Multivariable Calculus

Tornike Kadeishvili

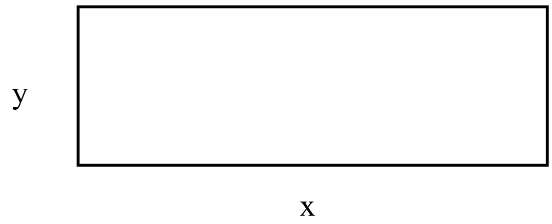
Functions $R^n \rightarrow R$

1. The area of a rectangle with dimensions x and y is a function of two variables $S : R^2 \rightarrow R$ given by *quadratic* function

$$S(x, y) = xy.$$

The perimeter of this rectangle is a *linear* function of two variables $P : R^2 \rightarrow R$ given by

$$P(x, y) = 2x + 2y.$$



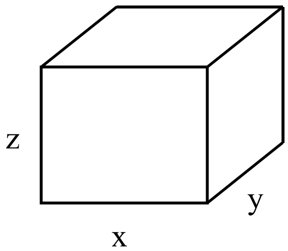
Functions $R^n \rightarrow R$

2. The volume of a box with dimensions x, y, z is a function of three variables $V : R^3 \rightarrow R$ given by *cubical* function

$$V(x, y, z) = xyz.$$

The area of the surface is a *quadratic* function of three variables

$$S(x, y, z) = 2xy + 2xz + 2yz.$$



3. The *amount* A is a function of three variables: P-*principal*, r-*annual rate*, t-*time* in years. The function $A : R^3 \rightarrow R$ is given by

$$A(P, r, t) = P(1 + rt).$$

4. For a *demand functions* $q = f(p)$ the quantity demanded q is a function of one variable: its own price p.

In reality the demanded quantity depends also on the prices of other goods in the market and on income y :

$$q_1 = f(p_1, p_2, y).$$

Computer – printer (complementary)

$$q_{comp}(p_{comp}, p_{print}) = A - k \cdot p_{comp} - s \cdot p_{print}$$

Computer –laptop (competitive)

$$q_{lap}(p_{lap}, p_{desk}) = A - k \cdot p_{lap} + s \cdot p_{desk}$$

5. Another example of multivariable function in economics is *production function.* Consider a firm which uses n inputs to produce a single output.

For $i = 1, ..., n$, let x_i denote the amount of input i. The vector $(x_1, ..., x_n)$ is called an *input bundle.* The firm's production function assigns to each input bundle $(x_1, ..., x_n)$ the amount of output $y = f(x_1, ..., x_n)$.

Functions $R^n \to R$

6. One more example is a *utility function.* Consider an economy with k commodities. Let x_i denote the amount of commodity i. The vector $(x_1, ..., x_k) \in R^k$ is called a *commodity bundle.*

Suppose two bundles $x = (x_1, ..., x_k)$ and $x' = (x'_1, ..., x'_k)$ are given.

Is it possible to say which from these two bundles is preferable?

A *utility function* is a function $u : R^k \longrightarrow R$ which assigns to a commodity bundle $(x_1, ..., x_k)$ a number $u(x_1, ..., x_k)$ which measures the consumer's degree of satisfaction or utility with the given commodity bundle. Utility function determines preferences: a commodity bundle $x = (x_1, ..., x_k)$ is *preferred* to another bundle $x' = (x'_1, ..., x'_k)$ if

$$u(x_1, ..., x_k) > u(x'_1, ..., x'_k),$$

and x and x' are called *indifferent* if $u(x_1, ..., x_k) = u(x'_1, ..., x'_k)$.

Partial Derivatives

Let $f : R^n \to R$. Then for each x_i at each point $x^0 = (x_1^0, ..., x_n^0)$ the ith *partial derivative* is defined as

$$\frac{\partial f}{\partial x_i}(x_1^0, ..., x_n^0) = \lim_{h \to 0} \frac{f(x_1^0, ..., x_i^0 + h, ..., x_n^0) - f(x_1^0, ..., x_i^0, ..., x_n^0)}{h}.$$

Notice that here only ith variable is changing, the others are treated as constants. Thus the partial derivative $\frac{\partial f}{\partial x_i}(x_1^0, ..., x_n^0)$ is the ordinary derivative of the function $f(x_1^0, ..., x_{i-1}^0, x, x_{i+1}^0, ..., x_n^0)$ of one variable x at the point $x = x_i^0$.

