ISET Math Camp 13

Tornike Kadeishvili

Functions and Graphs

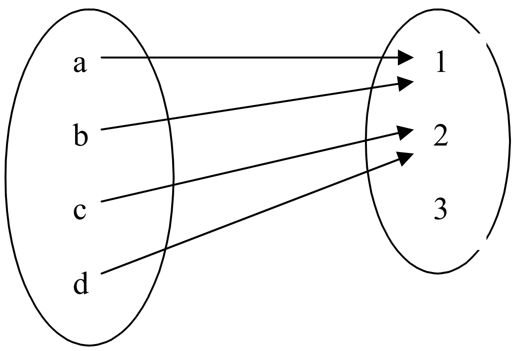
Definition of a Function

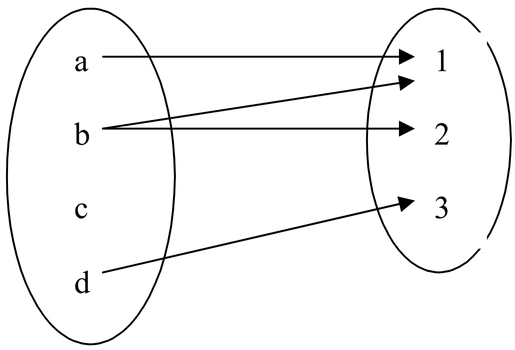
A function $f: X \to Y$ from X to Y consist of

(a) a set X called domain;
(b) a set Y called codomain or target;
(c) a rule which assigns to each element $x \in X$ exactly one element $y \in Y$.

Examples

1.





This is a function. This is not.

2. The correspondence $f: \{1,2,3,4\} \to \{1,2,3\}$ given by table

x	y
1	1
2	1
3	2
4	2

is a function,

x	y
1	1
2	1,2
3	
4	2

is not.

3. The correspondence $f: People \to People$ given by $f(x) = x's$ *brother* is not a function, but $f(x) = x's$ *mother* is.

4. The correspondence $f: R \to R$ given by $f(x) = \pm\sqrt{x}$ is not a function, but $f(x) = x^2$ is.

Graph of a Function

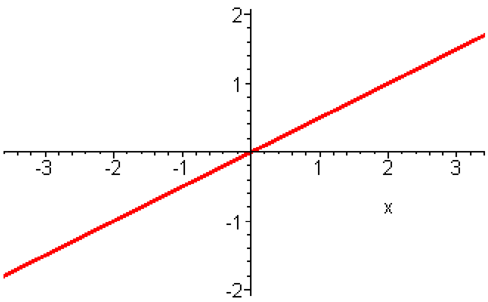
The graph of a function $f : R \to R$ is defined as a subset of the plane $\Gamma(f) \subset R^2$ given by

$$\Gamma(f) = \{(x, y) \in R^2,\ y = f(x)\}.$$

Examples

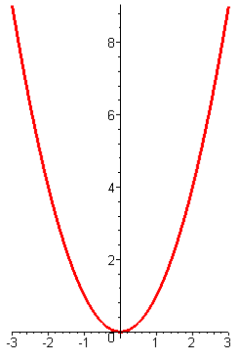
1. Suppose $f : R \to R$ is given by $f(x) = 0.5x$. Construct a table, indicate the points and join them

x	y
-3	-1.5
-2	-1
-1	-0.5
0	0
1	0.5
2	1
3	1.5



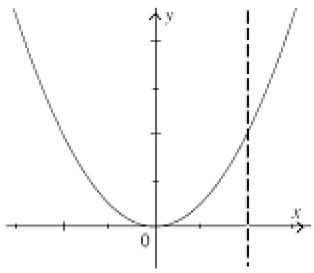
2. Suppose $f : R \to R$ is given by $f(x) = x^2$. Construct a table, indicate the points and join them

x	y
-3	-9
-2	-4
-1	-1
0	0
1	1
2	4
3	9

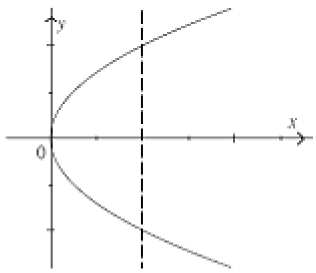


Vertical Line Test

Not all curves in the plane R^2 can be graphs of some function. Only those which pass the following Vertical Line Test: Any vertical line cuts the curve at most in 1 point.



This curve passes the test.



This fails.

Domain of a Function

Sometimes only the rule $y = f(x)$ is given. Then the domain of f consists of these x for which $y = f(x)$ is defined.

Examples

1. For $f(x) = \sqrt{x}$ the domain is $[0, +\infty) = \{x \geq 0\}$.

2. For $f(x) = \frac{1}{x}$ the domain is $(-\infty, 0) \bigcup (0, +\infty) = R \setminus 0$.

Range

For a function $f : X \to Y$ the range (or image) is the set of all $y \in Y$ such that $y = f(x)$ for some $x \in X$. In other words

$$\text{Im}(f) = \{y \in Y,\ \exists x \in X \ s.t.\ y = f(x)\}.$$

Examples

1. For the function $f : R \to R$ given by $f(x) = x^2$ the image is $[0, +\infty)$.

2. For the function $f(x) = \frac{1}{x-2}$ the domain is the set $(-\infty, 2) \bigcup (2, +\infty) = R \setminus \{2\}$ and the range is $(-\infty, 0) \bigcup (0, +\infty) = R \setminus \{0\}$.

Composition of Functions

Suppose $f : X \to Y$ and $g : Y \to Z$. Then the composition $g \circ f : X \to Z$ is defined as

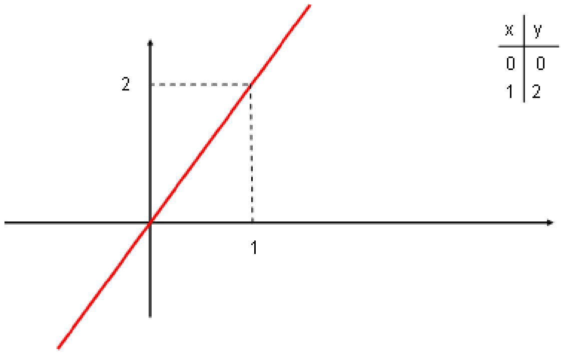
$$g \circ f(x) = g(f(x)).$$

Example

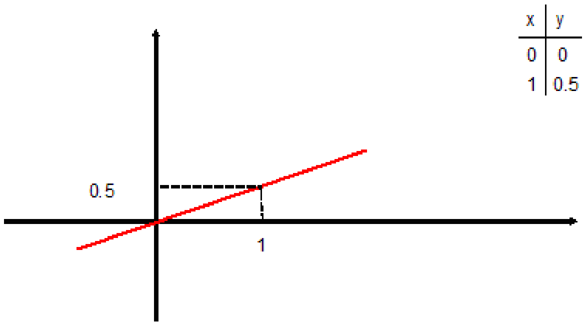
If $f(x) = x^2$ and $f(x) = x + 3$ then $g \circ f(x) = x^2 + 3$ and $f \circ g(x) = (x+3)^2$.

