ISET Math Camp 2012
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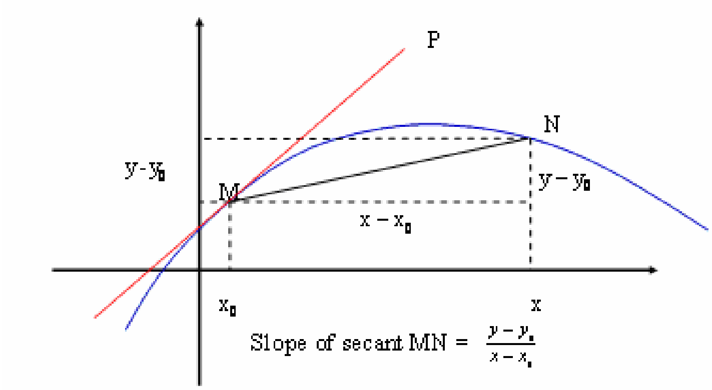
Derivative



Derivative of $y = f(x)$ at x_0 $= f'(x_0) =$ slope of tangent MP $= \lim_{N \to M}$ (slope of secant MN) =

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

Notation

Function	Derivative
$f(x)$	$f'(x)$ or $\frac{df(x)}{dx}$
f	f' or $\frac{df}{dx}$
y	y' or $\frac{dy}{dx}$
$y(x)$	$y'(x)$ or $\frac{dy(x)}{dx}$

Example. For $f(x) = x^2$

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0} \frac{(x_0 + h)^2 - x_0^2}{h} =$$

$$\lim_{h \to 0} \frac{x_0^2 + 2x_0 h + h^2 - x_0^2}{h} = \lim_{h \to 0}(2x_0 + h) = 2x_0.$$

Differentiation Rules

Rule 1 If k is a constant then $\dfrac{d}{dx}k = 0$.

Rule 2 If n is any number then $\dfrac{d}{dx}x^n = nx^{n-1}$.

Examples

If $f(x) = \dfrac{1}{\sqrt[3]{x}} = x^{-\frac{1}{3}}$ then $f'(x) = -\dfrac{1}{3}x^{-\frac{4}{3}}$.

If $y = \dfrac{1}{x\sqrt{x}} = x^{-\frac{3}{2}}$ then $\dfrac{dy}{dx} = -\dfrac{3}{2}x^{-\frac{5}{2}}$.

Example

Find the slope of the tangent to the graph of the function $g(t) = t^4$ at the point on the graph where $t = -2$.

Solution

The derivative is $g'(t) = 4t^3$, and so the slope of the tangent line at $t = -2$ is $g'(-2) = 4 \times (-2)^3 = -32$.

Example

Find the equation of the line tangent to the graph of $y = f(x) = x^{\frac{1}{2}}$ at the point $x = 4$.

Solution

$f(4) = 4^{\frac{1}{2}} = \sqrt{4} = 2$, so the coordinates of the point on the graph are $(4, 2)$. The derivative is

$$f'(x) = \frac{x^{-\frac{1}{2}}}{2} = \frac{1}{2\sqrt{x}}$$

and so the slope of the tangent line at $x = 4$ is $f'(4) = \frac{1}{4}$. We therefore know the slope of the line and we know one point through which the line passes.

Rule 3 If $f(x) = cg(x)$, where c is a constant, then $f'(x) = cg'(x)$.

Rule 4 If $f(x) = g(x) \pm h(x)$ then $f'(x) = g'(x) \pm h'(x)$.

Examples If $f(x) = 3x^2$ then $f'(x) = 3 \times \dfrac{d}{dx} x^2 = 6x$.

If $g(t) = 3t^2 + 2t^{-2}$ then $g'(t) = \dfrac{d}{dt} 3t^2 + \dfrac{d}{dt} 2t^{-2} = 6t - 4t^{-3}$.

If $y = \frac{3}{\sqrt{x}} - 2x\sqrt[3]{x} = 3x^{-\frac{1}{2}} - 2x^{\frac{4}{3}}$ then $\frac{dy}{dx} = -\frac{3}{2}x^{-\frac{3}{2}} - \frac{8}{3}x^{\frac{1}{3}}$.

