Optimization

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Recall one variable case

Let $f: R \to R$ be one variable function.
A point $x^* \in R$ is called stationary (critical) if $f'(x^*) = 0$.
This condition is necessary condition for local maximality or minimality:

$$x^* \; is \; \max \; or \; \min \; \Rightarrow \; f'(x^*) = 0.$$

But of course not sufficient, recall $f(x) = x^3$.

The sufficient is the following second order condition

$$\left.\begin{array}{l} f'(x^*) = 0 \\ f''(x^*) < 0 \end{array}\right\} \Rightarrow x^* \; is \; local \; \max, \qquad \left.\begin{array}{l} f'(x^*) = 0 \\ f''(x^*) > 0 \end{array}\right\} \Rightarrow x^* \; is \; local \; \min.$$

Two variable case

Definitions

For function $F(x_1, x_2)$ a point $x^* = (x_1^*, x_2^*)$ is

1. a global max if $F(x_1^*, x_2^*) \geq F(x_1, x_2)$ for all (x_1, x_2).

2. a local max if $F(x_1^*, x_2^*) \geq F(x_1, x_2)$ for all $(x_1, x_2) \in B_r(x_1^*, x_2^*)$ from some ball around (x_1^*, x_2^*).

3. a strict global max if $F(x_1^*, x_2^*) > F(x_1, x_2)$ for all (x_1, x_2).

4. a strict local max if $F(x_1^*, x_2^*) > F(x_1, x_2)$ for all $(x_1, x_2) \in B_r(x_1^*, x_2^*)$ from some ball around (x_1^*, x_2^*).

Similarly are defined min-s.

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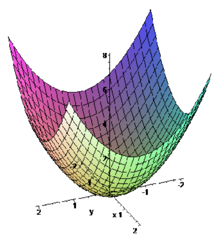
Two Variable Case

Examples

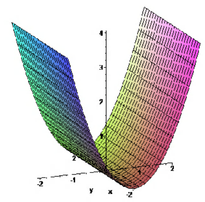
1. $F(x_1, x_2) = x_1^2 + x_2^2$ has strict global minimum at $x^* = (0, 0)$, check by plotting in MAPLE!

2. $F(x_1, x_2) = x_1^2$ has global minimums at each point $x^* = (0, x_2)$, check by plotting in MAPLE!

```
> plot3d(x^2+y^2,x=-2..2,y=-2..2);
```

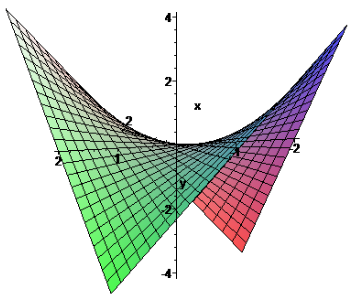


```
> plot3d(x^2,x=-2..2,y=-2..2);
```



Saddle Point

```
> plot3d(x*y,x=-2..2, y=-2..2);
```



For a multivariable function $F(x) = F(x_1, x_2, \dots, x_n)$ a point $x^* = (x_1^*, x_2^*, \dots, x_n^*) \in R^n$ is called stationary (well, critical) if all partial derivatives are zero at this point, i.e.

$$\frac{\partial F}{x_1}(x^*) = 0, \ \dots \ , \ \frac{\partial F}{x_n}(x^*) = 0,$$

in other words the gradient is 0-vector

$$Df(x^*) = (\frac{\partial F}{x_1}(x^*), \ \dots \ , \ \frac{\partial F}{x_n}(x^*)) = (0, \dots, 0) = \vec{0} \in R^n.$$

First order conditions for optimality

Criticality is necessary condition for local maximality or minimality:

$$x^* \ is \ \max \ or \ \min \ \Rightarrow \ DF(x^*) = 0.$$

Example. Find critical points for $F(x, y) = x^3 - y^3 - 9xy$.
Solution.

$$\frac{\partial F}{\partial x}(x, y) = 3x^2 + 9y, \quad \frac{\partial F}{\partial y}(x, y) = -3y^2 + 9x.$$

To find critical points solve the system

$$\begin{cases} \frac{\partial F}{\partial x}(x, y) = 0 \\ \frac{\partial F}{\partial y}(x, y) = 0 \end{cases} \left| \begin{array}{l} 3x^2 + 9y = 0 \\ -3y^2 + 9x = 0 \end{array} \right| -\frac{1}{3}x^4 + 9x = 0$$

the solutions are $(x = 0, y = 0)$, $(x = 3, y = -3)$. To determine whether either of these critical points is min max or neither we need *second order conditions* which involve second derivatives of F.

