Constrained Optimization

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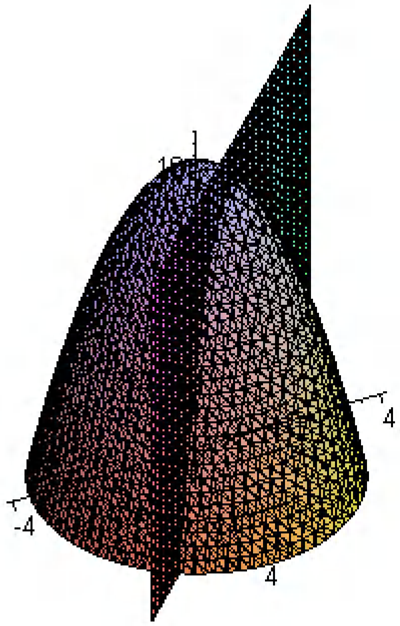
Example

Maximize $f(x,y)=10-x^2-y^2$ subject to $x+y=1$.



The absolute maximum of $f(x,y)=10-x^2-y^2$ is at $(0,0)$ and equals to $f(0,0)=10$. But what about the maximum along the line $x+y=1$?

There is "naive way" to solve this problem: just solve y from the constraint $y=1-x$, substitute to the function f

$$10-x^2-(1-x)^2$$

and maximize this function. The solution gives the critical point $x=1/2$, so the maximizer of f subject of constraint is the point $(1/2,1/2)$.

The maximum then is $f(1/2,1/2)=10-0.25-0.25=9.5$.

Constrained Optimization

General Problem

Maximize $f(x_1, ..., x_n)$ where $(x_1, ..., x_n) \in R^n$ must satisfy

$$g_1(x_1, ..., x_n) \leq b_1, \ \ldots \ , g_1(x_1, ..., x_n) \leq b_k;$$
$$h_1(x_1, ..., x_n) = c_1, \ \ldots \ , h_1(x_1, ..., x_n) = c_m.$$

The function $f(x_1, ..., x_n)$ is called objective function.

The functions $g_i(x_1, ..., x_n)$ are called inequality constraints.

The functions $h_i(x_1, ..., x_n)$ are called equality constraints.

Constrained Optimization

Two variables and One Equality Constraint

Theorem 1 *Suppose $x^* = (x_1^*, x_2^*)$ is a solution of the problem:*

maximize $f(x_1, x_2)$ subject to $h(x_1, x_2) = c$.

Suppose further that (x_1^*, x_2^*) is not a critical point of h:

$$\nabla h(x_1^*, x_2^*) = (\frac{\partial h}{\partial x_1}(x_1^*, x_2^*), \frac{\partial h}{\partial x_2}(x_1^*, x_2^*)) \neq (0, 0),$$

(this condition is called constraint qualification *at the point (x_1^*, x_2^*)).*

Then, there is a real number μ^* such that (x_1^*, x_2^*, μ^*) is a critical point of the Lagrangian function

$$L(x_1, x_2, \mu) = f(x_1, x_2) - \mu[h((x_1, x_2) - c].$$

In other words, at (x_1^*, x_2^*, μ^*) we have

$$\frac{\partial L}{\partial x_1} = 0, \quad \frac{\partial L}{\partial x_2} = 0, \quad \frac{\partial L}{\partial \mu} = 0.$$

Constrained Optimization

So a solution of the constrained optimization problem

$$maximize\ f(x_1, x_2)\ subject\ to\ h(x_1, x_2) = c$$

we must seek among solutions (x_1^*, x_2^*, μ^*) of the system of equations

$$\frac{\partial L}{\partial x_1} = 0, \quad \frac{\partial L}{\partial x_2} = 0, \quad \frac{\partial L}{\partial \mu} = 0$$

where

$$L(x_1, x_2, \mu) = f(x_1, x_2) - \mu[h((x_1, x_2) - c]$$

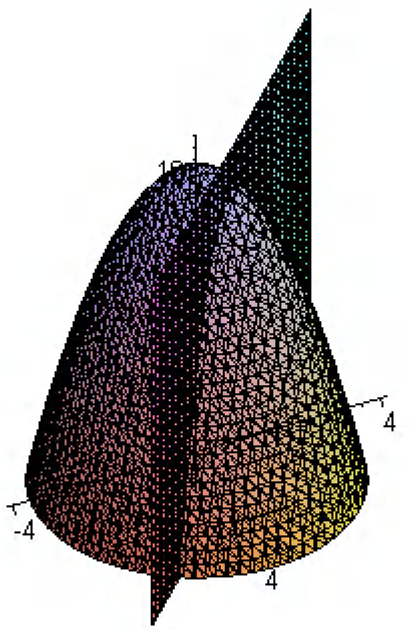
is the Lagrangin function.

Note that these conditions are just necessary conditions for constrained maximization.
If we want to minimize f instead of maximization, the same conditions are necessary too.
There exist more subtle second order sufficient conditions.

Back to our

Example

Maximize $f(x, y) = 10 - x^2 - y^2$ subject to $x + y = 1$.



Now let us solve the problem using Lagrange method. First we remark that qualification is satisfied: $h(x, y) = x + y$ has no critical points at all.

