Exercises for Functions

from Functions and their Graphs, MLC, Sidney

1.4 Exercises

1. a. State the domain and range of $f(x) = \sqrt{9 - x^2}$.
   b. Sketch the graph of $y = \sqrt{9 - x^2}$.

2. Given $\psi(x) = x^2 + 5$, find, in simplest form, $\dfrac{\psi(x+h) - \psi(x)}{h} \qquad h \neq 0.$

3. Sketch the following functions stating the domain and range of each:
   a. $y = \sqrt{x - 1}$
   b. $y = |2x|$
   c. $y = \frac{1}{x-4}$
   d. $y = |2x| - 1.$

4. a. Find the perpendicular distance from $(0, 0)$ to the line $x + y + k = 0$
   b. If the line $x + y + k = 0$ cuts the circle $x^2 + y^2 = 4$ in two distinct points, find the restrictions on k.

5. Sketch the following, showing their important features.
   a. $y = \left(\frac{1}{2}\right)^x$
   b. $y^2 = x^2.$

6. Explain the meanings of function, domain and range. Discuss whether or not $y^2 = x^3$ is a function.

7. Sketch the following relations, showing all intercepts and features. State which ones are functions giving their domain and range.
   a. $y = -\sqrt{4 - x^2}$
   b. $|x| - |y| = 0$
   c. $y = x^3$
   d. $y = \frac{x}{|x|}, x \neq 0$
   e. $|y| = x.$

8. If $A(x) = x^2 + 2 + \frac{1}{x^2}$, $x \neq 0$, prove that $A(p) = A(\frac{1}{p})$ for all $p \neq 0$.

9. Write down the values of x which are not in the domain of the following functio

a. $f(x) = \sqrt{x^2 - 4x}$

b. $g(x) = \frac{x}{x^2-1}$

10. If $\phi(x) = \log\left(\frac{x}{x-1}\right)$, find in simplest form:

a. $\phi(3) + \phi(4) + \phi(5)$

b. $\phi(3) + \phi(4) + \phi(5) + \cdots + \phi(n)$

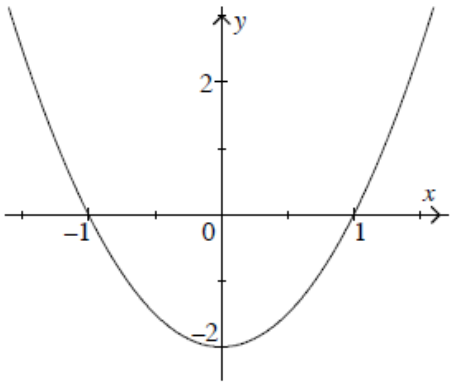
11. a. If $y = x^2 + 2x$ and $x = (z-2)^2$, find y when $z = 3$.

b. Given $L(x) = 2x + 1$ and $M(x) = x^2 - x$, find

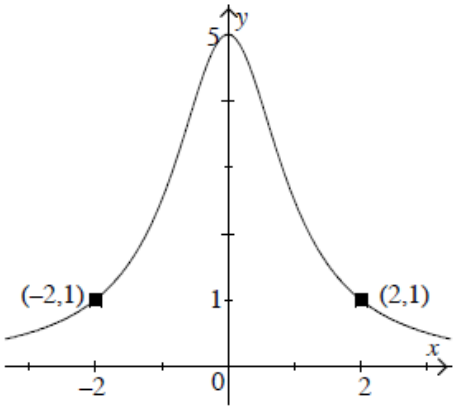
i $L(M(x))$

ii $M(L(x))$

12. Using the sketches, find the value(s) of the constants in the given equations:



$y = ax^2 - b$



$y = \frac{a}{bx^2+1}$

13. a. Define $|a|$, the absolute value of a, where a is real.

b. Sketch the relation $|x| + |y| = 1$.