Second order sufficient condition

We use the notation $\frac{\partial^2 F}{\partial x_i \partial x_j} = F_{x_i x_j}$. Let

$$HF(x^*) = \begin{pmatrix} F_{x_1x_1}(x^*) & F_{x_1x_2}(x^*) \\ F_{x_2x_1}(x^*) & F_{x_2x_2}(x^*) \end{pmatrix}$$

be the Hessian matrix of F at the critical point x^*.

This matrix has two *leading principal minors*

$$M_1(x^*) = F_{x_1x_1}(x^*)$$

$$M_2(x^*) = \begin{vmatrix} F_{x_1x_1}(x^*) & F_{x_1x_2}(x^*) \\ F_{x_2x_1}(x^*) & F_{x_2x_2}(x^*) \end{vmatrix} = F_{x_1x_1}(x^*) \cdot F_{x_2x_2}(x^*) - F_{x_1x_2}(x^*)^2$$

Suppose x^* is a critical point.

1. If $M_1(x^*) < 0$, $M_2(x) > 0$, then x^* is a strict local max.
2. If $M_1(x^*) > 0$, $M_2(x) > 0$, then x^* is a strict local min.
3. If either $M_1(x^*)$ or $M_1(x^*)$ violates this sign pattern, then x^* is a saddle point.

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Example. Now we can classify two critical points $(0,0)$ and $(3,-3)$ of the function $F(x,y) = x^3 - y^3 + 9xy$. The Hessian of F is

$$\begin{vmatrix} F_{xx} & F_{yx} \\ F_{xy} & F_{yy} \end{vmatrix} = \begin{vmatrix} 6x & 9 \\ 9 & -6y \end{vmatrix}.$$

The first leading principal minor is $F_{xx} = 6x$ and the second order principal leading minor is $-36xy - 81$.

At $(0,0)$ these two minors are 0 and -81, this is the situation 3, so $(0,0)$ is a saddle point.

At $(3,-3)$ these two minors are 18 and 24, this is the situation 2, so $(3,-3)$ is a strict local min point.

Note that this local min is not global:
$F(0,y)$ decreases to $-\infty$ when y increases to ∞.

n-variable case

1 *Suppose*

$$\frac{\partial F}{\partial x_i} = 0, \quad i = 1, 2, ..., n,$$

and n leading principal minors of $D^2F(x^*)$ alternate in sign

$$\begin{vmatrix} F_{x_1x_1} \end{vmatrix} < 0, \quad \begin{vmatrix} F_{x_1x_1} & F_{x_2x_1} \\ F_{x_1x_2} & F_{x_2x_2} \end{vmatrix} > 0, \quad \begin{vmatrix} F_{x_1x_1} & F_{x_2x_1} & F_{x_3x_1} \\ F_{x_1x_2} & F_{x_2x_2} & F_{x_3x_2} \\ F_{x_1x_3} & F_{x_2x_3} & F_{x_3x_3} \end{vmatrix} < 0, \quad ...$$

at x^*. Then x^* is a strict local max.

2 *Suppose*

$$\frac{\partial F}{\partial x_i} = 0, \quad i = 1, 2, ..., n,$$

and n leading principal minors of $D^2F(x^*)$ are positive

$$\begin{vmatrix} F_{x_1x_1} \end{vmatrix} > 0, \quad \begin{vmatrix} F_{x_1x_1} & F_{x_2x_1} \\ F_{x_1x_2} & F_{x_2x_2} \end{vmatrix} > 0, \quad \begin{vmatrix} F_{x_1x_1} & F_{x_2x_1} & F_{x_3x_1} \\ F_{x_1x_2} & F_{x_2x_2} & F_{x_3x_2} \\ F_{x_1x_3} & F_{x_2x_3} & F_{x_3x_3} \end{vmatrix} > 0, \quad ...$$

at x^*. Then x^* is a strict local min.