Rule 5 (The product rule) If $f(x) = u(x)v(x)$ then

$$f'(x) = u(x)v'(x) + u'(x)v(x).$$

Examples

If $y = (x+2)(x^2+3)$ then $y' = (x+2)2x + 1(x^2+3)$.

If $f(x) = \sqrt{x}(x^3 - 3x^2 + 7)$ then $f'(x) = \sqrt{x}(3x^2 - 6x) + \frac{1}{2}x^{-\frac{1}{2}}(x^3 - 3x^2 + 7)$.

If $z = (t^2+3)(\sqrt{t} + t^3)$ then $\frac{dz}{dt} = (t^2+3)(\frac{1}{2}t^{-\frac{1}{2}} + 3t^2) + 2t(\sqrt{t} + t^3)$.

Rule 6 (The quotient rule)

$$\begin{aligned} f(x) &= \frac{u(x)}{v(x)} \\ f'(x) &= \frac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2} \\ &= \frac{vu' - uv'}{} \end{aligned}$$

Examples

If $y = \dfrac{2x^2 + 3x}{x^3 + 1}$, then $\dfrac{dy}{dx} = \dfrac{(x^3+1)(4x+3) - (2x^2+3x)3x^2}{(x^3+1)^2}$.

If $g(t) = \dfrac{t^2 + 3t + 1}{\sqrt{t} + 1}$ then $g'(t) = \dfrac{(\sqrt{t}+1)(2t+3) - (t^2+3t+1)(\frac{1}{2}t^{-\frac{1}{2}})}{(\sqrt{t}+1)^2}$.

Rule 7 (The composite function rule (also known as the chain rule))

If $f(x) = h(g(x))$ then $f'(x) = h'(g(x)) \times g'(x)$.

Example

$$\frac{d(3x^2-5)^3}{dx} = 3(3x^2-5)^2 \times 6x = 18x(3x^2-5)^2.$$

Example

For $y = \sqrt{x^2+1}$ $\qquad y' = \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \times 2x.$

Derivatives of Exponential and Logarithmic Functions

$$\begin{aligned} \frac{d}{dx}e^x &= e^x \\ \frac{d}{dx}(\log_e x) &= \frac{1}{x}. \end{aligned}$$

Example

$$\begin{aligned} \frac{d}{dx}\log_e(x^2+3x+1) &= \frac{d}{dx}(\log_e u) \qquad (\text{where } u = x^2+3x+1) \\ &= \frac{d}{du}(\log_e u) \times \frac{du}{dx} \qquad (\text{by the chain rule}) \\ &= \frac{1}{u} \times \frac{du}{dx} \\ &= \frac{1}{x^2+3x+1} \times \frac{d}{dx}(x^2+3x+1) \\ &= \frac{1}{x^2+3x+1} \times (2x+3) \\ &= \frac{2x+3}{x^2+3x+1}. \end{aligned}$$

Example

$$
\begin{aligned}
\frac{d}{dx}(e^{3x^2}) &= \frac{de^u}{dx} \qquad \text{where } u = 3x^2 \\
&= \frac{de^u}{du} \times \frac{du}{dx} \qquad \text{by the chain rule} \\
&= e^u \times \frac{du}{dx} \\
&= e^{3x^2} \times \frac{d}{dx}(3x^2) \\
&= 6xe^{3x^2}.
\end{aligned}
$$

Example

$$
\begin{aligned}
\frac{d}{dx}(e^{x^3+2x}) &= \frac{de^u}{dx} \qquad (\text{where } u = x^3 + 2x) \\
&= e^u \times \frac{du}{dx} \qquad (\text{by the chain rule}) \\
&= e^{x^3+2x} \times \frac{d}{dx}(x^3 + 2x) \\
&= (3x^2 + 2) \times e^{x^3+2x}.
\end{aligned}
$$

Example

$$\frac{d}{dx}\ln(2x^3 + 5x^2 - 3)$$

$$
\begin{aligned}
&= \frac{1}{2x^3 + 5x^2 - 3} \times \frac{d}{dx}(2x^3 + 5x^2 - 3) \\
&= \frac{1}{2x^3 + 5x^2 - 3} \times (6x^2 + 10x) \\
&= \frac{6x^2 + 10x}{2x^3 + 5x^2 - 3}.
\end{aligned}
$$

