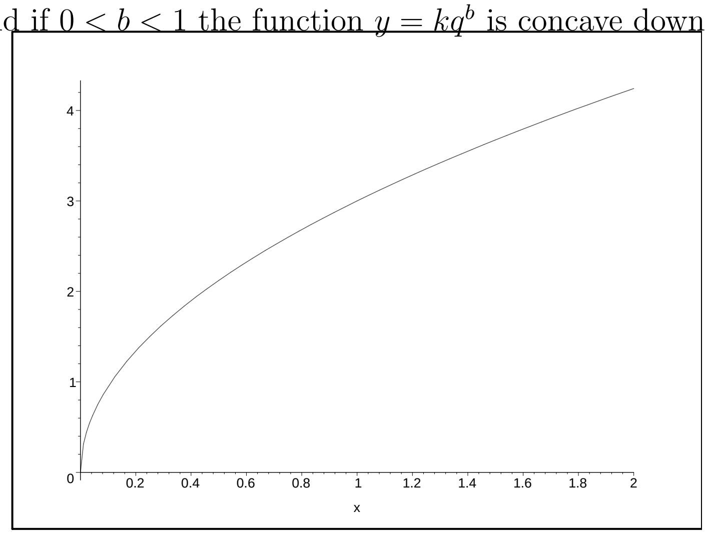
Often the production function is of type $y = kq^b$ where q is exogenous variable, and k and b are positive parameters.

Notice that if $b > 1$ the function $y = kq^b$ is concave up (**why?**):



$$f(q) = 3q^2$$

And if $0 < b < 1$ the function $y = kq^b$ is concave down (**why?**):



$$f(q) = 3q^{\frac{1}{2}}$$

1.2 Cost Function

A **cost function** $C(x)$ assigns to each output x the total cost of production. The value $C(0)$ is called **fixed cost**.

Such a function usually increases, $C'(x) > 0$.

The **marginal cost** $MC(x)$ measures the cost of $(x+1)$-th unit of output, that is

$$MC(x) = C(x+1) - C(x).$$

If x is "large" so that $h = 1$ is "small", then the marginal cost approximates the derivative

$$C'(x) = \lim_{h \to 0} \frac{C(x+h) - C(x)}{h} \approx \frac{C(x+1) - C(x)}{1} = MC(x).$$

The **average cost** $AC(x)$ is defined as

$$AC(x) = \frac{C(x)}{x},$$

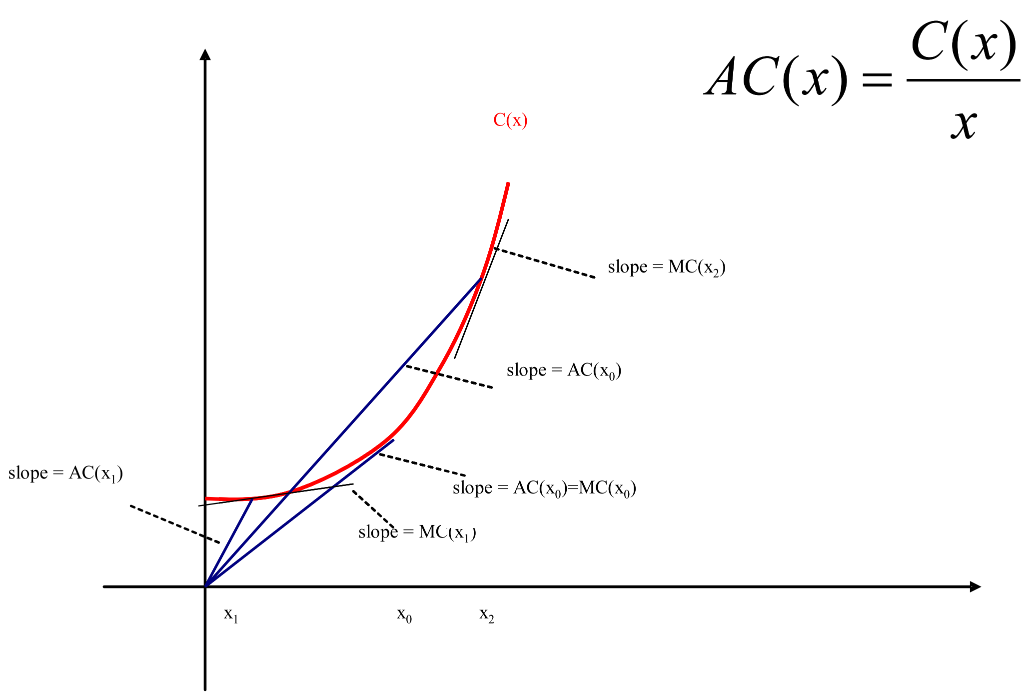
it is the cost per unit when x units are produced.