The Lagrangian here is

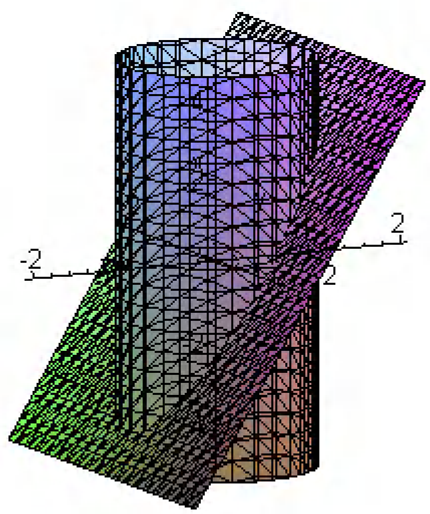
$$L(x, y, \mu) = 10 - x^2 - y^2 - \mu(x + y - 1).$$

So we have the system

$$\begin{aligned} \frac{\partial L}{\partial x} &= -2x - \mu = 0 \\ \frac{\partial L}{\partial y} &= -2y - \mu = 0 \\ \frac{\partial L}{\partial \mu} &= -(x + y - 1) = 0, \end{aligned}$$

solution gives $x = 0.5$, $y = 0.5$, $\mu = -1$. So the only candidate is the point $(0.5, 0.5)$.

Example. Maximize $f(x, y) = 4x$ subject to $h(x, y) = x^2 + y^2 - 1 = 0$.



The Lagrangian here is

$$L(x, y, \mu) = 4x - \mu(x^2 + y^2 - 1).$$

So we have the system

$$\begin{aligned}\frac{\partial L}{\partial x} &= 4 - 2\mu x = 0\\ \frac{\partial L}{\partial y} &= -2\mu y = 0\\ \frac{\partial L}{\partial \mu} &= -(x^2 + y^2 - 1) = 0,\end{aligned}$$

solution gives

$$x = 1, y = 0, \mu = 2, f(1, 0) = 4,$$

$$x = -1, y = 0, \mu = -2, f(-1, 0) = -4.$$

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> with(Student[MultivariateCalculus]):
> LagrangeMultipliers(4*x,[x^2+y^2-1],[x,y],output=detailed);
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$[x = 1, y = 0, \lambda_1 = 2, 4\,x = 4], [x = -1, y = 0, \lambda_1 = -2, 4\,x = -4]$

Constrained Optimization

Economical example

A manufacturing firm has budgeted \$60,000 per month for labor and materials. If \$x thousand is spent on labor and \$y thousand is spent on materials, and if the monthly output (in units) is given by $N(x, y) = 4xy - 8x$ how should the \$60,000 be allocated to labor and materials in order to maximize N? What is the maximum N?

Constrained optimization problem:

maximize N(x,y) = 4xy – 8x subject to x + y = 60 000.

Good lack!

Exercises

1. Minimize $f(x, y, z) = x^2+y^2+z^2$ subject to $h(x, y) = 2x-y+3z = -28$.

2. Maximize and minimize $f(x, y, z) = 2x + 4y + 4z$ subject to $h(x, y) = x^2 + y^2 + z^2 = 9$.

3. Maximize $f(x, y) = 4 - x^2 - y^2$ subject to the constraint $h(x, y) = y - x^2 + 1 = 0$.

4. A manufacturing firm has budgeted \$60,000 per month for labor and materials. If \$x thousand is spent on labor and \$y thousand is spent on materials, and if the monthly output (in units) is given by $N(x, y) = 4xy - 8x$ how should the \$60,000 be allocated to labor and materials in order to maximize N? What is the maximum N?

5. The Cobb-Douglas production function for a new product is given by

$$f(x, y) = 16x^{0.25}y^{0.75}.$$

where x is the number of units of labor and y is the number of units of capital required to produce $f(x, y)$ units of the product. Each unit of labor costs \$50 and each unit of capital costs \$100. If \$500,000 has been budgeted for the production, how should this amount be allocated between labor and capital in order to maximize production? What is the maximum number of units that can be produced?

6. A consulting firm for a manufacturing company arrived at the following Cobb-Douglas production function for a particular product: $N(x, y) = 50x^{0.8}y^{0.2}$ where x is the number of units of labor and y is the number of units of capital required to produce $N(x, y)$ units of the product. Each unit of labor costs \$40 and each unit of capital costs \$80.

(A) If \$400,000 is budgeted for production of the product, determine how this amount should be allocated to maximize production, and find the maximum production.

(B) Find the marginal productivity of money in this case, and estimate the increase in production if an additional \$50,000 is budgeted for the production of this product.

7. The research department for a manufacturing company arrived at the following Cobb-Douglas production function for a particular product: $N(x, y) = 10x^{0.6}y^{0.4}$ where x is the number of units of labor and y is the number of units of capital required to produce $N(x, y)$ units of the product. Each unit of labor costs \$30 and each unit of capital costs \$60.

(A) If \$300,000 is budgeted for production of the product, determine how this amount should be allocated to maximize production, and find the maximum production.

(B) Find the marginal productivity of money in this case, and estimate the increase in production if an additional \$80,000 is budgeted for the production of this product.

8. Find the maximum and minimum distance from the origin to the ellipse $x^2 + xy + y^2 = 3$.

9. Find the point on the parabola $y = x^2$ that is closest to the point $(2, 1)$. (Estimate the solution of the cubic equation which results.)

10. The standard beverage can has a volume 12 oz, or 21.66 in^3. What dimension yield the minimum surface area? Find the minimum surface area.