3 *Suppose*

$$\frac{\partial F}{\partial x_i} = 0, \quad i = 1, 2, ..., n,$$

and some nonzero leading principal minors of $D^2F(x^*)$ *violate the sign pattern of 1 and 2. Then* x^* *is a saddle point.*

Example. A monopolist producing a single output has two types of customers. If it produces Q_1 units for customers of type 1, then these customers are willing to pay a price of $50 - 5Q_1$ dollars per unit. If it produces Q_2 units for customers of type 2, then these customers are willing to pay a price of $100 - 10Q_2$

Solution. The profit function is

$$F(Q_1, Q_2) =$$

$$Q_1(50-5Q_1)+Q_2(100-lOQ_2)-(90+20(Q_1+Q_2)).$$

The critical points of F satisfy

$$\frac{\partial F}{\partial Q_1} = 50 - 10Q_1 - 20 = 0, \quad Q_1 = 3,$$
$$\frac{\partial F}{\partial Q_2} = 100 - 20Q_2 - 20 = 0, \quad Q_2 = 4.$$

So the critical point is $(3, 4)$.

Now check the second order conditions. Since

$$F_{Q_1Q_1} = -10, \quad F_{Q_2Q_2} = -20, \quad F_{Q_1Q_2} = 0,$$

the Hessian looks as

$$D^2(Q_1, Q_2) = \begin{pmatrix} F_{Q_1Q_1} & F_{Q_2Q_1} \\ F_{Q_1Q_2} & F_{Q_2Q_2} \end{pmatrix} = \begin{pmatrix} -10 & 0 \\ 0 & -20 \end{pmatrix}.$$

The first order leading principal minor of $D^2F(3, 4)$ is -10 and the second leading principal minor is 200. Therefore $(3, 4)$ is strict local max.

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Exercises

1. For each of the following functions find the critical points and classify these as local max, local min, saddle point, or "can't tell":
(a) $F(x,y) = x^2 + xy + y^2 - 3x$, (b) $F(x,y) = xy - x^3 - y^2$;
(c) $F(x,y) = xy^2 + x^3y - xy$, (d) $F(x,y) = 3x^4 + 3x^2y - y^3$.

2. A firm produces two kind of golf ball, one that sells for \$3 and one for \$2. The total cost, in thousands of dollars, of producing of x thousand balls at \$3 each and y thousand balls at \$2 each is given by

$$C(x,y) = 2x^2 - 2xy + y^2 - 9x + 6y + 7.$$

Find the amount of each type of ball that must be produced and sold in order to maximize profit.

3. A one-product company finds that its profit, in millions of dollars, is a function P given by

$$P(a,p) = 2ap + 80p - 15p^2 - 1/10 \cdot a^2p - 100,$$

where a is the amount spent on advertising, in millions of dollars, and p is the price charged per item of the product, in dollars. Find the maximum value of P and the values of a and p at which it is attained.

4. A one-product company finds that its profit, in millions of dollars, is a function P given by

$$P(a,n) = -5a^2 - 3n^2 + 48a - 4n + 2an + 300,$$

where a is the amount spent on advertising, in millions of dollars, and n is the number of items sold, in thousands. Find the maximum value of P and the values of a and n at which it is attained.

5. A trash company is designing an open-top, rectangular container that will have a volume of 320 ft^3. The cost of making the bottom of the container is \$5 per square foot, and the cost of the sides is \$4 per square foot. Find the dimensions of the container that will minimize total cost. (Hint: Make a substitution using the formula for volume.)

6. A computer firm, markets two kinds of electronic calculator that compete with one another. Their demand functions are expressed by the following relationships:

$$q_1 = 78 - 6p_1 - 3p_2,$$
$$q_2 = 66 - 3p_1 - 6p_2,$$

where p_1 and p_2 are the price of each calculator, in multiples of \$10, and q_l and q_2 are the quantity of each calculator demanded, in hundreds of units.

a) Find a formula for the total-revenue function R in terms of the variables p_1 and p_2.

b) What prices p_1 and p_2 should be charged for each product in order to maximize total revenue?

c) How many units will be demanded?

d) What is the maximum total revenue?