Theorem 1 *1. If $MC > AC$ then AC is increasing;*
2. If $MC < AC$ then AC is decreasing;
3. At an interior minimum of AC we have $AC = MC$.

This theorem can be justified by:

1. Economical intuition. If the cost of one new unit (the marginal cost MC) is more then average cost of one unit AC, then the new average cost will be bigger.

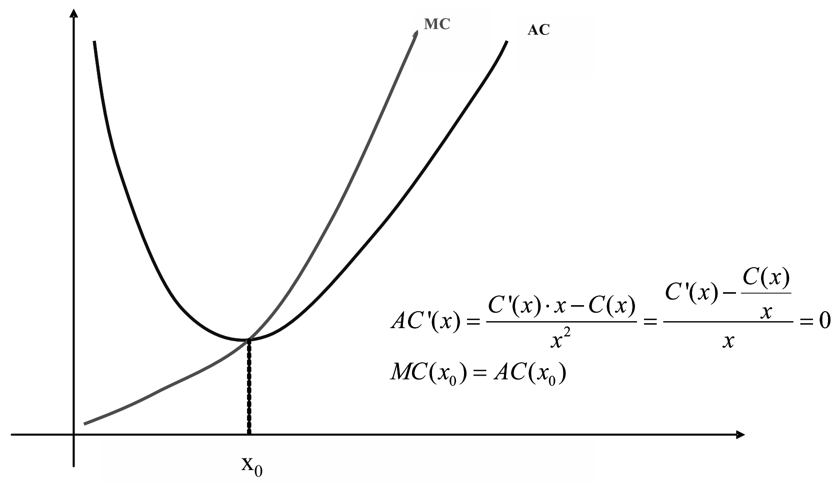
2. Geometrical observation.



3. Mathematical proof. Just calculate the derivative

$$AC'(x) = \left(\frac{C(x)}{x}\right)' = \frac{C'(x)x - C(x)}{x^2} = \frac{x(C'(x) - \frac{C(x)}{x})}{x^2} = \frac{MC - AC}{x}.$$

AC minimizer

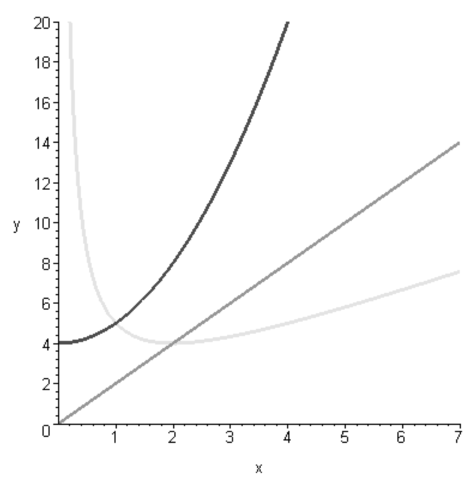


So the critical point of AC is the solution x_0 of the equation $MC = AC$. But is it max or min for AC? Just check AC'' at x_0.

Example

Let $C(x) = 4 + x^2$.

