Tornike Kadeishvili

Linear Combinations

A *linear combination* of vectors $v_1, v_2, \ldots, v_m \in R^n$ with scalar coefficients $\alpha_1, \alpha_2, \ldots, \alpha_m \in R$ is the vector

$$\alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + \ldots + \alpha_m \cdot v_m.$$

The set of all linear combinations of vectors $v_1, v_2, \ldots, v_m \in R^n$ is denoted as

$$L[v_1, v_2, \ldots, v_m] = \{\alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + \ldots + \alpha_m \cdot v_m, \quad \alpha_i \in R\}.$$

Span

A sequence of vectors $v_1, v_2, \ldots, v_m \in R^n$ spans R^n if each vector $u \in R^n$ is their linear combination, i.e. there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_m \in R$ s.t.

$$u = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_n.$$

In other words it means that $L(v_1, v_2, \ldots, v_m) = R^n$

Example. The vectors $v_1 = (1, 0, 0)$, $v_2 = (0, 1, 0)$ span the xy plane (the plane given by the non-parameterized equation $z = 0$) of R^3. Indeed, any point $p = (a, b, 0)$ of this plane is the following linear combination

$$av_1 + bv_2 = a(1, 0, 0) + b(0, 1, 0) = (a, 0, 0) + (0, b, 0) = (a, b, 0).$$

Example. The vectors $v_1 = (1, 2)$, $v_2 = (3, 4)$ span whole R^2. Indeed, let's take any vector $v = (a, b)$. Our aim is to find c_1, c_2 s.t.

$$c_1 \cdot v_1 + c_2 \cdot v_2 = v.$$

In coordinates this equation looks as a system

$$\begin{cases} c_1 \cdot 1 + c_2 \cdot 3 = a \\ c_1 \cdot 2 + c_2 \cdot 4 = b \end{cases}.$$

The determinant of this system $\neq 0$, so this system has a solution for each a and b.

Linear Independence

Definition 1. A sequence of vectors $v_1, v_2, \ldots, v_n$ is called *linearly dependent* if one of these vectors is linear combination of others. That is

$$\exists i,\ v_i \in L(v_1, \ldots, \hat{v}_i, \ldots, v_n).$$

Definition 1'. A sequence of vectors $v_1, v_2, \ldots, v_m$ is linearly dependent if there exist $\alpha_1, \ldots, \alpha_m$ with at last one nonzero α_k s.t.

$$\alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + \ldots + \alpha_m \cdot v_m = 0.$$

Why these definitions are equivalent?

Example. Any sequence of vectors which contains the zero vector is linearly dependent. (Why?)

Example. Any sequence of vectors which contains two collinear vectors is linearly dependent. (Why?)

Example. Any sequence of vectors of R^2 which consists of more then two vectors is linearly dependent. (Why?)

Example. A sequence consisting of two vectors v_1, v_2 is linearly dependent if and only if these vectors are collinear (proportional), i.e. $v_2 = k \cdot v_1$. (Why?)

Definition 2. A sequence of vectors $v_1, v_2, \ldots, v_n$ is called *linearly independent* if it is not linearly dependent.

Example. The vectors $e_1 = (1, 0, 0),\ e_2 = (0, 1, 0),\ e_3 = (0, 0, 1) \in R^3$ are linearly independent.

Basis

A sequence of vectors $v_1, \ldots, v_n \in R^n$ forms a *basis* of R^n if
(1) they are linearly independent;
(2) they span R^n.

Example. The vectors

$$e_1 = (1, 0, \ \dots, 0), \ e_2 = (0, 1, \ \dots, 0), \ \dots, e_n = (0, 0, \ \dots, 1)$$

form a basis of R^n.

Indeed, firstly they are linearly independent since the $n \times n$ matrix

$$(e_1 \ \ e_2 \dots \ e_n)$$

is the identity, thus it's determinant is $1 \neq 0$.

Secondly, they span R^2: any vector $v = (x_1, \ \dots, x_n)$ is the following linear combination

$$v = x_1 \cdot e_1 + \ \dots \ + x_n \cdot e_n.$$

A basis $v_1, \ \dots, v_n \in R^n$ is called *orthogonal* if $v_i \cdot v_j = 0$ for $i \neq j$. This means that all vectors are perpendicular to each other: $v_i \cdot v_j = 0$ for $i \neq j$.

An orthogonal basis $v_1, \ \dots, v_n \in R^n$ is called *orthonormal* if $v_i \cdot v_i = 1$. This means that each vector of this basis has the length 1. In other words: $v_i \cdot v_j = \delta_{i,j}$ where δ_{ij} is famous Kroneker's symbol

$$\delta_{ij} = \begin{cases} 1 & if \quad i = j \\ 0 & if \quad i \neq j \end{cases}.$$

The basis $e_1, \ \dots, e_n$ is orthonormal.