Linear Function

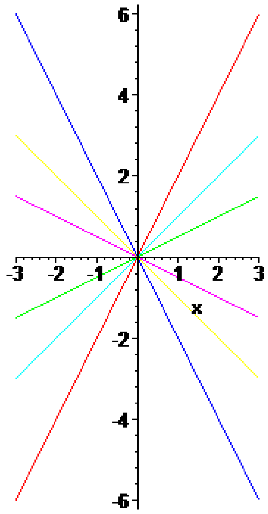
$y = 2x$



$y = \frac{1}{2}x$



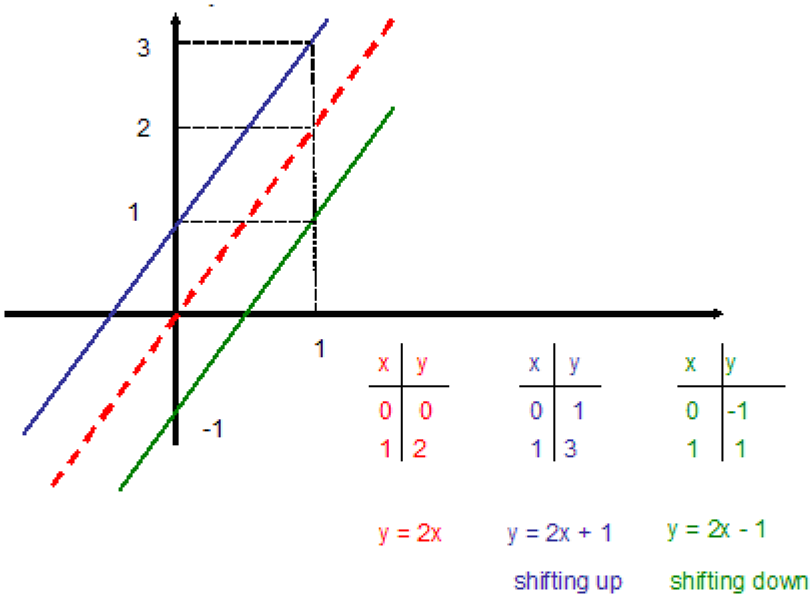

```
> plot({x,2*x,0.5*x,-x,-2*x,-0.5*x},x=-3..3);
```



Conclusion

The graph of linear function $y = k \cdot x$ is a straight line which passes trough the origin, the slope depends on the coefficient k : (a) if $k > 0$ passes trough I and III quadrant, (b) if $k < 0$ passes trough II and IV quadrants, (c) if $k = 0$ coincides with x –axes.

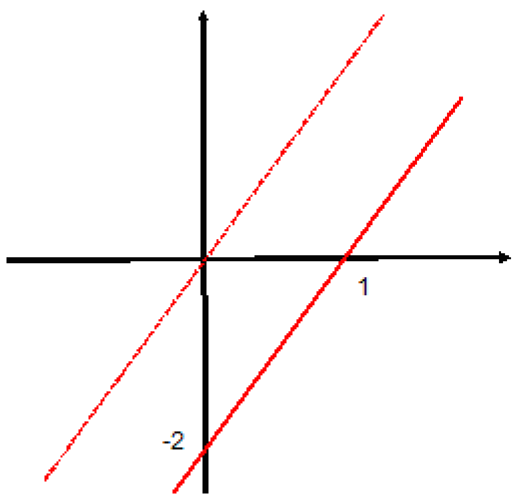
Affine Function $f(x) = kx + b$



The graph of $kx + b$ can be obtained from the graph of kx by sifting by $|b|$ units up if $b > 0$ and down if $b < 0$.

Intercepts

Where the graph of $kx + b$ intersects x and y axes?



f(x) = 2x - 2

y intercept: f(0) = -2

x intercept: f(x) = 0

2x – 2 = 0, 2x = 2, x = 1

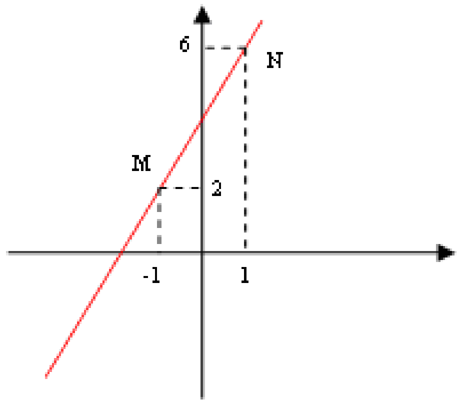
Conclusion

The y intercepts can be fount by substitution $x = 0$, thus it is $y = k \cdot 0 + b = b$.

The x intercept can be found by substitution $y = 0$, i.e. from $0 = kx + b$, thus it is $x = -\frac{b}{k}$.

Equation of a Line

Write the equation of the line which passes trough the points M(-1,2) and N(1,6).



$y = kx + b,\ k = ?,\ \ b = ?$

$x = -1 \Longrightarrow y = 2,\ 2 = k\cdot(-1) + b$

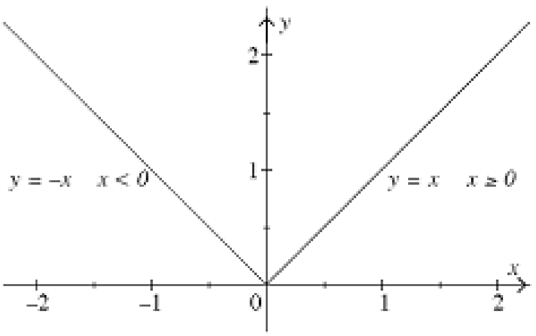
$x = 1 \Longrightarrow y = 6,\ 6 = k\cdot 1 + b$

$$\begin{cases} 2 = -k + b \\ 6 = k + b \end{cases}$$

Solution gives $k = 2$ and $b = 4$.
Thus the equation of this line is $y = 2x + 4$

Absolute Value Function

$$y = |x| = \begin{cases} +x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

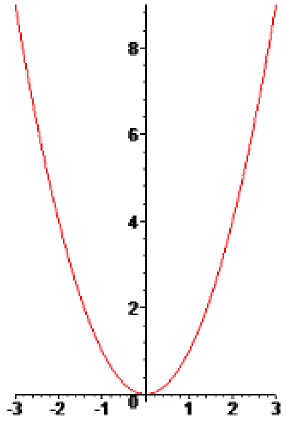


The graph of $y = |x|$.

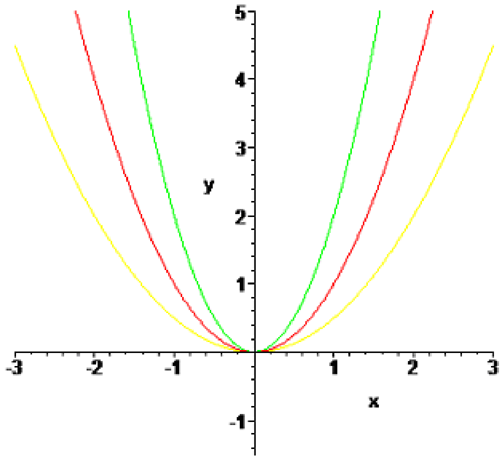
Quadratic Functions

$y = x^2$

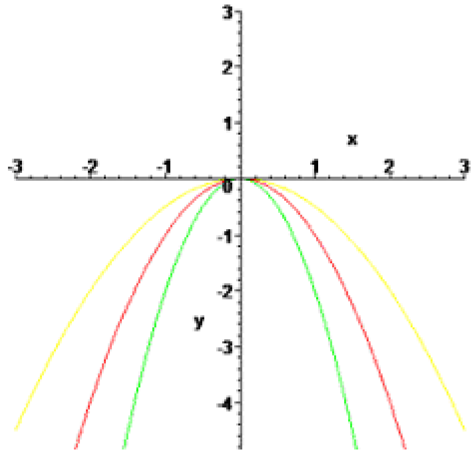
x	y
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9



Quadratic functions $y = x^2$, $y = 2x^2$, $y = \frac{1}{2}x^2$. Who is who?