14. Given that $S(n) = \frac{n}{2n+1}$, find an expression for $S(n-1)$.

Hence show that $S(n) - S(n-1) = \frac{1}{(2n-1)(2n+1)}$.

2.8 Exercises

1. Sketch the following:

a. $y = x^2$ **b.** $y = \frac{1}{3}x^2$ **c.** $y = -x^2$ **d.** $y = (x+1)^2$

2. Sketch the following:

a. $y = \frac{1}{x}$ **b.** $y = \frac{1}{x-2}$ **c.** $y = \frac{-2}{x}$ **d.** $y = \frac{1}{x+1} + 2$

3. Sketch the following:

a. $y = x^3$ **b.** $y = |x^3 - 2|$ **c.** $y = 3 - (x-1)^3$

4. Sketch the following:

a. $y = |x|$ **b.** $y = 2|x-2|$ **c.** $y = 4 - |x|$

5. Sketch the following:

a. $x^2 + y^2 = 16$ **b.** $x^2 + (y+2)^2 = 16$ **c.** $(x-1)^2 + (y-3)^2 = 16$

6. Sketch the following:

a. $y = \sqrt{9 - x^2}$ **b.** $y = \sqrt{9 - (x-1)^2}$ **c.** $y = \sqrt{9 - x^2} - 3$

7. Show that $\dfrac{x-1}{x-2} = \dfrac{1}{x-2} + 1$.

Hence sketch the graph of $y = \dfrac{x-1}{x-2}$.

8. Sketch $y = \frac{x+1}{x-1}$.

9. Graph the following relations in the given interval:

a. $y = |x| + x + 1$ for $-2 \leq x \leq 2$ [Hint: Sketch by adding ordinates]

b. $y = |x| + |x-1|$ for $-2 \leq x \leq 3$

c. $y = 2^x + 2^{-x}$ for $-2 \leq x \leq 2$

d. $|x - y| = 1$ for $-1 \leq x \leq 3$.

10. Sketch the function $f(x) = |x^2 - 1| - 1$.

11. Given $y = f(x)$ as sketched below, sketch

a. $y = 2f(x)$

b. $y = -f(x)$

c. $y = f(-x)$

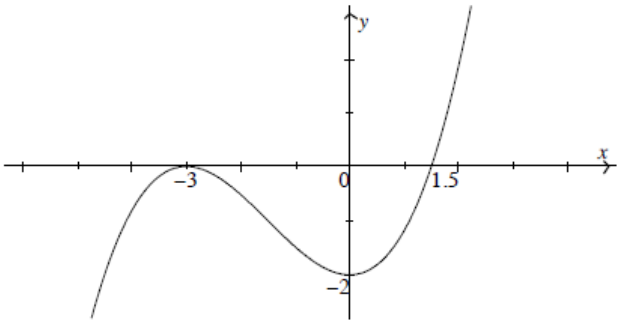
d. $y = f(x) + 4$

e. $y = f(x - 3)$

f. $y = f(x + 1) - 2$

g. $y = 3 - 2f(x - 3)$

h. $y = |f(x)|$



12. By sketching graphs solve the following equations:

a. $|2x| = 4$

b. $\frac{1}{x-2} = -1$

c. $x^3 = x^2$

d. $x^2 = \frac{1}{x}$

13. Solve $|x - 2| = 3$.

a. algebraically

b. geometrically.

14. The parabolas $y = (x - 1)^2$ and $y = (x - 3)^2$ intersect at a point P. Find the coordinates of P.

15. Sketch the circle $x^2 + y^2 - 2x - 14y + 25 = 0$. [Hint: Complete the squares.] Find the values of k, so that the line $y = k$ intersects the circle in two distinct points.