Chain Rule

$$\frac{d}{dx}(e^{f(x)}) = f'(x)e^{f(x)}$$

$$\frac{d}{dx}(\ln f(x)) = \frac{f'(x)}{f(x)}.$$

$(e^x)^` = e^x$, but what is $(a^x)^`$?

$$(a^x)' = ((e^{\ln a})^x)' = (e^{\ln a \cdot x})' = e^{\ln a \cdot x} \cdot (\ln a \cdot x)' = e^{\ln a \cdot x} \cdot \ln a = a^x \cdot \ln a.$$

$(\ln x)` = 1/x$, but what is $(\log_a x)`$?

$$(\log_a x)' = (\frac{\log_e x}{\log_e a})' = (\frac{1}{\ln a} \cdot \ln x)' = \frac{1}{\ln a} \cdot (\ln x)' = \frac{1}{x \cdot \ln a}.$$

Just for fun

$$\frac{d}{dx}\sin x = \cos x$$
$$\frac{d}{dx}\cos x = -\sin x.$$

Implicit Function

Sometimes a function y(x) is hidden in an equation F(x,y) = 0:
the equation 2y + 6x – 8 = 0 determines the function y = -3x +4.

But often it is impossible to solve y from the equation, for example for
$x^2 + y^2 = 1$.
Nevertheless sometimes, it is possible to calculate the derivative of this implicit function.

Implicit Function

Example. Find the derivative of the function $y(x)$ defined by the equation $y^3 - x^2y + 6 = 0$.

Solution. Let us differentiate both sides of the given equation with respect to x, using the chain rule:

$$\frac{d}{dx}\left(y^3 - x^2y + 6\right) = \frac{d}{dx}(0)$$

$$3y^2\frac{dy}{dx} - x^2\frac{dy}{dx} - 2xy = 0$$

solving for dy/dx we obtain

$$\frac{dy}{dx}\left(3y^2 - x^2\right) = 2xy$$

$$\frac{dy}{dx} = \frac{2xy}{3y^2 - x^2}$$

Suppose $y = f(x)$ and $z = g(x)$. Sometimes one needs the derivative

$$\frac{dy}{dz}$$

Example.

If $u = (x^2+9)^{1/2}$ and $v = 3x^2-2x$, then what is du/dv as a function of x?

Solution.

$$\frac{du}{dx} = \frac{1}{2}\left(x^2+9\right)^{-\frac{1}{2}}(2x) = \frac{x}{\sqrt{\left(x^2+9\right)}} \quad \Rightarrow \quad du = \frac{x}{\sqrt{\left(x^2+9\right)}}dx$$

$$\frac{dv}{dx} = 6x - 2 \quad \Rightarrow \quad dv = 2(3x-1)dx$$

thus

$$\frac{du}{dv} = \frac{\frac{x}{\sqrt{\left(x^2+9\right)}}dx}{2(3x-1)dx} = \frac{x}{2(3x-1)\sqrt{\left(x^2+9\right)}}$$