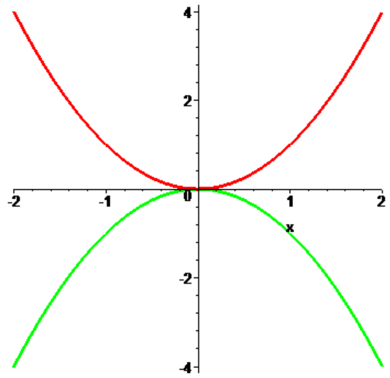
Quadratic functions $y = -x^2$, $y = -2x^2$, $y = -\frac{1}{2}x^2$. Who is who?



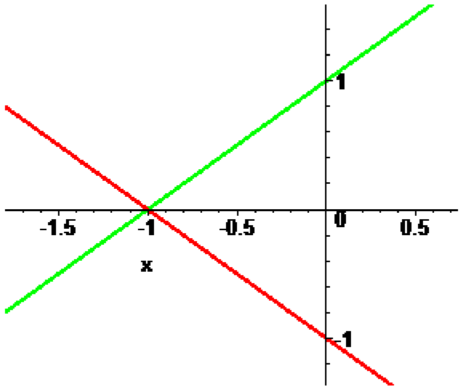
Reflection About x – axes

How the graphs of functions $y = f(x)$ and $y = -f(x)$ are related?

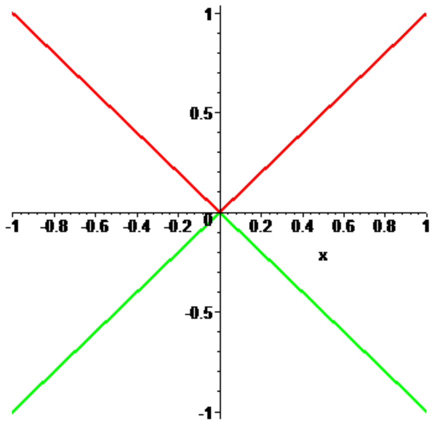
1. $y = x^2$ and $y = -x^2$



2. $y = x+1$ and $y = -x-1$



3. $y = |x|$ and $y = -|x|$



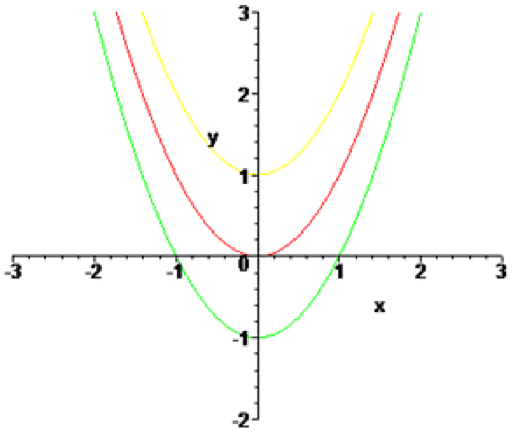
Conclusion

The graph of $y = -f(x)$ can be obtained from the graph of $y = f(x)$ by reflection about the x-axes, i.e. these graphs are symmetric with respect to x-axes.

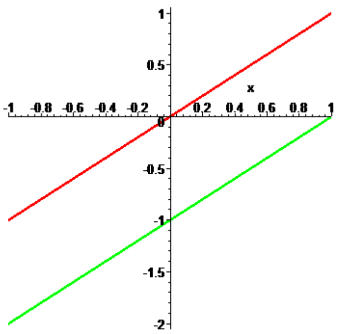
Vertical Shift

How the graphs of functions $y = f(x)$ and $y = f(x) + b$ are related?

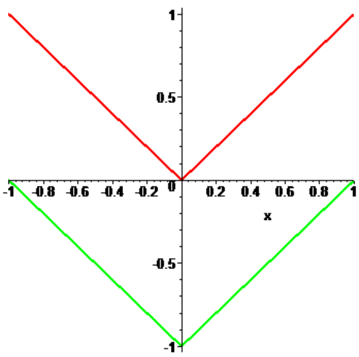
1. $y = x^2$, $y = x^2 + 1$, $y = x^2 - 1$



2. $y = x$ and $y = x - 1$



3. $y = |x|$ and $y = |x - 1|$



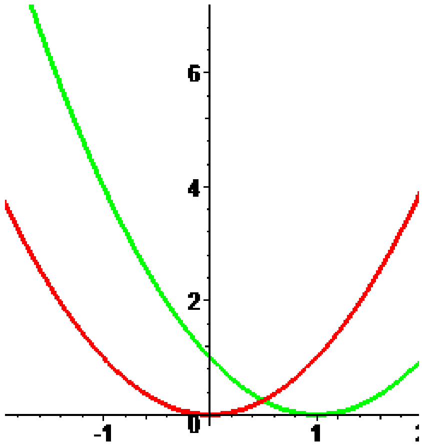
Conclusion

The graph of $y = f(x) + b$ can be obtained from the graph of $y = f(x)$ by shifting by $|b|$ units up if $b > 0$ and down if $b < 0$.

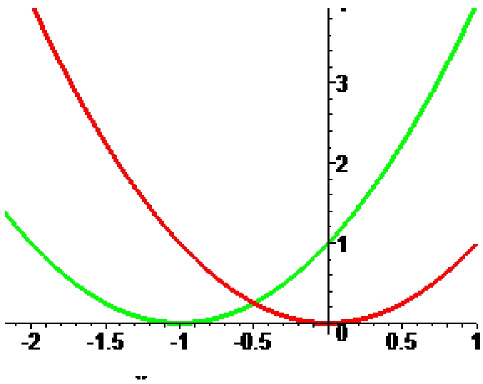
Horizontal shift

How the graphs of functions $y = f(x)$ and $y = f(x+a)$ are related?

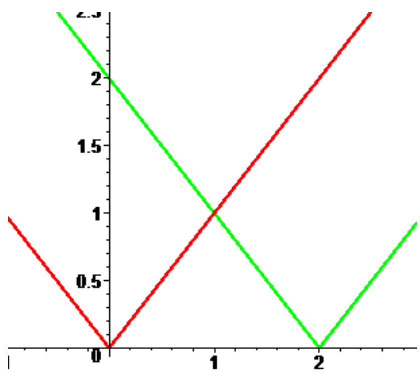
1. $y = x^2$, $y = (x-1)^2$



2. $y = x^2$, $y = (x+1)^2$



3. $y = |x|$, $y = |x-2|$



Conclusion

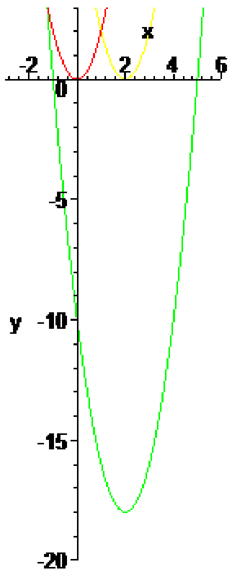
The graph of $y = f(x+a)$ can be obtained from the graph of $y = f(x)$ by shifting by $|a|$ units left if $a > 0$ and right if $a < 0$.