1. Draw the graphs of $C(x),\ MC(x),\ AC(x)$.



2. Find a point x_0 which minimizes $AC(x)$, show this point on your drawing, what is minimal average cost? $AC(2) = 4$.

1.3 Revenue and Profit

A **revenue function** $R(x)$ indicates how much money a firm receives for selling of x units of output. This function is also increasing. $MR(x)$ denotes **marginal revenue** $MR(x) \approx R'(x)$.

Let $p(x)$ be the price of one unit when the total output is x. Then $R(x) = x \cdot p(x)$.

A **profit function** is the difference

$$P(x) = R(x) - C(x).$$

The **optimal value** is the value x which maximizes the profit $P(x)$.

Theorem 2 *At optimal value the marginal revenue equals to the marginal cost, that is if x_0 is the profit maximizer, then $MR(x_0) = MC(x_0)$.*

Again this can be proved by economical intuition or using calculus.

"This principal, that marginal revenue equals marginal cost at the optimal output, is one of the cornerstones of economic theory".

Intuitive explanation:

When $MR > MC$, that is the revenue from producing one more unit is more then the cost of this additional unit, the producing of this one additional unit increases its profit.

When $MR < MC$, that is the revenue from producing one more unit is less then the cost of this additional unit, the firm decreases its profit.

So at profit maximizer x_0 we have $MR(x_0) = MC(x_0)$.

Mathematical proof:

$$P'(x_0) = 0 \;\Rightarrow\; (R(x_0) - C(x_0))' = R'(x_0) - C'(x_0) = 0 \;\Rightarrow R'(x_0) = C'(x_0).$$

1.3.1 Perfect Competition Case

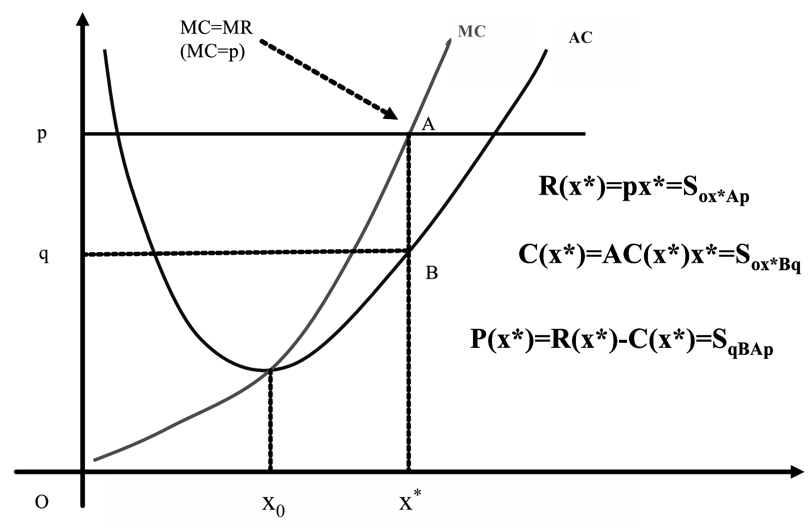
In the case of **perfect competition** the price function is assumed constant $p(x) = p$.

In this case $R(x) = p \cdot x$ is linear and $MR(x) = AR(x) = p$ (**why?**).