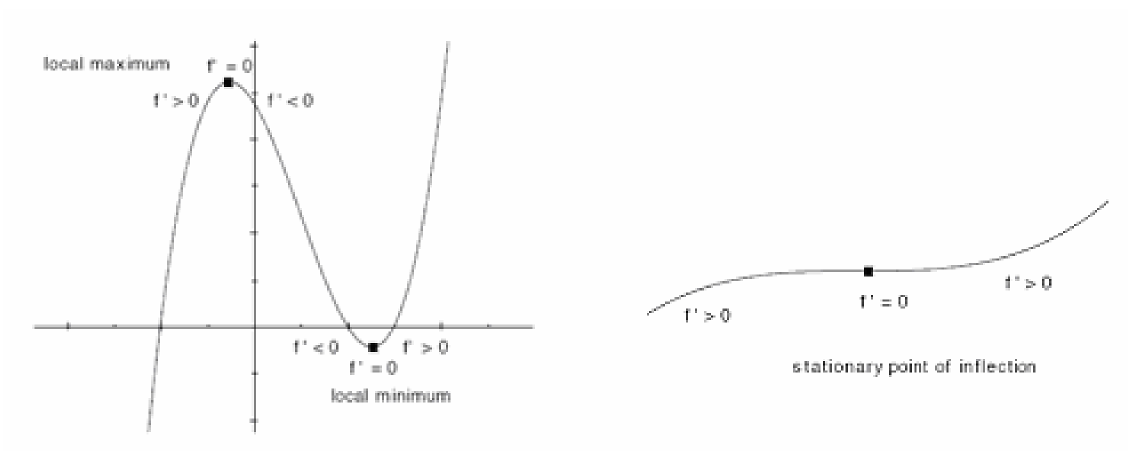
Optimization

If $f'(x_0) > 0$ the function increases in some neighborhood $(x_0 - \varepsilon, x_0 + \varepsilon)$ of x_0.

If $f'(x_0) < 0$ the function decreases in some neighborhood $(x_0 - \varepsilon, x_0 + \varepsilon)$ of x_0.

Definition. A stationary point for a function $y = f(x)$ is a point where the derivative $f'(x)$ equals to zero (tangent has zero slope = tangent is horizontal).

Definition. A critical point for a function $y = f(x)$ is a point where the derivative $f'(x)$ equals to zero or does not exist.
Types of stationary points



Example Find the stationary points of the function $f(x) = 2x^3 + 3x^2 - 12x + 17$.

Solution $f'(x) = 6x^2 + 6x - 12$. Setting $f'(x) = 0$ and solving we obtain

$$\begin{aligned} 6x^2 + 6x - 12 &= 0 \\ x^2 + x - 2 &= 0 \\ (x-1)(x+2) &= 0 \\ x &= 1, -2. \end{aligned}$$

This gives us the values of x for which the function f is stationary. The corresponding values of the function are found by substituting 1 and -2 into the function.

They are $f(1) = 2 \times 1^3 + 3 \times 1^2 - 12 \times 1 + 17 = 10$ and

$f(-2) = 2 \times (-2)^3 + 3 \times (-2)^2 - 12 \times (-2) + 17 = 37$. The stationary points are therefore $(1, 10)$ and $(-2, 37)$.

Example Find the stationary points of the function $g(t) = e^{t^2}$.

Solution Differentiating and setting the derivative equal to zero we obtain the equation $g'(t) = 2te^{t^2} = 0$. Since e^{t^2} is never zero, the only solution to this equation is where $2t = 0$, ie $t = 0$. Substituting into the formula for g we obtain the function value $g(0) = e^{0^2} = 1$. Thus the stationary point is $(0, 1)$.

First Derivative Test

Example. Locate minimums and maximums of $f(x) = x^3 - 6x^2 + 9x + 1$.

Solution

1. Derivative $f'(x) = 3x^2 - 12x + 9$.
2. Critical point points $f'(x) = 0,\quad 3x^2 - 12x + 9 = 0,\quad x^2 - 4x + 3 = 0,\quad x_1 = 1,\ x_2 = 3.$
3. Sign chart

First some calculations:
$f(1) = 5,\quad f(3) = 1$.

Take any test point from the interval $(-\infty, 1)$, say $x = 0$, then $f'(0) = 9 > 0$, so put + and ↗ in corresponding boxes.

Take any test point from the interval $(1, 3)$, say $x = 2$, then $f'(2) = 3 < 0$, so put - and ↘ in corresponding boxes.

Take any test point from the interval $(3, +\infty)$, say $x = 4$, then $f'(4) = 9 > 0$, so put + and ↗ in corresponding boxes.

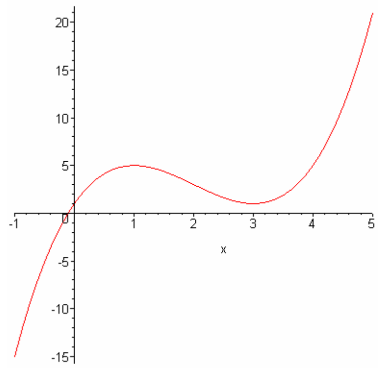
Finally we get the chart

x	$(-\infty, 1)$	1	$(1, 3)$	3	$(3, +\infty)$
y'	+	0	-	0	+
y	↗	5	↘	1	↗
		max		min	

5. y-intercept: substitute in $y = f(x)$ the value $x = 0$, then $y = f(0) = 1$.