General quadratic function $y = ax^2 + bx + c$

$y = 2x^2 - 8x - 10 = 2(x^2 - 4x) - 10 = 2(x^2 - 2\cdot x\cdot 2 + 2^2 - 4) - 10 = 2[(x-2)^2 - 4] - 10 =$

$2(x-2)^2 - 8 - 10 = 2(x-2)^2 - 18$

Shift $y = 2x^2$ right by 2 and down by 14:



Intercepts

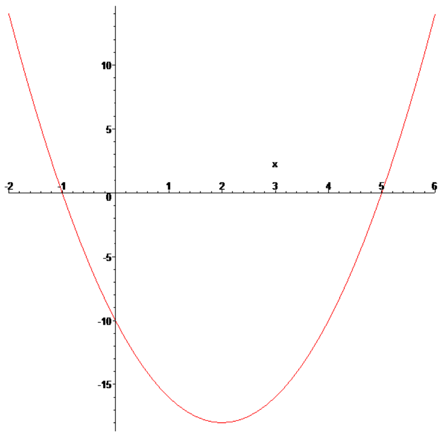
For y intercept find f(0).

For x intercepts solve f(x) = 0.

Example $f(x) = 2x^2 - 8x - 10$

y intercept $f(0) = -6$.

x intercept $2x^2 - 8x - 6 = 0$, $x_1 = -1$, $x_2 = 5$.



Coordinates of the pole

$x_0 = (x_1+x_2)/2 = 2$, $y_0 = f(x_0) = -18$.

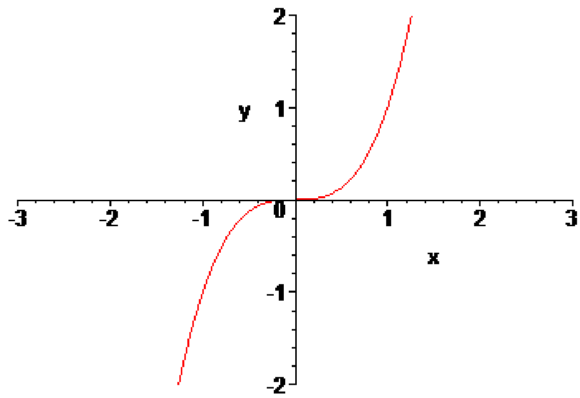
Genarally for $y = ax^2 + bx + c$ the coordinates of the pole are

$x_0 = -b/2a$, $y_0 = (-b^2 + 4ac)/4a$.

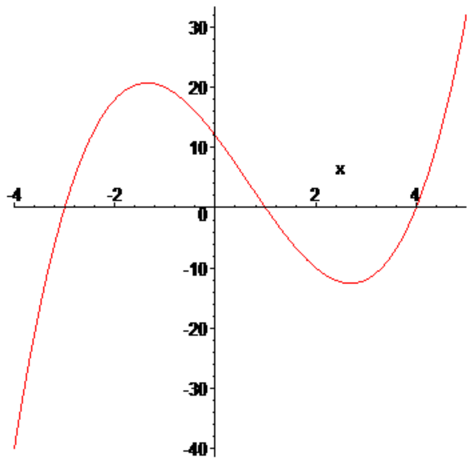
> **plot(2*x^2-8*x-10,x=-2..6);**

Cubical function $y = x^3$

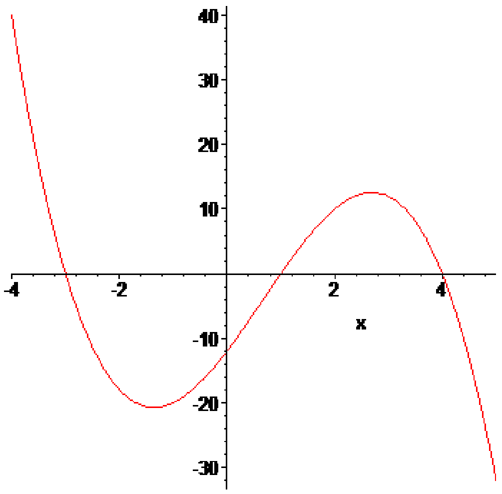
When $x \to -\infty$ then $y \to -\infty$, when $x \to +\infty$ then $y \to +\infty$



Cubical function $y = x^3 - 2x^2 - 11x + 12 = (x+3)(x-1)(x-4)$

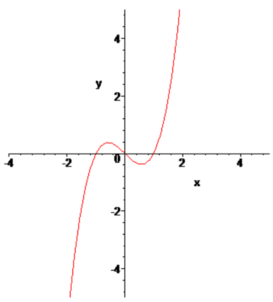


Cubical function $y = -x^3 + 2x^2 + 11x - 12 = -(x+3)(x-1)(x-4)$

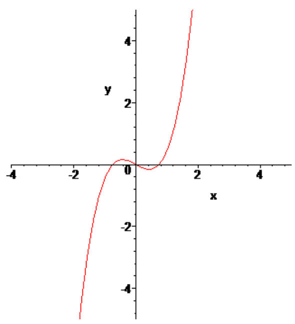


Cubical function $y = (x+a)\cdot x\cdot(x-a)$

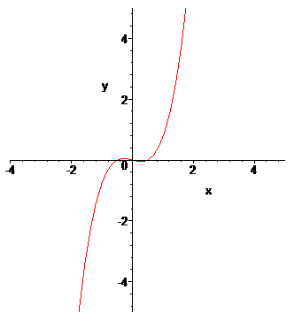
a = 1



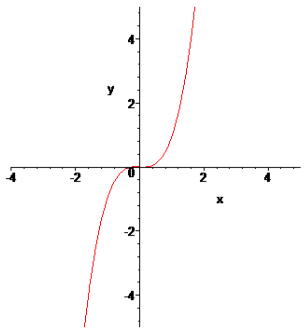
a = 0.75



a = 0.5

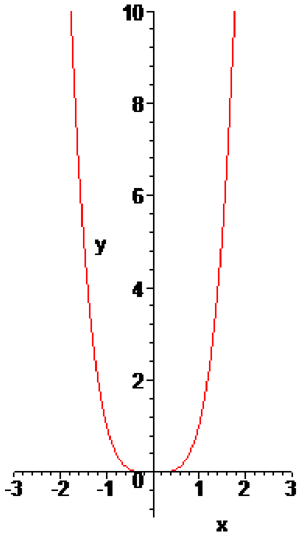


a = 0.25



Monomial of degree 4 $y = x^4$

When $x \to -\infty$ then $y \to +\infty$ when $x \to +\infty$ then $y \to +\infty$