The optimal value of x is a solution of the equation

$$MC(x) = p.$$

Perfect Competition p = const   Profit maximizer



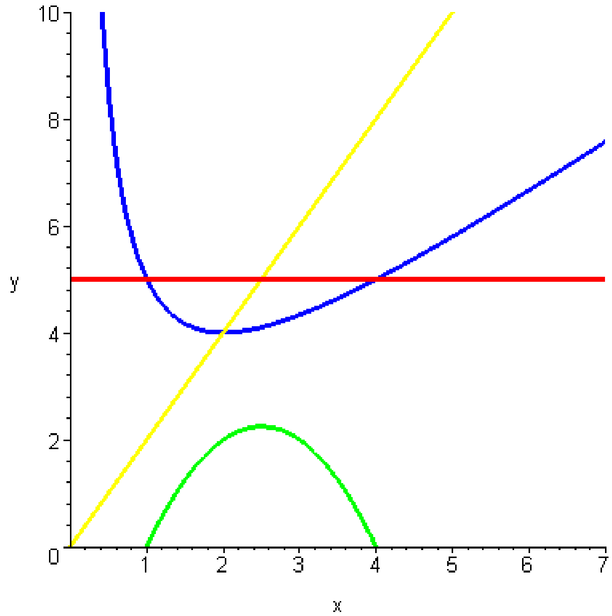
What happens with optimal value if the price p increases?

Since the marginal cost $MC(x) = C'(x)$ is an increasing function, then if the market price p increases, the corresponding optimal output x increases too.

Can you show on this picture the brake-even points, i.e. the points where $P(x) = 0$??

Exercise

1. Let again $C(x) = 4 + x^2$ and the selling price of one unit is $p(x) = 5$ (perfect competition case).
1. Write $R(x)$ and $P(x)$. $R(x) = 5x, \quad P(x) = 5x - 4 - x^2$.
2. Draw the graphs of $MC(x),\ AC(x),\ p(x),\ P(x)$.



3. Find a point x^* which maximizes the profit, show this point on your drawing. Calculate $R(x^*),\ C(x^*),\ P(x^*)$.

$$MC = MR,\ 2x = 5,\ x_0 = 2.5$$

$$R(2.5) = 12.5,\ C(2.5) = 10.25,\ P(2.5) = 2.25$$

4. Show on your drawing the rectangles whose area exhibits $R(x^*),\ C(x^*)$, $P(x^*)$ respectively.

Exercise

2. Replace in the previous exercise the cost function $C(c) = 4 + x^2$ by new one $C(x) = 4 + \sqrt{x}$ and observe what happens.

1.3.2 Pure Monopolist Case

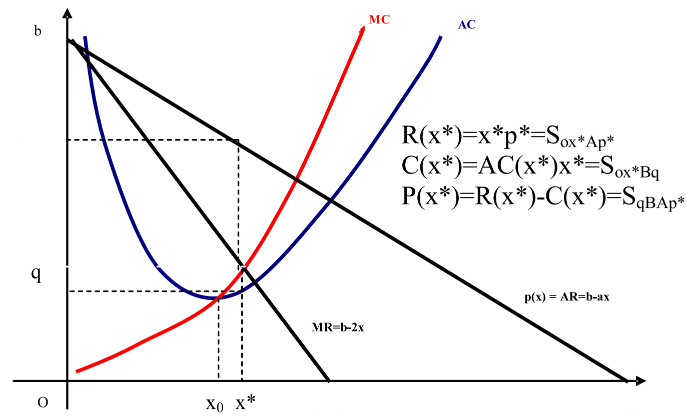
In this case price $p(x)$ depends on the demand x. Assume that this dependence is linear

$$p(x) = p - ax.$$

In this case $R(x) = xp(x) = px - ax^2$ and $MR(x) = R'(x) = p - 2ax$. From this follows that $MR(x)$ and $p(x)$ have the same p-intercepts, but the slope of $MR(x)$ is twice the slope of $p(x)$. Besides, the x-intercepts of $p(x) = p - ax$ and $MR(x) - p - 2ax$ are respectively $\frac{p}{a}$ and $\frac{p}{2a}$.

The profit maximizer is a solution of $MR(x) = MC(x)$, i.e. the intersection point of MR and MC.

Pure Monopolist p(x) = b - ax Profit



Exercise

3. Let again $C(x) = 4+x^2$ and the selling price of one unit is $p(x) = 10-x$ (pure monopolist case).
1. Write $R(x)$ and $P(x)$.
2. Draw the graphs of $MC(x),\ AC(x),\ p(x),\ MR(x)$.
3. Find a point x^* which maximizes the profit, show this point on your drawing. Calculate $R(x^*),\ C(x^*),\ P(x^*)$.
4. Show on your drawing the rectangles whose area exhibit $R(x^*),\ C(x^*),\ P(x^*)$.