6 x-intercept: substitute in $y = f(x)$ the value $y = 0$, and solve $0 = f(x)$: $x^3 - 6x^2 + 9x + 1 = 0$, well, not easy, using maple you obtain $x = -0.104$

Now you are ready to plot the graph



The Location of Maxima and Minima

1. At the endpoints (if they exist) of the region under consideration.
2. Inside the region at a stationary point.
3. Inside the region at a point where the derivative does not exist.

Procedure for finding the maximum or minimum values of a function.

1. Find the endpoints of the region under consideration (if there are any).
2. Find all the stationary points in the region.
3. Find all points in the region where the derivative does not exist.
4. Substitute each of these into the function and see which gives the greatest (or smallest) function value.

Example Find the maximum and minimum values of the function $g(t) = \frac{1}{3}t^3 - t + 2$ for $0 \leq t \leq 3$.

Solution The endpoints are $t = 0$ and $t = 3$. Differentiating and equating to zero we get $g'(t) = t^2 - 1 = (t-1)(t+1) = 0$ so the stationary points are at $t = -1, 1$. Since -1 is not in the region, the possible locations of the maximum and the minimum are $t = 0, 1, 3$. Substituting into g we obtain $g(0) = 2$, $g(1) = \frac{4}{3}$ and $g(3) = 8$. The maximum is therefore $g(3) = 8$ and the minimum is $g(1) = \frac{4}{3}$.

Example Find the minimum value and the maximum value of the function $f(x) = x^2 e^x$ for $-4 \le x \le 1$.

Solution We will follow the procedure outlined above. The endpoints are -4 and 1. Differentiating we obtain $f'(x) = x^2e^x + 2xe^x = x(x+2)e^x$. Setting $f'(x) = 0$ and solving we get stationary points at $x = 0$ and $x = -2$. There are no points where the derivative does not exist. Therefore the maximum and minimum values will be found at one of the points $x = -4, -2, 0, 1$. Substituting we obtain $f(-4) \approx 0.29$, $f(-2) \approx 0.54$, $f(0) = 0$ and $f(1) = e \approx 2.7$. therefore the maximum value occurs at $x = 1$ and is equal to e, and the minimum value occurs at $x = 0$ and is 0.

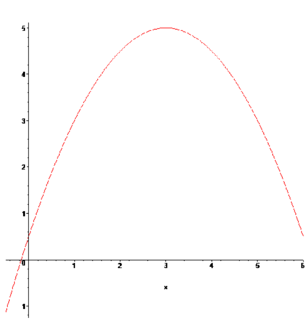
Example A farmer is to make a rectangular paddock. The farmer has 100 metres of fencing and wants to make the rectangle that will enclose the greatest area. What dimensions should the rectangle be?

Solution There are many rectangular paddocks that can be made with 100 metres of fencing. If we call one side of the rectangle x, then because the perimeter is 100, the other side of the rectangle is $50 - x$. The area of the paddock is then $A(x) = x(50 - x)$. We must maximise the function $A(x)$ for $0 \le x \le 50$ (since the sides of the rectangle cannot

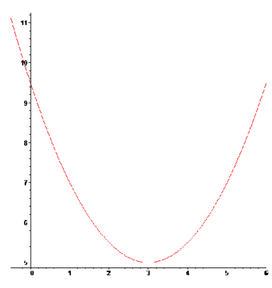
Assume that $f(x)$ is differentiable in a neighborhood of x_0 and it has a stationary point at x_0, i.e. $f'(x_0) = 0$.