Notation

$$\frac{\partial f}{\partial x_i}(x_1^0, ..., x_n^0) = f_{x_i} \ (= f'_{x_i}).$$

Examples. 1. Find partial derivatives for $f(x, y) = 3x^2y^2 + 4xy^3 + 7y$.

For $\frac{\partial f}{\partial x}$ only x is considered as a *variable* and y is treated as a *constant*:

$$\frac{\partial f}{\partial x} = 3 \cdot 2x \cdot y^2 + 4 \cdot 1 \cdot y^3 + 0 = 6xy^2 + 4y^3.$$

For $\frac{\partial f}{\partial y}$ only y is considered as a variable and x is treated as a constant:

$$\frac{\partial f}{\partial y} = 3x^2 \cdot 2y + 4x \cdot 3y^2 + 7 \cdot 1 = 6x^2y + 12xy^2 + 7.$$

2. Find partial derivatives for $f(x, y) = \sqrt{xy}$.

$$\frac{\partial f}{\partial x} = \frac{1}{2\sqrt{xy}} \cdot \frac{\partial \ (xy)}{\partial x} = \frac{y}{2\sqrt{xy}}.$$

$$\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{xy}} \cdot \frac{\partial \ (xy)}{\partial y} = \frac{x}{2\sqrt{xy}}.$$

Chain Rule

If $y = f(x_1, \ldots, x_n)$, $x_i = x_i(t_1, \ldots, t_m)$, $i = 1, 2, \ldots, n$, then

$$\frac{\partial y}{\partial t_j} = \sum_{i=1}^{n} \frac{\partial f(x_1, \ldots, x_n)}{\partial x_i} \frac{\partial x_i}{\partial t_j}.$$

Second Partial Derivatives

Second order partial derivative of $y = f(x_1, \ldots, x_n)$ is defined as

$$\frac{\partial^2 y}{\partial x_j \partial x_i} = f_{x_i x_j}(x_1, \ldots, x_n) = \frac{\partial}{\partial x_j} f_{x_i}(x_1, \ldots, x_n), \quad i, j = 1, 2, \ldots, n.$$

Example.

Let $f(x,y) = x^2y^3$. Then

$$f(x,y) = x^2 y^3$$

$$f_x = 2xy^3 \qquad f_y = 3x^2y^2$$

$$f_{xx} = 2y^3 \qquad f_{xy} = 6xy^2 \;=\; f_{yx} = 6xy^2 \qquad f_{yy} = 6x^2y$$

Young's Theorem

$$\frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j}, \quad i, j = 1, 2, \ldots, n.$$

Gradient

Gradient of a function $f(x_1, \ldots, x_n)$ is the vector

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right) = (f_{x_1}, \ldots, f_{x_n})$$

Sometimes the gradient vector is denoted as $D^1 f$.

Example. For $f(x,y) = x^2y^3$ the gradient is $D^1 f(x,y) = (2xy^3, 3x^2y^2)$.

Hessian

Hessian of a function $f(x_1, \dots, x_n)$ is the matrix

$$H(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1^2} \end{pmatrix}$$

Sometimes $H(f)$ is denoted as $D^2 f$.

Hessian

Example. For $f(x,y) = x^2y^3$ the Hessian is

$$D^2 f(x,y) = \begin{pmatrix} 2y^3 & 6xy^2 \\ 6xy^2 & 6x^2y \end{pmatrix}$$

Linear Approximation

Recall that for a function of one variable $f(x)$ the derivative

$$f'(x^*) = \frac{df}{dx}(x^*)$$

allows to approximate $f(x^* + \Delta\, x)$ as a linear function of $\Delta\, x$

$$f(x^* + \Delta\, x) \approx f(x^*) + f'(x^*) \cdot \Delta\, x.$$

Similarly, for a function of two variables $F(x,y)$ the partial derivatives $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ allow to approximate $F(x,y)$ in the neighborhood of a given point (x^*, y^*):

$$F(x^* + \Delta x, y^* + \Delta y) \approx F(x^*, y^*) + \frac{\partial F}{\partial x}(x^*, y^*) \cdot \Delta x + \frac{\partial F}{\partial u}(x^*, y^*) \cdot \Delta y.$$

For a function of n variables similar linear approximation looks as

$$F(x_1^* + \Delta x_1, ..., x_n^* + \Delta x_n) \approx$$
$$F(x_1^*, ..., x_n^*) + \frac{\partial F}{\partial x_1}(x_1^*, ..., x_n^*) \cdot \Delta x_1 + ... + \frac{\partial F}{\partial x_n}(x_1^*, ..., x_n^*) \cdot \Delta x_n.$$

The expression

$$dF = \frac{\partial F}{\partial x_1}(x_1^*, ..., x_n^*) \cdot dx_1 + ... + \frac{\partial F}{\partial x_n}(x_1^*, ..., x_n^*) \cdot dx_n$$

is called the *total differential*, and it approximates the actual change $\Delta F = F(x_1^* + \Delta x_1, ..., x_n^* + \Delta x_n) - F(x_1^*, ..., x_n^*)$.