16. Solve $\frac{4}{5-x} = 1$, using a graph.

17. Find all real numbers x for which $|x - 2| = |x + 2|$.

18. Given that $Q(p) = p^2 - p$, find possible values of n if $Q(n) = 2$.

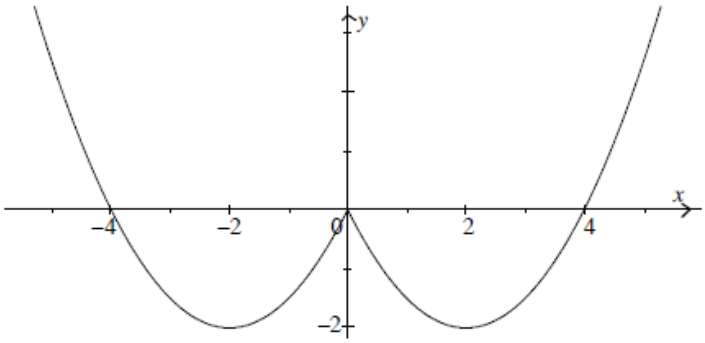
19. Solve $|x - 4| = 2x$.

a. algebraically

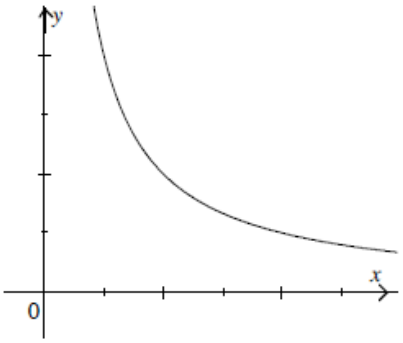
b. geometrically.

2.11 Exercises

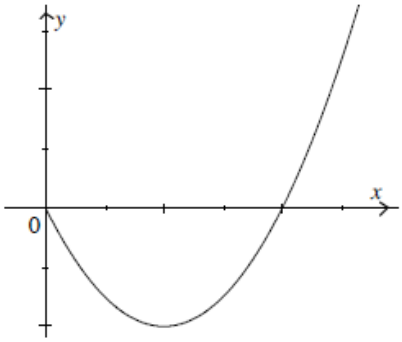
1. Given the graph below of $y = f(x)$:
   - **a.** State the domain and range.
   - **b.** Where is the graph
     - i increasing?
     - ii decreasing?
   - **c.** if k is a constant, find the values of k such that $f(x) = k$ has
     - i no solutions
     - ii 1 solution
     - iii 2 solutions
     - iv 3 solutions
     - v 4 solutions.
   - **d.** Is $y = f(x)$ even, odd or neither?



2. Complete the following functions if they are defined to be (a) even (b) odd.



$y = f(x)$



$y = g(x)$

3. Determine whether the following functions are odd, even or neither.

   a. $y = x^4 + 2$ b. $y = \sqrt{4 - x^2}$ c. $y = 2^x$ d. $y = x^3 + 3x$

   e. $y = \frac{x}{x^2}$ f. $y = \dfrac{1}{x^2 - 4}$ g. $y = \dfrac{1}{x^2 + 4}$ h. $y = \dfrac{x}{x^3 + 3}$

   i. $y = 2^x + 2^{-x}$ j. $y = |x - 1| + |x + 1|$

4. Given $y = f(x)$ is even and $y = g(x)$ is odd, prove

   a. if $h(x) = f(x) \cdot g(x)$ then $h(x)$ is odd

   b. if $h(x) = (g(x))^2$ then $h(x)$ is even

   c. if $h(x) = \dfrac{f(x)}{g(x)}$, $g(x) \neq 0$, then $h(x)$ is odd

   d. if $h(x) = f(x) \cdot (g(x))^2$ then $h(x)$ is even.

5. Consider the set of all odd functions which are defined at $x = 0$. Can you prove that for every odd function in this set $f(0) = 0$? If not, give a counter-example.