1.4 Elasticity

Example 1. Suppose the function $s(t) = 30 \cdot t$ describes the distance (in kilometers) which a car goes in time t (in hours). Then the speed of the car is the derivative

$$v = s'(t) = 30.$$

Quite low speed! Now let us measure the time again in hours, but the distance in meters, then the same motion is described by the function $s(t) = 30000 \cdot t$, so the speed in this case is the derivative

$$v = s'(t) = 30000.$$

Is it much bigger? No, the speed is the same: $30\ km/h = 30000\ m/h$.

So we see that the derivative (which describes the rate of change) heavily depends on measurement units.

Example 2. For a demand function $x(p)$ the derivative $x'(p) = \frac{dx}{dp}$ measures the price sensitivity of the demand x.

Suppose the demand function is linear

$$x(p) = 10 - 2p$$

(notice demand x should be a decreasing function of the price p).

The marginal demand in this case in **constant** $x'(p) = -2$:
For example at $p = 1$ the marginal demand $Mx(p) = x(p+1) - x(p)$ is

$$x(2) - x(1) = 6 - 8 = -2,$$

and at $p = 4$ is

$$x(5) - x(4) = 0 - 8 = -2.$$

OK, but now let us check the following: what is the **percent** of change the demand when we increase the price by 10%?

In other words, suppose we step from p_0 to $p_1 = 1.1 \cdot p_0$, then what is the percent of change of demand $x(p)$? This is

$$100 \cdot \frac{\Delta x}{x} = 100 \cdot \frac{x(p_1) - x(p_0)}{x(p_0)}.$$

(a) Let us first check it for $p_0 = 1$:

$$100 \cdot \tfrac{\Delta x}{x} = 100 \cdot \tfrac{x(p_1)-x(p_0)}{x(p_0)} =$$
$$100 \cdot \tfrac{x(1.1\cdot 1))-x(1)}{x(1)} = 100 \cdot \tfrac{x(1.1)-x(1)}{x(1)} = 100 \cdot \tfrac{(10-2.2)-(10-2)}{10-2} = 100 \cdot \tfrac{7.8-8}{8} =$$
$$-100 \cdot \tfrac{0.2}{8} = -100 \cdot 0.025 = -2,5\%,$$

so at $p_0 = 1$ when the price increases by 10% then the demand decreases by 2.5%.

(b) Let us check it for $p_0 = 2.5$:

$$100 \cdot \tfrac{\Delta x}{x} = 100 \cdot \tfrac{x(p_1)-x(p_0)}{x(p_0)} =$$
$$100 \cdot \tfrac{x(1.1\cdot 2.5))-x(2.5)}{x(2.5)} = 100 \cdot \tfrac{x(2.75)-x(2.5)}{x(2.5)} = 100 \cdot \tfrac{(10-2.75)-(10-5)}{10-5} =$$
$$100 \cdot \tfrac{4.5-5}{10-5} = -100 \cdot \tfrac{0.5}{5} = -100 \cdot 0.1 = -10\%,$$

so at $p_0 = 2.5$ when the price increases by 10% then the demand decreases also by 10%.

(c) Now let us check it for $p_0 = 4$:

$$100 \cdot \tfrac{\Delta x}{x} = 100 \cdot \tfrac{x(p_1)-x(p_0)}{x(p_0)} =$$
$$100 \cdot \tfrac{x(1.1\cdot 4)-x(4)}{x()} = 100 \cdot \tfrac{x(9.9)-x(9)}{x(4)} = 100 \cdot \tfrac{(10-8.8)-(10-8)}{10-8} = 100 \cdot \tfrac{1.2-2}{2} =$$
$$-100 \cdot \tfrac{0.8}{2} = -100 \cdot 0.4 = -40\%,$$

so at $p_0 = 4$ when the price increases by 10% then the demand decreases by 40%!