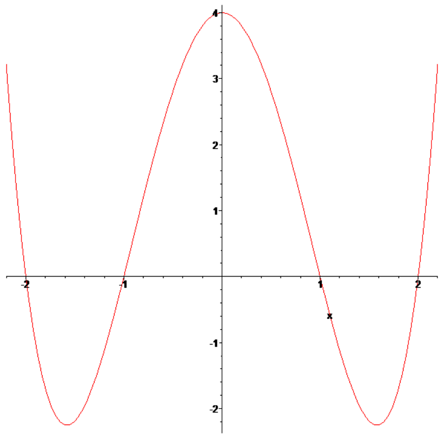


Polynomial of degree 4

$y = (x + 2)(x + 1)(x - 1)(x - 2) = x^4 - 5x^2 + 4$

When $x \to -\infty$ then $y \to +\infty$ when $x \to +\infty$ then $y \to +\infty$

4 roots (4 x intercepts) $x_1 = -2$, $x_2 = -1$, $x_3 = 1$, $x_4 = 2$

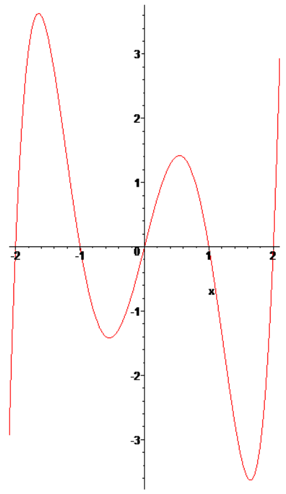


Polynomial of degree 5

$y = (x + 2)(x + 1)x(x - 1)(x - 2) = x^5 - 5x^3 + 4x$

When $x \to -\infty$ then $y \to -\infty$ when $x \to +\infty$ then $y \to +\infty$

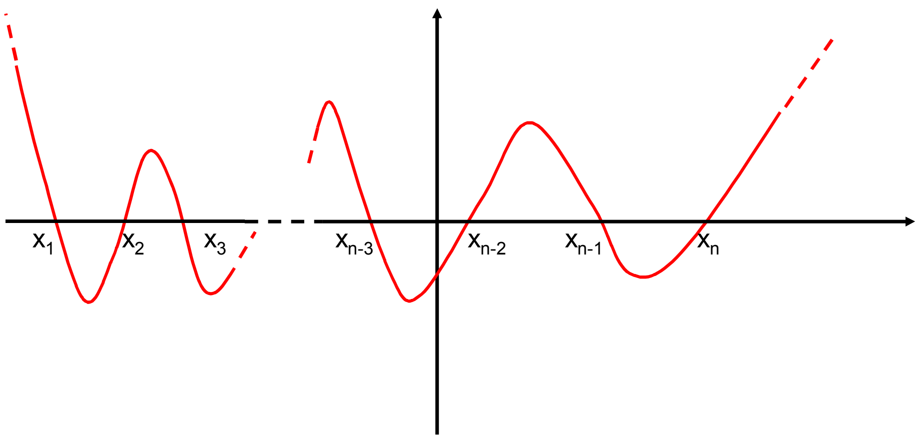
4 roots (4 x intercepts) $x_1 = -2$, $x_2 = -1$, $x3 = 0$, $x_4 = 1$, $x_5 = 2$



General polynomial

$y = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-2}x^2 + a_{n-1}x + a_n = \sum_{k = 0,1,\dots,n} a_kx^{n-k}$

Assume $a_0 > 0$ and n is even ($n = 2k$), then:
when $x \to -\infty$ then $y \to +\infty$ when $x \to +\infty$ then $y \to +\infty$
And generally n roots (n x-intercepts) $x_1, x_2, \dots, x_n$

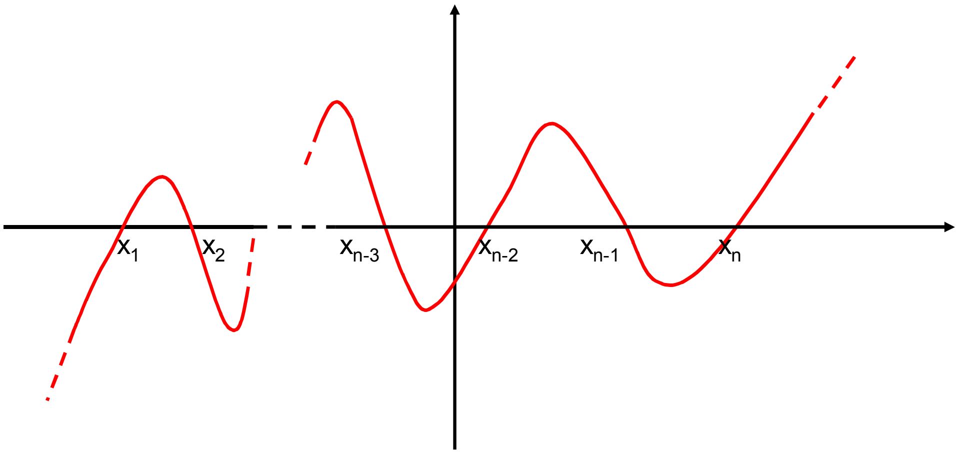


General polynomial

$y = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-2}x^2 + a_{n-1}x + a_n =$

$$\sum_{k = 0,1,\dots,n} a_k x^{n-k}$$

Assume $a_0 > 0$ and n is odd $(n = 2k + 1)$, then:
when $x \to -\infty$ then $y \to -\infty$ when $x \to +\infty$ then $y \to +\infty$
And generally n roots (n x-intercepts) $x_1, x_2, \dots, x_n$



Example

The graph of the polynomial $y = f(x)$ is given below. It has a local maximum and minimum as marked. Use the graph to answer the following questions.

a. State the roots of $f(x) = 0$. $x = -2, 0, 2$

b. What is the value of the repeated root. $x = 2$

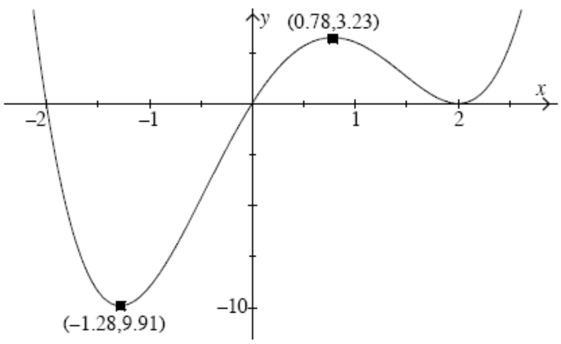
c. For what values of k does the equation $f(x) = k$ have exactly 3 solutions. $k = 0, 3.23$

d. Solve the inequality $f(x) < 0$. $-2 < x < 0$

e. What is the *least* possible degree of $f(x)$? 4

f. State the value of the constant of $f(x)$. 0

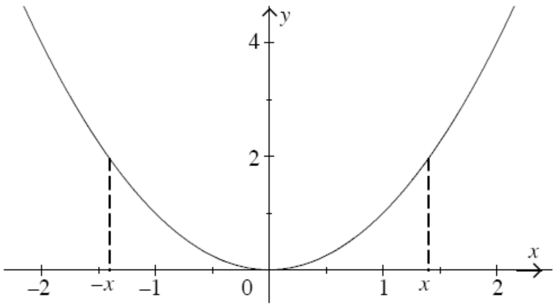
g. For what values of k is $f(x) + k \geq 0$ for all real x. $k \geq 9.91$



Even and odd functions

A function y = f(x) is even if f(-x) = f(x) for all x in the domain of f.