3.2 Exercises

1. For the function

$$f(x) = \begin{cases} 1 - x^2 & \text{for } x \geq 0 \\ 1 - x & \text{for } x < 0 \end{cases}$$

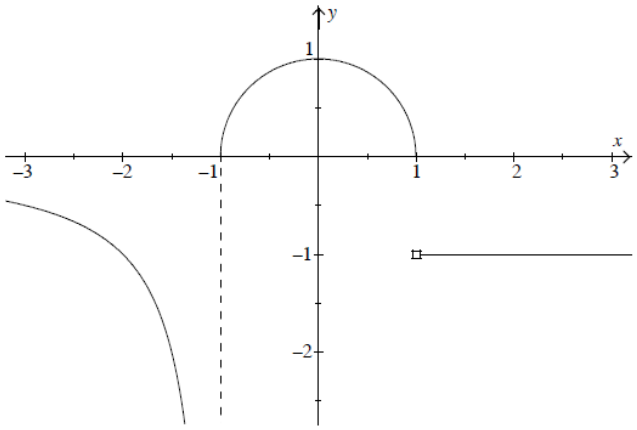
evaluate

a. $2f(-1) + f(2)$

b. $f(a^2)$

2. For the function given in **1**, solve $f(x) = 2$.

3. Below is the graph of $y = g(x)$. Write down the rules which define $g(x)$ given that its pieces are hyperbolic, circular and linear.



4. **a.** Sketch the graph of $y = f(x)$ if

$$f(x) = \begin{cases} -\sqrt{4 - x^2} & \text{for } -2 \leq x \leq 0 \\ x^2 - 4 & \text{for } x > 0 \end{cases}$$

b. State the range of f.

c. Solve

i $f(x) = 0$

ii $f(x) = -3$.

d. Find k if $f(x) = k$ has

i 0

ii 1

iii 2 solutions.

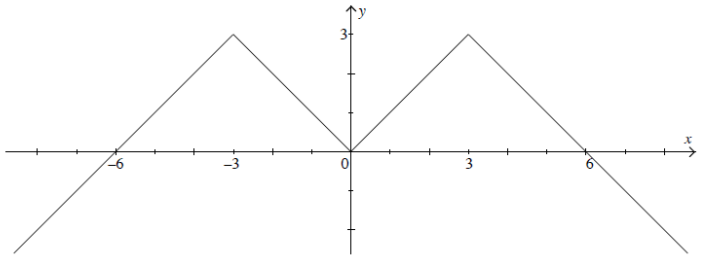
5. Sketch the graph of $y = f(x)$ if

$$f(x) = \begin{cases} 1 - |x - 1| & \text{for } x \geq 0 \\ |x + 1| & \text{for } x < 0 \end{cases}$$

6. Sketch the graph of $y = g(x)$ if

$$g(x) = \begin{cases} \frac{2}{x+2} & \text{for } x < -1 \\ 2 & \text{for } -1 \leq x < 1 \\ 2^x & \text{for } x \geq 1 \end{cases}$$

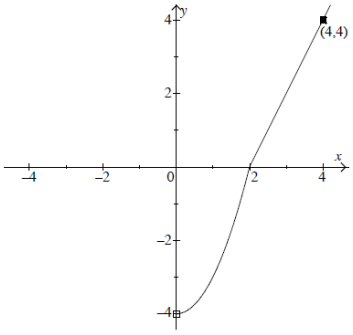
7. McMaths burgers are to modernise their logo as shown below.



Write down a piecewise function that represents this function using (a) 4 (b) 3 (c) 2 pieces (i.e. rules that define the function).

8. **a.** The following piecewise function is of the form

$$f(x) = \begin{cases} ax^2 + b & \text{for } 0 < x \leq 2 \\ cx + d & \text{for } x > 2 \end{cases}$$



Determine the values of a, b, c and d.

b. Complete the graph so that $f(x)$ is an odd function defined for all real x, $x \neq 0$.

c. Write down the equations that now define $f(x)$, $x \neq 0$.

3.4 Exercises

1. Solve

a. $x^2 \leq 4x$

b. $\frac{4p}{p+3} \leq 1$

c. $\frac{7}{9-x^2} > -1$

2. **a.** Sketch the graph of $y = 4x(x-3)$.

b. Hence solve $4x(x-3) \leq 0$.