The First-Derivative Test for Local Extremum

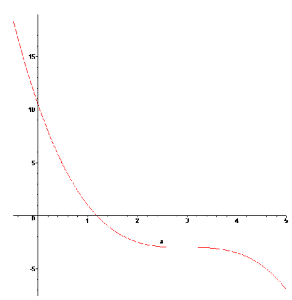
If at a stationary point x_0 the first derivative of a function f	*a) changes its sign from positive to negative,* *b) changes its sign from negative to positive,* *c) does not change its sign,*	*then the value of the function at x_0, $f(x_0)$, will be*	*a) a local maximum;* *b) a local minimum;* *c) neither a local maximum nor a local minimum.*



case a



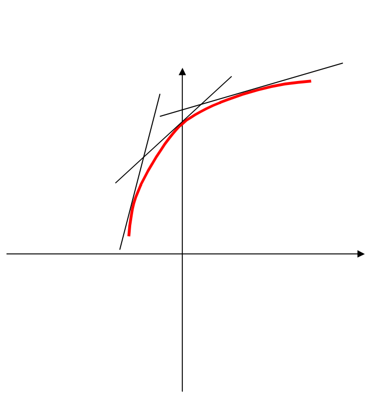
case b

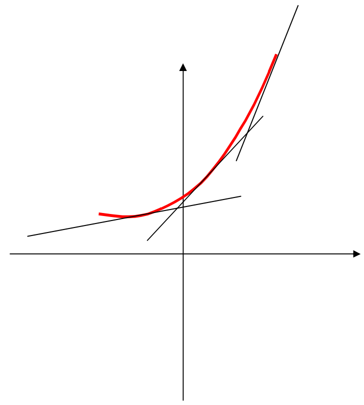


case c

Recall the meaning of the first and the second derivatives of a function f. The sign of the first derivative tells us whether the *value* of the function increases ($f' > 0$) or decreases ($f' < 0$), whereas the sign of the second derivative tells us whether the *slope* of the function increases ($f'' > 0$) or decreases ($f'' < 0$). This gives us an insight into how to verify that at a stationary point we have a maximum or minimum.

Concave and convex functions





Concave $f''(x) < 0$

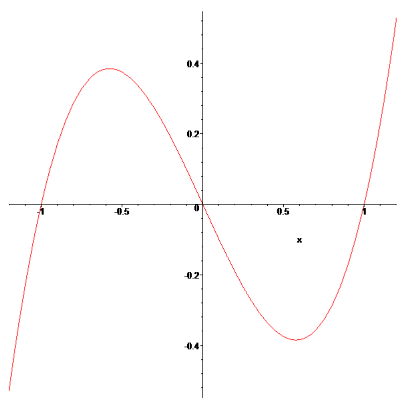
slope of tangent decreases

Convex $f''(x) > 0$

slope of tangent increases

Concave and convex functions

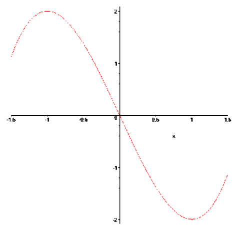
The function $f(x) = x^3 - x$ is concave on $(-\infty,0)$ and is convex on $(0,+\infty)$: $f'(x) = 3x^2$, $f''(x) = 6x$. $x = 0$ is an inflection point.



Optimization

Second-Derivative Test for Local Extremum

A stationary point x_0 of $f(x)$ will be a local maximum if $f''(x_0) < 0$ and a local minimum if $f''(x_0) > 0$.



Example

$f(x) = x^3 - 3x$, $f'(x) = 3x^2 - 3x$, $f''(x) = 6x$

Stationary points $f'(x) = 0$, $3x^2 - 3 = 0$, $x^2 - 1 = 0$, $x_1 = -1$, $x_2 = 1$

Second derivative test:

$f''(-1) = -6 < 0$, so $x_1 = -1$ is a local max,

$f''(1) = 6 > 0$, so $x_2 = 1$ is a local min

However, it may happen that $f''(x_0) = 0$, therefore the second-derivative test is not applicable. To compensate for this, we can extend the latter result and to apply the following general test:

If $f'(x_0) = f''(x_0) = \ldots = f^{(n-1)}(x_0) = 0$, $f^{(n)}(x_0) \neq 0$ and $f^{(n)}$ is continuous at x_0 then at point x_0 $f(x)$ has

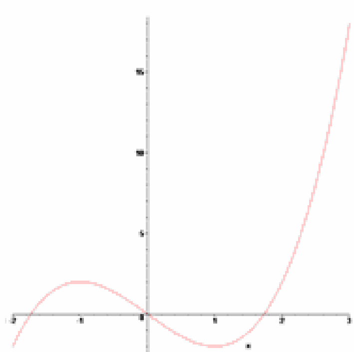
a) an inflection point if n is odd;

b) a local maximum if n is even and $f^{(n)}(x_0) < 0$;

c) a local minimum if n is even and $f^{(n)}(x_0) > 0$.