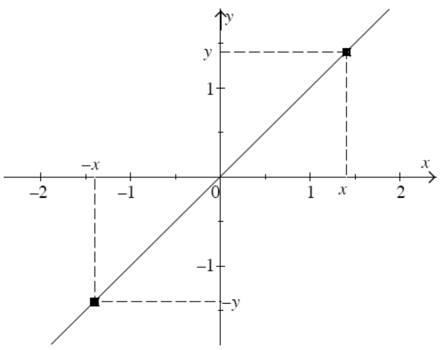
Geometrically, an even function is symmetrical about the y-axis (it has line symmetry). The function $f(x) = x^2$ is an even function as $f(-x) = (-x)^2 = x^2 = f(x)$ for all values of x. We illustrate this on the following graph.



The graph of $y = x^2$.

A function, $y = f(x)$, is *odd* if $f(-x) = -f(x)$ for all x in the domain of f.

Geometrically, an odd function is symmetrical about the origin (it has rotational symmetry).
The function $f(x) = x$ is an odd function as $f(-x) = -x = -f(x)$ for all values of x. This is illustrated on the following graph.



The graph of $y = x$.

All monomials of even degree

$y = c$, $y = x^2$, $y = x^4$, … , $y = x^{2k}$, … are even functions.

More examples $y = \cos x$, $y = 1/(x^2 - 1)$.

All monomials of odd degree

$y = x$, $y = x^3$, $y = x^5$, … , $y = x^{2k+1}$, … are odd functions.

More examples $y = \sin x$, $y = 1/(x^3 - x)$.

The function $y = x + 1$ is neither even nor odd.

Example. Determine whether the following functions are odd, even or nither.

a. $f(x) = x^4 + 2$

$f(-x) = (-x)^4 + 2 = x^4 + 2 = f(x)$ **even**

b. $g(x) = x^3 + 3x$

$g(-x) = (-x)^3 + 3(-x) = -x^3 - 3x = -(x^3 + 3x) = -g(x)$ **odd**

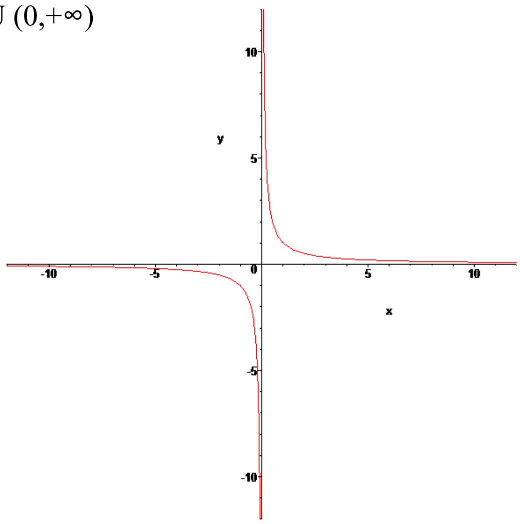
c. $h(x) = 2^x$

$h(-x) = 2^{-x} = 1/2^x$ **neither**

Inverse Proportionality (Hyperbolic Function)

$$y = \frac{1}{x}$$ Domain (-∞,0) U (0,+∞)

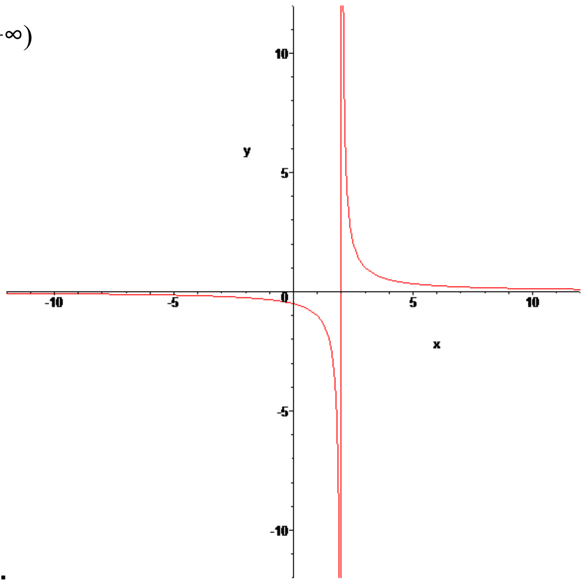
Range (-∞,0) U (0,+∞)



> **plot(1/x,x=-12..12,y=-12..12);**

$$y = \frac{1}{x-2}$$ Domain (-∞,2) U (2,+∞)

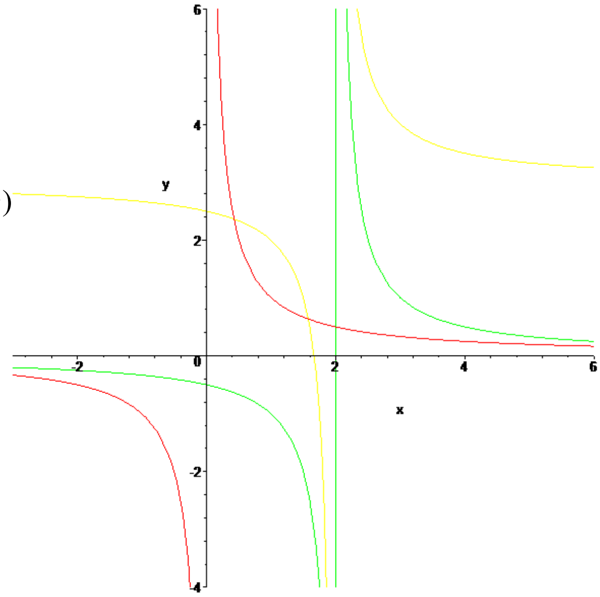
Range (-∞,0) U (0,+∞)



> **plot(1/(x-2),x=-12..12,y=-12..12);**

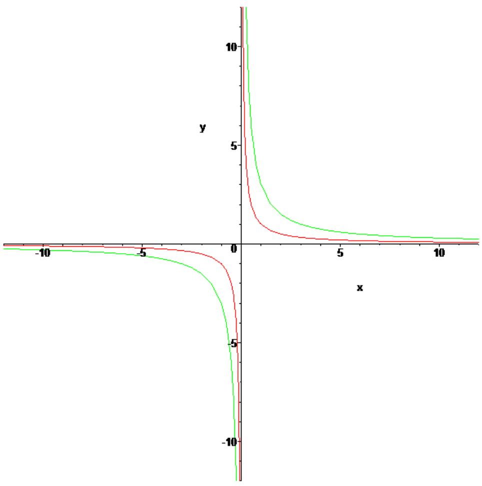
$y = \frac{1}{x-2} + 3$ Domain (-∞,2) U (2,+∞)

Range (-∞,3) U (3,+∞)



> **plot({1/x,1/(x-2),1/(x-2)+3},x=-3..6,y=-4..6);**

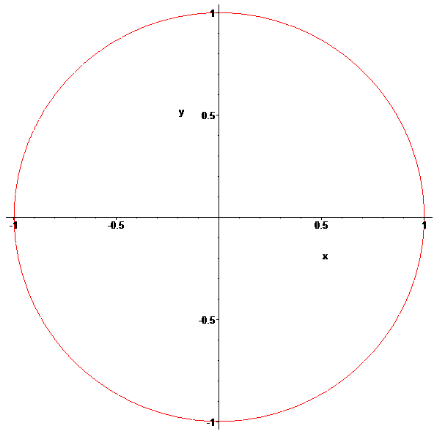
$y = \frac{1}{x}$ $y = \frac{3}{x}$ Domain (-∞,2) U (2,+∞)