3. **a.** Find the points of intersection of the graphs $y = 5 - x$ and $y = \frac{4}{x}$.

b. On the same set of axes, sketch the graphs of $y = 5 - x$ and $y = \frac{4}{x}$.

c. Using part (ii), or otherwise, write down all the values of x for which

$$5 - x > \frac{4}{x}$$

4. **a.** Sketch the graph of $y = 2^x$.

b. Solve $2^x < \frac{1}{2}$.

c. Suppose $0 < a < b$ and consider the points $\mathrm{A}(a, 2^a)$ and $\mathrm{B}(b, 2^b)$ on the graph of $y = 2^x$. Find the coordinates of the midpoint M of the segment AB.

Explain why

$$\frac{2^a + 2^b}{2} > 2^{\frac{a+b}{2}}$$

5. **a.** Sketch the graphs of $y = x$ and $y = |x-5|$ on the same diagram.

b. Solve $|x-5| > x$.

c. For what values of m does $mx = |x-5|$ have exactly

i two solutions

ii no solutions

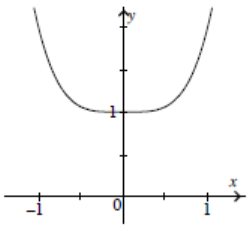
6. Solve $5x^2 - 6x - 3 \leq |8x|$.

4.3 Exercises

1. Sketch the graphs of the following polynomials if $y = P(x)$ is:
   a. $x(x+1)(x-3)$
   b. $x(x+1)(3-x)$
   c. $(x+1)^2(x-3)$
   d. $(x+1)(x^2-4x+5)$

2. The graphs of the following quartic polynomials are sketched below. Match the graph with the polynomial.
   a. $y = x^4$ b. $y = x^4 - 1$ c. $y = x^4 + 1$ d. $y = 1 - x^4$ e. $y = (x-1)^4$ f. $y = (x+1)^4$

i



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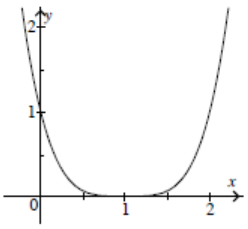

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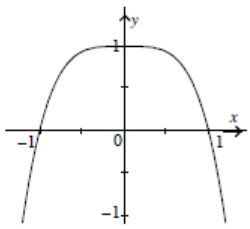

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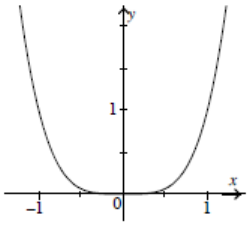

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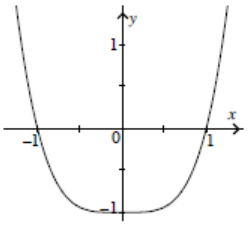

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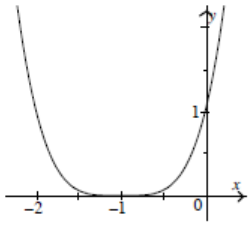


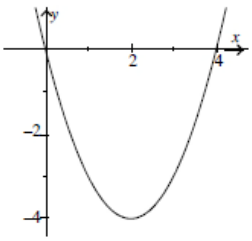
3. Sketch the graphs of the following quartic polynomials if $y = C(x)$ is:
   a. $x(x-1)(x+2)(x+3)$
   b. $x(x-1)(x+2)(3-x)$
   c. $x^2(x-1)(x-3)$
   d. $(x+1)^2(x-3)^2$
   e. $(x+1)^3(x-3)$
   f. $(x+1)^3(3-x)$
   g. $x(x+1)(x^2-4x+5)$
   h. $x^2(x^2-4x+5)$.

4. By sketching the appropriate polynomial, solve:
   a. $x^2 - 4x - 12 < 0$
   b. $(x+2)(x-3)(5-x) > 0$
   c. $(x+2)^2(5-x) > 0$
   d. $(x+2)^3(5-x) \geq 0$.