Example. Consider the Cobb-Douglas production function $Q = 4K^{\frac{3}{4}}L^{\frac{1}{4}}$. For $K = 10000,\ L = 625$ the output is $Q = 20000$. We want to use marginal analysis to estimate (a) $Q(10010, 625)$, (b) $Q(10000, 623)$, (c) $Q(10010, 623)$.

Step 1. The partial derivatives of Q are:
the *marginal product of capital* is $\frac{\partial Q}{\partial K} = 3K^{-\frac{1}{4}}L^{\frac{1}{4}}$
the *marginal product of labor* is $\frac{\partial Q}{\partial K} = 3K^{\frac{3}{4}}L^{-\frac{3}{4}}$.

Step 2. Calculate these partial derivatives on $(10000, 625)$:
$\frac{\partial Q}{\partial K}(1000, 625) = 1.5, \quad \frac{\partial Q}{\partial L}(1000, 625) = 8.$

Step 3. (a) $Q(10010, 625) = Q(1000, 625) + \frac{\partial Q}{\partial K}(1000, 625) \cdot 10 = 20000 + 1.5 \cdot 10 = 20015.$

(b) $Q(10000, 623) = Q(1000, 625) + \frac{\partial Q}{\partial L}(1000, 625) \cdot (-2) = 20000 + 8 \cdot (-2) = 19984.$

(c) $Q(10000, 623) = Q(1000, 625) + \frac{\partial Q}{\partial K}(1000, 625) \cdot 10 + \frac{\partial Q}{\partial L}(1000, 625) \cdot (-2) = 20000 + 1.5 \cdot 10 + 8 \cdot (-2) = 19999.$

Exercises

1. Compute the partial derivatives of the following functions

$a)\ 4x^2y - 3xy^3 + 6x;\quad b)\ xy;\quad c)\ xy^2;\quad d)\ e^{2x+3y};$
$e)\ \frac{x+y}{x-y};\quad f)\ 3x^2y - 7x\sqrt{y};\quad g)\ (x^2 - y^3)^3;\quad h)\ \sqrt{2x - y^2};$
$i)\ ln(x^2 + y^2);\quad j)\ y^2e^{xy^2};\quad k)\ \frac{x^2-y^2}{x^2+y^2}.$

2. Find an example of a function $f(x, y)$ such that $\frac{\partial f}{\partial x} = 3$ and $\frac{\partial f}{\partial y} = 2$. How many such a functions are there?

6. A firm has the Cobb-Douglas production function $y = 10x_1^{\frac{1}{3}}x_2^{\frac{1}{2}}x_3^{\frac{1}{6}}$. Currently it is using the input bundle $(27, 16, 64)$.

a) How much is producing?

b) Use differentials to approximate its new output when x_i increases to 27.1, x_2 decreases to 15.7, and x_3 remains the same.

c) Compare the answer in part b with the actual output.

d) Do b and c for $\Delta x_1 = \Delta x_2 = 0$ and $\Delta x_3 = -0.4$.

7. Use differentials to approximate each of the following:

$a)\ f(x, y) = x^4 + 2x^2y^2 + xy^4 + 10y\ \ at\ \ x = 10.36\ \ and\ \ y = 1.04;$
$b)\ f(x, y) = 6x^{\frac{2}{3}}y^{\frac{1}{2}}\ \ at\ \ x = 998\ \ and\ \ y = 101.5;$
$c)\ f(x, y) = \sqrt{x^{\frac{1}{2}} + y^{\frac{1}{3}} + 5x^2}\ \ at\ \ x = 4.2\ \ and\ \ y = 1.02.$

8. . Use calculus and no calculator to estimate the output given by the production function $Q = 3K^{\frac{2}{3}}L^{\frac{1}{3}}$ when

a) $K = 1000$ and $L = 125$;

b) $K = 998$ and $L == 128$.

9 . Estimate $\sqrt{(4.1)^3 - (2.95)^3 - (1.02)^3}$.