> **plot({1/x,3/x},x=-12..12,y=-12..12);**

CIrcle

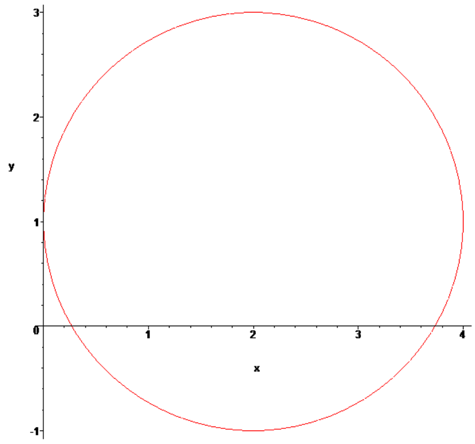
$x^2 + y^2 = 1$



> with(plots):
implicitplot(x^2+y^2=1, x=-1..1, y=-1..1);

CIrcle

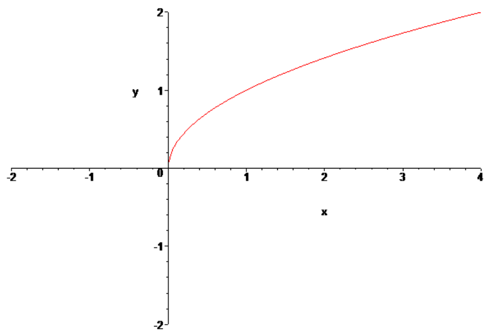
$(x-2)^2 + (y-1)^2 = 4$



> with(plots):
implicitplot((x-2)^2+(y-1)^2=4, x=0..4, y=-1..3);

Squaire root

$y = \sqrt{x}$ Domain (0, +∞)

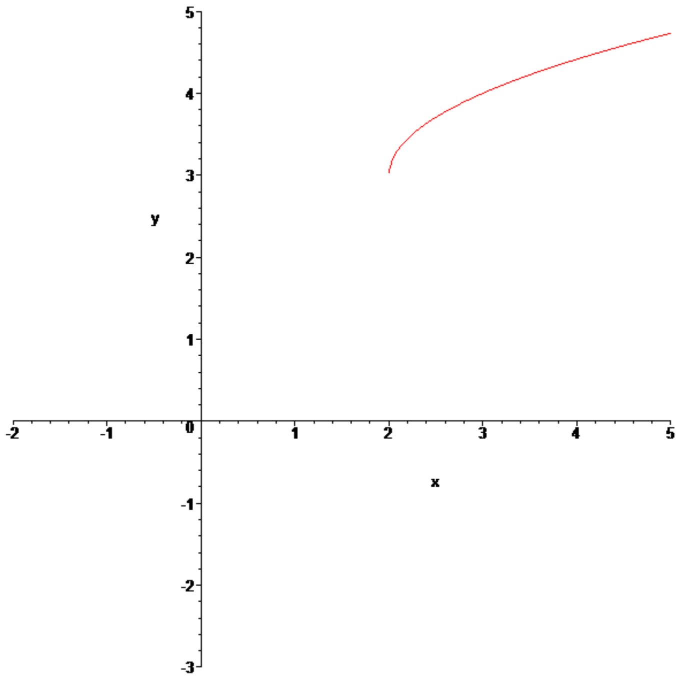



```
> plot(sqrt(x),x=0..4,y=0..2);
```

Squaire root

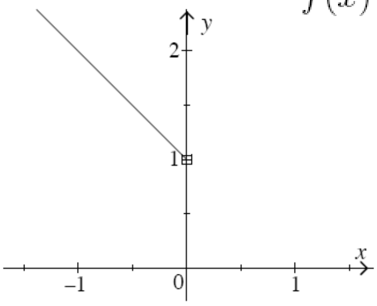
$y = \sqrt{x-2} + 3$ Domain (2, +∞)

Range (3, +∞)

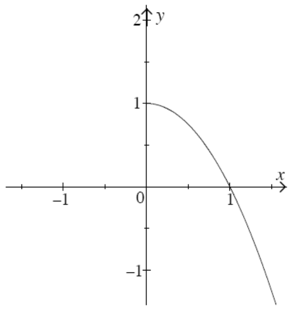


Piecewise Function

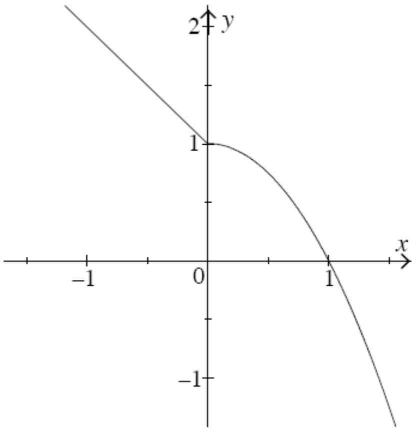
$$f(x) = \begin{cases} 1 - x^2 & \text{for } x \geq 0 \\ 1 - x & \text{for } x < 0 \end{cases}$$



The graph of $y = 1 - x$ for $x < 0$.

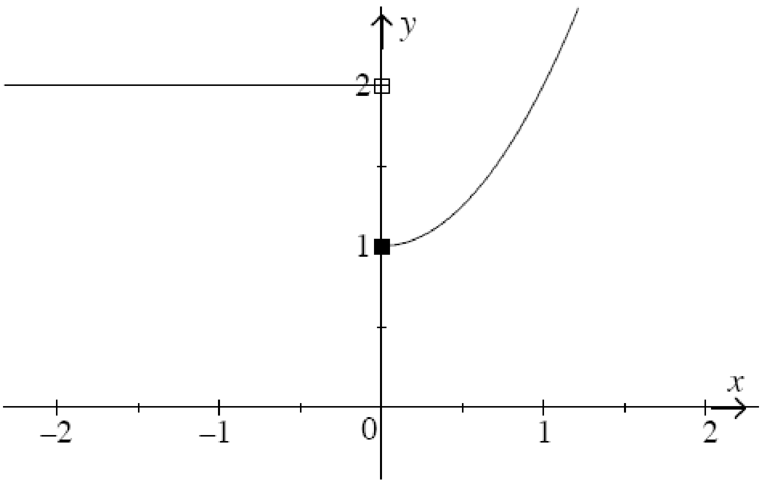


The graph of $y = 1 - x^2$ for $x \geq 0$.