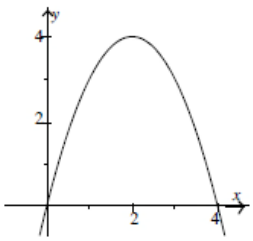
5. For what values of k will $P(x) \geq 0$ for all real x if $P(x) = x^2 - 4x - 12 + k$?

6. The diagrams show the graph of $y = P(x)$ where $P(x) = a(x-b)(x-c)^d$.
   In each case determine possible values for a, b, c and d.

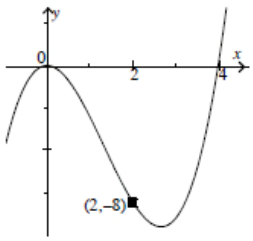
a.



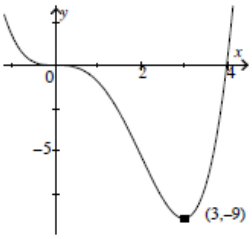
b.



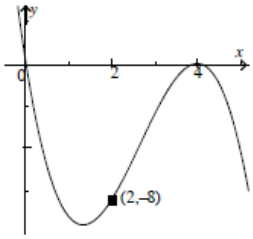
c.



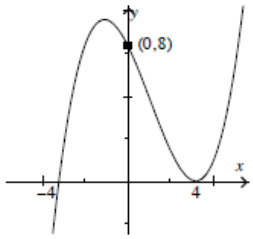
d.



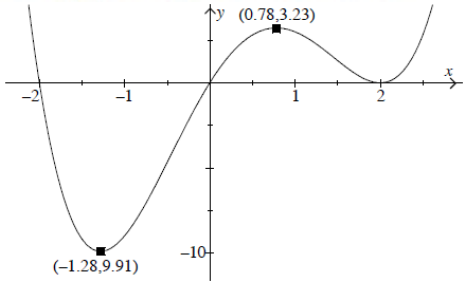
e.



f.



7. The graph of the polynomial $y = f(x)$ is given below. It has a local maximum and minimum as marked. Use the graph to answer the following questions.
   a. State the roots of $f(x) = 0$.
   b. What is the value of the repeated root.
   c. For what values of k does the equation $f(x) = k$ have exactly 3 solutions.
   d. Solve the inequality $f(x) < 0$.
   e. What is the *least* possible degree of $f(x)$?
   f. State the value of the constant of $f(x)$.
   g. For what values of k is $f(x) + k \geq 0$ for all real x.



Exercises for exp and log
from Introduction to Exponents and Logarithms, MLC, Sydnay

1.6 Exercises

The following expressions evaluate to quite a 'simple' number. If you leave some of your answers in fractional form you won't need a calculator.

36. $9^{\frac{1}{2}}$ **37.** $16^{\frac{3}{4}}$ **38.** $(\frac{1}{5})^{-1}$ **39.** $(3^{-1})^2$ **40.** $(\frac{5}{2})^{-2}$

41. $(-8)^{\frac{3}{2}}$ **42.** $(\frac{-27}{8})^{\frac{2}{3}}$ **43.** $5^{27}5^{-24}$ **44.** $8^{\frac{1}{2}}2^{\frac{1}{2}}$ **45.** $(-125)^{\frac{2}{3}}$

These look a little complicated but are equivalent to simpler ones. 'Simplify' them. Again, you won't need a calculator.

46. $\dfrac{3^{n+2}}{3^{n-2}}$ **47.** $\sqrt{\dfrac{16}{x^6}}$ **48.** $(a^{\frac{1}{2}}+b^{\frac{1}{2}})^2$

49. $(x^2+y^2)^{\frac{1}{2}}-x^2(x^2+y^2)^{-\frac{1}{2}}$ **50.** $\dfrac{x^{\frac{1}{2}}+x}{x^{\frac{1}{2}}}$ **51.** $(u^{\frac{1}{3}}-v^{\frac{1}{3}})(u^{\frac{2}{3}}+(uv)^{\frac{1}{3}}+v^{\frac{2}{3}})$

Write these numbers in scientific notation.

