ISET MATH I Term Final Exam

Answers without work or justification will not receive credit

Problem 1.

Let start with definition: For a real number $x \in R$ its "integer part" ("floor", "greatest integer" in other terms) is defined as

$$[x] = largest\ integer\ not\ greater\ than\ x.$$

For example $[5.25] = 5$, $[1.001] = 1$. Got it?

There is another notion: The "fractional part" of a real number $x \in R$ is defined as $\{x\} = x - [x]$. For example $\{5.25\} = 0.25$, $\{1.001\} = 0.001$.

(a) Plot the graph of $f(x) = [x]$ for $x \in [-4, 4]$.
(b) Compute $\int_0^4 f(x)dx$
(c) Plot the graph of $g(x) = \{x\}$ for $x \in [-4, 4]$.
(d) Compute $\int_0^4 g(x)dx$.

Answer

(a)	
(b)	
(c)	
(d)	

Solution

Problem 2. Prove the following theorem and deduce from it the following 3 corollaries:
Theorem. $(ln\ x)' = \frac{1}{x}$.
Corollary 1. $(log_a\ x)' = \frac{1}{x \cdot ln\ a}$.
Corollary 2. $(e^x)' = e^x$.
Corollary 3. $(a^x)' = a^x \cdot ln\ a$.

Proofs

Solution

Problem 3.

(a) For $f(x) = \frac{1}{1-x}$ find a polynomial of degree three $P(x) = ax^3 + bx^2 + cx + d$ such that $f^{(n)}(0) = P^{(n)}(0)$ for $n = 0, 1, 2, 3$ (here $f^{(k)}(x)$ denotes the k^{th} derivative of $f(x)$ and $f^{(0)}(x) = f(x)$).

(b) More generally, for a given function $f(x)$ (OK, enough differentiable) find a polynomial of degree three $Q(x) = ax^3 + bx^2 + cx + d$ such that $f^{(n)}(0) = P^{(n)}(0)$ for $n = 0, 1, 2, 3$. Does it remaind you something?

(c_1) Write the MacLaurin polynomial (that is Taylor at $x = 0$) of degree 2 for the function $g(x) = x + 2x^2 + 3x^3$.

(c_2) Write the MacLaurin polynomial of degree 3 for the same function $g(x) = x + 2x^2 + 3x^3$.

(c_3) Write the MacLaurin polynomial of degree 4 for the same function $g(x) = x + 2x^2 + 3x^3$.

(c_4) Using c_1, c_2, c_3 conclude *something* about MacLaurin polynomials $P_k(x)$ of various degrees k for a *polynomial function* $G(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ of degree n.

Answer

(a)	$P(x) =$
(b)	$Q(x) =$
(c_1)	
(c_2)	
(c_3)	
(d)	

Solution

Problem 4.

(a_1) Consider all triangles formed by lines passing through the point $(1,1)$ and both the x-and y-axes. Find the cathets of the triangle with *smallest area*, that is minimize the function $S(x)$, where x is the x-cathet of triangle and $S(x)$ is its area. Do not forget the second order condition!

(a_2) Plot the graph of the function $S(x)$. Indicate on that graph all critical points, intercepts and asymptotes if any.

(b) More generally, consider all triangles formed by lines passing through the point (p,q), which lays in the first quadrant, and both the x-and y-axes. Find the cathets of the triangle with smallest area. Do not forget the second order condition!

Answer

(a_1)	$S(x) =$ $x_{cath} =$, $y_{cath} =$, $S_{min} =$ *second order condition* :
(a_2)	
(b)	

Solution

Problem 5.

(a) The **Mean Value Theorem** claims that for a function f which is continuous on a closed interval $[a, b]$ and is differentiable on the open interval (a, b) there exists a point $c \in (a, b)$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$. In other words there exists a point $c \in (a, b)$ at which the tangent is parallel to . . . (finish yourself in the box and sketch a graph which illustrates this theorem).

(b) Find a point c for $f(x) = x^2, \quad a = 1, \quad b = 3.$

(c) More generally, express c for $f(x) = x^2$ in terms of endpoints a and b.

(d) Express c for $f(x) = x^3$ in terms of endpoints a and b.

(e) Express c for $f(x) = \sqrt{x}$ and $a = 0$ in terms of b.

(f) There exist another famous **Roll's Theorem**, which is the direct consequence of previous Mean Value Theorem for a function $f(x)$ which satisfies the condition $f(a) = f(b)$. Give the formulation of Roll's Theorem in the box and also and sketch a graph which illustrates this theorem.

Answer

(a)	*at which the tangent is parallel to*
(b)	$c =$
(c)	$c =$
(d)	$c =$
(e)	$c =$
(f)	*Roll's Theorem :*

Solution

ADDITIONAL PAPER

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