Picewise fuction

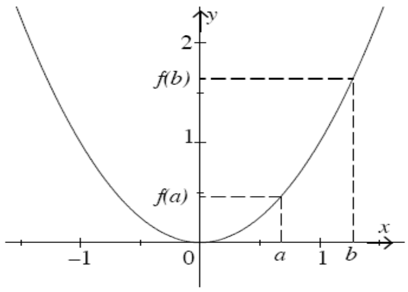
$$f(x) = \begin{cases} x^2 + 1 & \text{for } x \geq 0 \\ 2 & \text{for } x < 0 \end{cases}$$



Increasing and decreaing functions

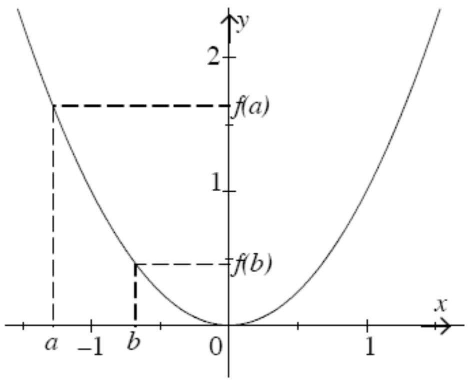
A function *f* is increasing on an interval *I* , if for all *a* and *b* in *I* such that *a* < *b* => *f(a) > f(b).*

The function $y = x^2$ is increasing on the interval $I = [0,+\infty)$



A function f is decreasing on an interval I , if for all a and b in I such that a < b => f(a) > f(b).

The function $y = x^2$ is decreasing on the interval I = $(-\infty,0]$



Example

Given the graph below of $y = f(x)$:

a. State the domain and range. Domain R, range $y \geq -2$

b. Where is the graph

i increasing? $-2 < x < 0$ or $x > 2$

ii decreasing? $x < -2$ or $0 < x < 2$

c. if k is a constant, find the values of k such that $f(x) = k$ has

i no solutions $k < -2$

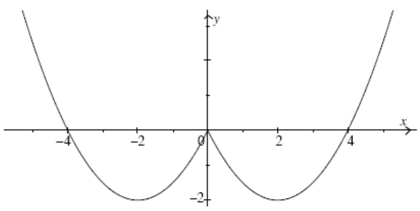
ii 1 solution no such k

iii 2 solutions $k = 2$ or $k > 0$

iv 3 solutions $k = 0$

v 4 solutions. $-2 < k < 0$

d. Is $y = f(x)$ even, odd or neither? Looks even



Exponential and Logarithmic Functions

Properties of exponent

$a^m \cdot a^n = a^{m+n};$

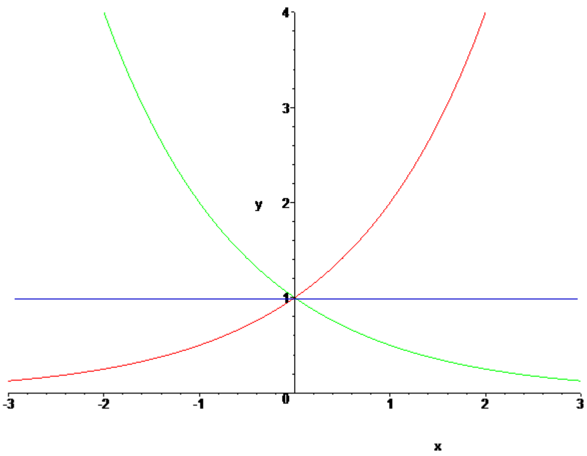
$a^{-n} = \frac{1}{a^m};$

$\frac{a^m}{a^n} = a^{m-n};$

$(a^m)^n = a^{m \cdot n};$

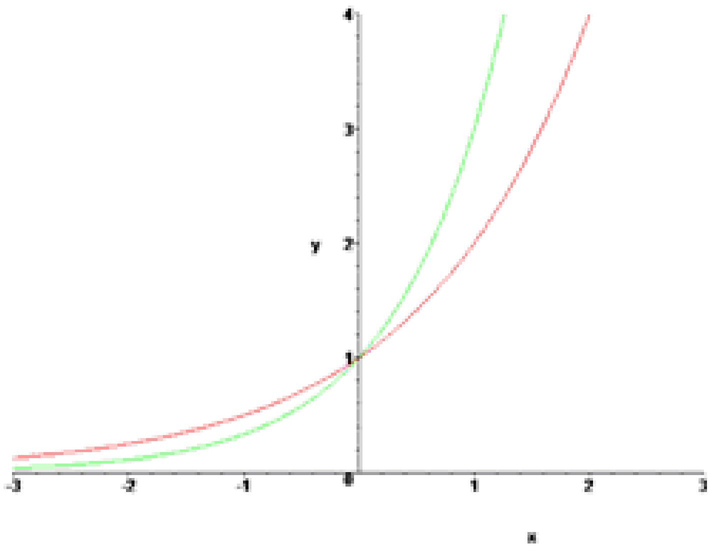
$a^0 = 1.$

Exponential Function



$y = 2^x$ $y = 1^x$ $y = 2^{-x} = (0.5)^x$

Exponential function $y = a^x$ is increasing for $a > 1$, is decreasing for $0 < a < 1$, and is constant for $a = 1$.



$y = 2^x$ $y = 3^x$

Nepper Number

The Nepper Number e ≈ 2.7181693 … (an important irrational number, as π = 3.141516…) is defined as $e = \lim_{n\to\infty}(1+\frac{1}{n})^n$. By the way, $\lim_{n\to\infty}(1+\frac{k}{n})^n = e^k$. The function e^x often is denoted as exp(x).

Logarithms

For a > 0, a ≠ 1, b >0,

$\log_a b = n$ if $a^n = b$. That is

$$a^{\log_a b} = b$$

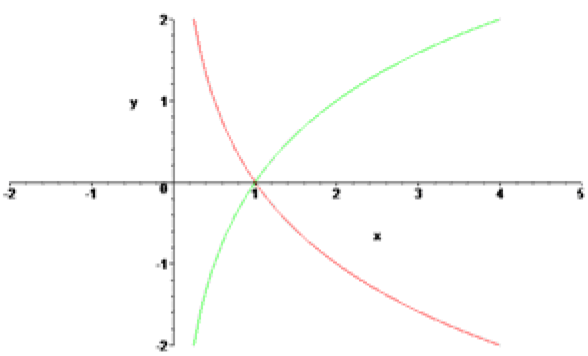
Properties of Logarithm

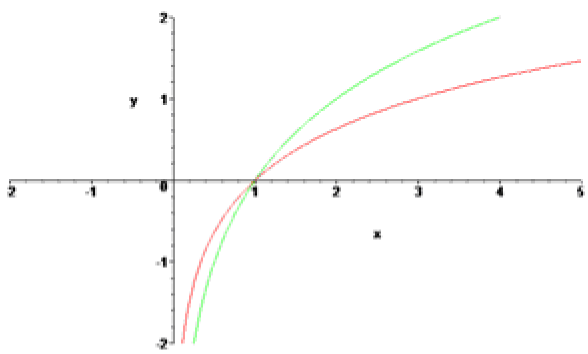
$$\log_a(r \cdot s) = \log_a r + \log_a s;$$
$$\log_a \tfrac{1}{r} = -\log_a r;$$
$$\log_a \tfrac{r}{s} = \log_a r - \log_a s;$$
$$\log_a r^s = s \cdot \log_a r;$$
$$\log_a 1 = 0;$$
$$\log_r s = \frac{1}{\log_s r};$$
$$\log_r s = \frac{\log_a s}{\log_a r}.$$

Notation:

Decimal logarithm lg x := log10 x.

Natural logarithm ln x := $\log_e$ x.





$y = \log_2 x$ **$y = \log_{0.5} x$** **$y = \log_2 x$** **$y = \log_3 x$**