52. 0.00317 **53.** $2.15\times10^7\times3.54\times10^{-1}$ **54.** $3.47\times10^{17}\times7.4\times10^{-3}$

55. $(2.7\times10^{65})^{\frac{1}{3}}$ **56.** $5.98\times10^6-3.7\times10^5$ **57.** $\dfrac{3.8\times10^{27}}{2.45\times10^{-8}}$

Don't bother working these ones out, just decide whether or not the expressions are defined.

58. $(-1.7)^{\frac{1}{8}}$ **59.** $(-3)^{\frac{2}{19}}$ **60.** $(-4.8)^{-\frac{6}{14}}$ **61.** $(\pi)^{\sqrt{2}}$ **62.** $(-\pi)^{-\frac{8}{14}}$

2.5 Exercises

3. Which of the following functions are increasing and which are decreasing? If you have understood this section fully you will be able to answer this question without graphing the functions or substituting any values.

a. $f(x)=(\frac{5}{3})^x$ b. $f(x)=(\frac{5}{3})^{-x}$ c. $f(x)=(\frac{3}{5})^{-x}$ d. $f(x)=(\frac{3}{5})^x$

4. Sketch the graphs of the functions $f(x)=3^x$ and $f(x)=3^{-x}$. On the same diagrams mark in roughly the graphs of $f(x)=2.9^x$ and 2.9^{-x}.

5. It is true that $e^{1.09861}\approx3$. Try it for yourself on a calculator if you have one. How do you think the functions $y=3^x$ and $y=e^{1.09861x}$ compare? Why? If you cannot solve this otherwise, you might like to try substituting in a few numbers for x in both of the functions and comparing the values.

3.7 Exercises

Without using a calculator, find the following numbers.

8. $\log_{10} 10^{-19}$ **9.** $\log_e e\sqrt[5]{e}$ **10.** $\log_2 16$

11. $\log_{17} \frac{17^3}{\sqrt{17}}$ **12.** $\ln \frac{e^2}{e^{21}}$ **13.** $\frac{\ln e^7}{\log_{11} 121}$

14. $5^{\log_5 32.7}$ **15.** $e^{\ln \frac{9}{2}}$ **16.** $e^{\ln \sqrt[3]{27}}$

Rewrite the following expressions using the rules of logarithms, and simplify where possible.

17. $\log_{10} \frac{100x^2}{9y}$ **18.** $\ln \frac{xy^{-3}}{e^{1.37}}$ **19.** $\log_4 \frac{4^{-1.3}z^7}{x^2y^3}$

20. $\log_3 \frac{x^3y^2}{27z^{\frac{1}{2}}}$ **21.** $\ln(e^{-2.4}x^6)$ **22.** $\log_5 \frac{125x^3}{0.2y^2}$

Using the rules of logarithms, rewrite the following expressions so that just one logarithm appears in each.

23. $3\log_2 x + \log_2 30 + \log_2 y - \log_2 w$ **24.** $2\ln x - \ln y + a\ln w$

25. $12(\ln x + \ln y)$ **26.** $\log_3 e \times \ln 81 + \log_3 5 \times \log_5 w$

27. $\log_7 10 \times \log_{10} x^2 - \log_7 49x$ **28.** $\log_{10} 0.1 \times \log_6 x - 2\log_6 y + \log_6 4 \times \log_4 e$

Given that $\log_e 5 \approx 1.6094$, and $\log_e 7 \approx 1.9459$, find the following numbers without using a calculator except to perform multiplication or division.

29. $\log_5 e$ **30.** $\log_5 7$ **31.** $\log_5 7^2$ **32.** $\log_{49} 5$ **33.** $\log_{49} 25$ **34.** $\log_e 25$