So we conclude that for our demand function

$$x(p) = 10 - 2p$$

the derivative $x'(p) = -2$ is *constant*, but but the *percent of change* of demand when the price is changed by 10% is:
At $p_0 = 1$ is -2.5% (smaller than the percent of change of p);
At $p_0 = 2.5$ is -10% (same as the percent of change of p);
At $p_0 = 4$ is -40% (bigger than the percent of change of p).

Again we see that the derivative does not work properly.

More subtle tool to measure the price sensitivity of x is **price elasticity** which is defined as the **ratio of the percent change in endogenous variable (x in our case) to the percent change in exogenous variable (p in our case)**:

$$\epsilon(p) = \frac{\%\ chnge\ in\ demand}{\%\ chnge\ in\ price} = \frac{100 \cdot \frac{\Delta x}{x}}{100 \cdot \frac{\Delta p}{p}} = \frac{\frac{\Delta x}{x}}{\frac{\Delta p}{p}} = \frac{p}{x} \cdot \frac{\Delta x}{\Delta p} \approx \frac{p}{x} \cdot \frac{dx}{dp} = \frac{p}{x} \cdot x'(p).$$

Example. For our demand function $x(p) = 10 - 2p$ (with constant derivative!) the elasticity

$$\epsilon(p) = \frac{p}{x} \cdot x'(p) = -\frac{2p}{10 - 2p}$$

is nonconstant! Notice that when p changes from $p = 0$ to $p = 5$ the elasticity changes from 0 to $-\infty$:

Consider the following 3 cases:

(a) The demand at p is **elastic** if $\epsilon(p) \in (-\infty, -1)$. In this case a "small" change of p induces "big" change of x.

(b) The demand at p is **of unit elasticity** if $\epsilon(p) = -1$.

(c) The demand at p is **inelastic** if $\epsilon(p) \in (-1, 0)$. In this case a "big" change of p induces "small" change of x.

Examples

1. Back to our demand function $x(p) = 10 - 2p$.
(a) At the point $p = 1$ the elasticity is $\epsilon(p) = -0.25$ (inelastic): a 10% increase in price produces a 2.5% decrease in demand.

(b) At the point $p = 2.5$ the elasticity is $\epsilon(p) = -1$ (unit elasticity): a 10% increase in price produces a 10% decrease in demand.
(c) At the point $p = 4$ the elasticity is $\epsilon(p) = -4$ (elastic): a 10% increase in price produces a 40% decrease in demand.

2. Suppose $x(p) = 120 - 20p$. Then the derivative is constant $\frac{dx}{dp} = -20$, but

$$\epsilon(p) = \frac{p}{x} \cdot \frac{dx}{dp} = \frac{-20p}{120 - 20p}.$$

(a) The interval of inelasticity is the solution of inequality $\epsilon(p) > -1$. This gives $p \in (0, 3)$.
(b) The point of unit elasticity is the solution of the equation $\epsilon(p) = -1$. This gives $p = 3$.
(c) The interval of elasticity is the solution of inequality $\epsilon(p) < -1$. This gives $p \in (3, 6)$.

3. Suppose $x(p) = 3p^{-2}$. Then the derivative $x'(p) = -6p^{-3}$ is nonconstant, but

$$\epsilon(p) = \frac{p}{x} \cdot \frac{dx}{dp} = \frac{p}{3p^{-2}} \cdot (-6p^{-3}) = -3$$

is constant. Generally, a function of type $x(p) = k \cdot p^r$ is called the constant elasticity function (why?).