Example. *Find maximum of $f(x) = x^3 - 3x$ in $[-2, 3]$.*

The first-derivative test gives two stationary points: $f'(x) = 3x^2 - 3 = 0$ at $x = 1$ and $x = -1$. The second-derivative test guarantees that $x = -1$ is a local maximum. However, $f(-1) = 2 < f(3) = 18$. Therefore the global *maximum of f in $[-2, 3]$ is reached at the border point $x = 2$.*



How to sketch the graph of a function

1. Determine the domain of f.
2. Find the x and y intercepts.
3. End behavior ($\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$)
4. Find the stationary points $f'(x) = 0$.
5. Determine the intervals where f(x) is increasing or decreasing.
6. Find local minimums and maximums.
7. Determine the intervals of concavity and convexity, and inflections points.
8. Sketch the graph.

Graphing Strategy

Step 1. Analyze $f(x)$.
(A) Find the domain of f.
(B) Find intercepts.

Step 2. Analyze $f'(X)$.
(A) Find all critical points for $f(x)$.
(B) Construct a sign chart for $f'(x)$.
(C) Determine the intervals where f is increasing and decreasing.
(D) Find local minima and maxima.

Step 3. Analyze $f''(x)$.
(A) Find all critical points for $f'(x)$.
(B) Construct a sign chart for $f''(x)$.
(C) Determine where the graph of f is concave up and concave down.

Step 4. Sketch the graph.

Example. $f(x) = x^3 - 3x$.

Domain: $(-\infty,\infty)$.

y-intercept $f(0) = 0$. x-intercepts $x^3 - 3x = 0$, $x = -\sqrt{3}$, $x = 0$, $x = \sqrt{3}$.

$f'(x) = 3x^2 - 3$. Critical (stationery) points

$3x^2 - 3x = 0$, $x = -1$, $x = 1$.

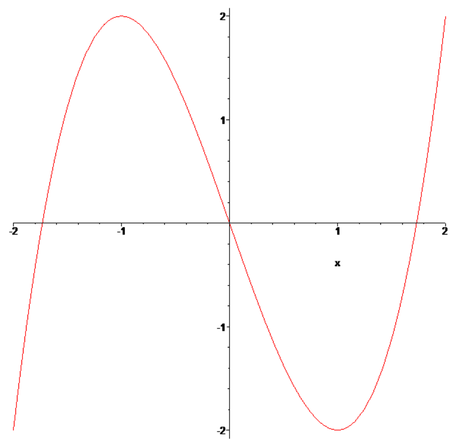
Chart

x	(-∞,-1)	-1	(-1,1)	1	(1, ∞)
f '(x)	+	0	-	0	+
f(x)	↑	2	↓	-2	↑
		min		max	

$f''(x) = 6x$.

Inflection points: $6x = 0$, $x = 0$.

x	(-∞,0)	0	(0,∞)
f "(x)	-	0	+
f(x)	conc	inf	conv



Taylor series: If f is continuously differentiable

it can be expanded around a point x_0, i.e. this function can be transformed

into a polynomial form in which the coefficient of the various terms are expressed in terms of the derivatives values of f, all evaluated at the point of expansion x_0. More precisely,

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(x_0)}{i!}(x - x_0)^i, \quad \text{where } k! = 1 \cdot 2 \cdot \ldots \cdot k,\ 0! = 1,\ f^{(0)}(x_0) = f(x_0).$$

The Maclaurin series is the special case of the Taylor series when we set $x_0 = 0$.

Example. Expand $f(x) = e^x$ around $x = 0$.

$$\textit{Since } \frac{d^n}{dx^n}e^x = e^x \textit{ for all } n \textit{ and } e^0 = 1,\ e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{i=0}^{\infty} \frac{x^i}{i!}.$$