Exercise

4. Suppose the demand function is given by $x(p) = 12 - 3p, \quad 0 < p < 4$.
1. Find the point of unit elasticity, interval of elasticity and interval of inelasticity.
2. Find maximal revenue.
3. Plot the graphs of $x(p),\ \epsilon(p),\ R(p)$.
4. Calculate percent of change of demand if the price $p_0 = 0.5$ increases by 7%? Give the answer first by direct calculation, then using elasticity, and compare the results.
5. Find a price p_* whose doubling decreases the demand by 10%.

1.4.1 Elasticity and revenue

If the demand function is given by $x(p)$ the total revenue at price p is $R(p) = p \cdot x(p)$.

Usually $x'(p) < 0$, that is the demand $x(p)$ is decreasing function of price p. But what about the $R(p)$, is it increasing or decreasing?

Theorem 3 *(1) If for a price p the demand $x(p)$ is inelastic, that is $-1 < \epsilon(p) < 0$, then the revenue $R(p)$ is increasing.*

(2) If for a price p the demand $x(p)$ is elastic, that is $\epsilon(p) < -1$, then the revenue $R(p)$ is decreasing.

(3) If for a price p the demand $x(p)$ has unit elasticity, that is $\epsilon(p) = -1$, then the revenue $R(p)$ is at maximum.

Proof. Let us first calculate the derivative of the revenue

$$\begin{gathered} R'(p) = (p \cdot x(p))' = x(p) + p \cdot x'(p) = \\ x(p) \cdot (1 + \tfrac{p \cdot x'(p)}{x(p)}) = x(p)(1 + \epsilon(p)). \end{gathered}$$

(1) If $-1 < \epsilon(p) < 0$, then

$$R'(p) = x(p)(1 + \epsilon(p)) > 0,$$

thus $R(p)$ is increasing.
(2) If $\epsilon(p) < -1$, then

$$R'(p) = x(p)(1 + \epsilon(p)) < 0,$$

thus $R(p)$ is decreasing.
(3) If $\epsilon(p) = -1$, then

$$R'(p) = x(p)(1 + \epsilon(p)) = 0,$$

thus p is a critical point of $R(p)$.

1.4.2 Linear Demand

Let us study the elasticity of linear demand function

$$x = F(p) = 6 - 2p, \quad p \in (0, 3)$$

(a decreasing linear function).

For this function

$$\epsilon(p) = \frac{F'(p) \cdot p}{F(p)} = \frac{-2 \cdot p}{6 - 2p}.$$

1. The demand is **inelastic** if $\epsilon(p) > -1$, that is

$$\frac{-2 \cdot p}{6 - 2p} > -1,$$

the solution of this inequality gives $p \in (0, 1.5)$.

2. The demand is **unit elastic** if $\epsilon(p) = -1$, that is

$$\frac{-2 \cdot p}{6 - 2p} = -1,$$

the solution of this equation gives $p = 1.5$.

3. The demand is **elastic** if $\epsilon(p) < -1$, that is

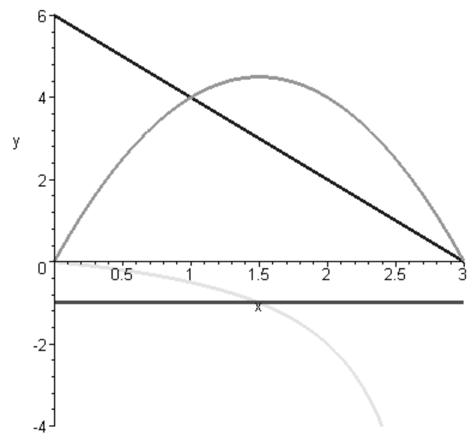
$$\frac{-2 \cdot p}{6 - 2p} < -1,$$

the solution of this inequality gives $p \in (1.5, 3)$.

The revenue function here is $R(p) = x(p) \cdot p = (6 - 2p) \cdot p = -2p^2 + 6p$. Critical point of $R(p)$ is the solution of the equation

$$R'(p) = -4p + 6 = 0,$$

so the critical point is the unit elasticity point $x^* = 1.5$. Is it a maximum point? Yes: $R''(1.5) = -4 < 0$.



Similarly for a general linear demand function

$$x = F(p) = a - bp, \quad a > 0,\ b > 0,\ x \in (0, \frac{a}{b})$$

(a decreasing linear function) the elasticity is given by

$$\epsilon(p) = \frac{F'(p) \cdot p}{F(p)} = \frac{-b \cdot p}{a - bp}.$$

1. The demand is **inelastic** if $\epsilon(p) > -1$, that is

$$\frac{-b \cdot p}{a - bp} > -1,$$

the solution of this inequality gives $p \in (0, \frac{a}{2b})$.

2. The demand is **unit elastic** if $\epsilon(p) = -1$, that is

$$\frac{-b \cdot p}{a - bp} = -1,$$

the solution of this equation gives $p = \frac{a}{2b}$, i.e. the midpoint of the interval $(0, \frac{a}{b})$.

3. The demand is **elastic** if $\epsilon(p) < -1$, that is

$$\frac{-b \cdot p}{a - bp} < -1,$$

the solution of this inequality gives $p \in (\frac{a}{2b}, \frac{a}{b})$.

The revenue is maximal at the unit elasticity point $x^* = \frac{a}{b}$ (check this).

1.4.3 Constant Elasticity Demand

Now let us study the elasticity if the demand function

$$x = 6p^{-2}.$$

For this function

$$\epsilon(p) = \frac{F'(p) \cdot p}{F(p)} = \frac{6 \cdot (-2) \cdot p^{-3} \cdot p}{6p^{-2}} = -2,$$

so the elasticity of this function is constant $\epsilon(p) = -2$ and the demand is elastic.

Generally for

$$x = kp^{-r}, \quad k, r > 0$$

the elasticity is

$$\epsilon(p) = \frac{F'(p) \cdot p}{F(p)} = \frac{k \cdot (-r) \cdot p^{-r+1} \cdot p}{kp^{-r}} = -r,$$

so the elasticity of this function is constant $\epsilon(p) = -r$ and

- The demand is elastic if $r > 1$.
- The demand is of unit elasticity if $r = 1$.
- The demand is inelastic if $0 < r < 1$.

Exercises

5. Let $C(x) = 4 + x^2$ and the selling price of one unit is $p(x) = 6$ (perfect competition case).
(a) Write $MC(x),\ AC(x),\ R(x),\ P(x)$.
(b) Draw the graphs of $MC(x),\ AC(x),\ p(x),\ MR(x)$.
(c) Find a point x_0 which minimizes $AC(x)$, show this point on your drawing, what is minimal average cost?
(d) Find a point x^* which maximizes the profit, show this point on your drawing. Calculate $R(x^*),\ C(x^*),\ P(x^*)$.

(e) Show on your drawing the rectangles whose area exhibits $R(x^*),\ C(x^*),\ P(x^*)$ respectively.

6. Solve the previous problem replacing $p(x) = 6$ by $p(x) = 15 - x$ (pure monopolist case).

7. Suppose the demand function is given by $x(p) = 10 - 2p, \quad 0 < p < 5.$
(a) Find the point of unit elasticity, interval of elasticity and interval of inelasticity.
(b) Find maximal revenue.
(c) Plot the graphs of $x(p),\ \epsilon(p),\ R(p)$.
(d) Calculate percent of change of demand if the price $p_0 = 4$ increases by 10%? Give the answer first by direct calculation, then using elasticity, and compare the results.

8. Suppose the demand function is given by $x(p) = 27 - p^2,\ 0 < p < 3\sqrt{3}.$
(a) Find the point of unit elasticity, interval of elasticity and interval of inelasticity.
(b) Find maximal revenue.
(c) Plot the graphs of $x(p),\ \epsilon(p),\ R(p)$.
(d) Calculate percent of change of demand if the price $p_0 = 1$ increases by 13%? Give the answer first by direct calculation, then using elasticity, and compare the results.