Theorem *Let $v_1, \ \dots, v_k \in R^n$ and A be the matrix whose columns are v_j's:*

$$A = \begin{pmatrix} v_{11} & v_{21} & \dots & v_{n1} \\ v_{12} & v_{22} & \dots & v_{n1} \\ \dots & \dots & \dots & \\ v_{1n} & v_{2n} & \dots & v_{nn} \end{pmatrix}.$$

Then the following statements are equivalent

(a) $v_1, \ \dots, v_n$ are linearly independent;

(b) $v_1, \ \dots, v_n$ span R^n;

(c) $v_1, \ \dots, v_n$ is a basis of R^n;

(d) det $A \neq 0$.

Linear Transformations

A linear transformation (map, function) $f : R \to R$ has the form
$f(x) = a \cdot x$,
so it is just the multiplication with some scalar a. Note that $a = f(1)$.

A linear transformation (map, function) $f : R^2 \to R$ has the form

$$f(x_1,x_2) = a_1 \cdot x_1 + a_2 \cdot x_2 ,$$

so it is just the inner product

$$f(x_1,x_2) = (a_1,a_2) \cdot (x_1,x_2).$$

Note that $a_1 = f(1,0)$ and $a_2 = f(0,1)$.

A linear transformation (map, function) $f : R^2 \to R^2$ has the form

$$f(x_1,x_2) = (a_{11} \cdot x_1 + a_{12} \cdot x_2 , a_{21} \cdot x_1 + a_{22} \cdot x=)$$

so it is just the matrix product

$$f(x_1,x_2) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Note that the first column of this matrix is f(1,0) written as a column vector, the second column of this matrix is f(0,1) written as a column vector.

Linear Transformations

A *linear function* $f : R^k \to R^m$ is a function that preserves the vector space structure

$$f(x+y) = f(x) + f(y), \quad f(kx) = kf(x).$$

Such a function is determined by a $m \times k$ matrix

$$A = \begin{pmatrix} a_{11} & \dots & a_{1k} \\ \dots & & \\ a_{m1} & \dots & a_{mk} \end{pmatrix}$$

and $f(x) = A \cdot x$ where $x \in R^k$ and $f(x) \in R^m$ are written as column vectors

the i-th column of A is the column vector $f(e_i)$

where e_i is the i-th ort $e_i = (0, \dots ,0,1,0, \dots ,0)$.

Example.

Let $f : R^2 \to R^2$ be the linear map which is

rotation of the plane by 90° clockwise. Thus

$$f(1,0) = (0,-1), \quad f(0,1) = (1,0),$$

so the matrix of this linear map is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Exercises

Vrite the following linear functions in matrix torm

(a) $f : R^3 \to R$ given by $f(x_1, x_2, x_3) = 2x_1 - 3x_2 + 5x_3$.

(b) $f : R^2 \to R^?$ given by $f(x_1, x_2) = (2x_1 - 3x_2, x_1 - 4x_2, x_1)$.

(c) $f : R^? \to R^?$ given by $f(x_1, x_2, x_3) = (x_1 - x_3, 2x_1 + 3x_2 - 6x_3, x_3 - 2x_2)$.

Let F: $R^3 \to R^3$ be a linear map given by

F(1,0,0) = (1,1,0), F(0,1,0) = (0,10,-2), F(0,0,1) = (-1,-6,1),

and v = (-2,3,-1).

(a) Find a vector u of length 1 such tat F(u) = v.

(b) Find a vector w in R^3 which is NOT in the image of F, that is there exists no vector p such that w = F(p).

Eigenvalues and Eigenvectors

If $A \cdot x = \lambda \cdot x$ where

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

is a matrix, λ is a number and $x = \begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix} \in R^n$ is a nonzero vector, then λ is called an *eigenvalue* of the matrix A and x is called a *eigenvector* of A corresponding to the eigenvalue λ.

Note that if x is an eigenvector corresponding to an eigenvalue λ then kx is an eigenvector too: $A \cdot (kx) = kA \cdot x = k\lambda x = \lambda(kx)$.

The *specter* of A (denoted by $spec(A)$) is defined as the set of all eigenvalues $\lambda_1, ..., \lambda_k$

Eigenvalues and Eigenvectors

How to Find Eigenvalues and Eigenvectors

These can be found solving the matrix equation $A \cdot x = \lambda \cdot x$, equivalently $(A - \lambda I)x = 0$, which in its turn is equivalent to the system

$$\begin{cases} (a_{11} - \lambda)x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n = 0 \\ a_{21}x_1 & + & (a_{22} - \lambda)x_2 & + & \dots & + & a_{2n}x_n = 0 \\ \dots & \dots & \dots & \dots & & & \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \dots & + & (a_{nn} - \lambda)x_n = 0 \end{cases}.$$

This is homogenous system so it has a nonzero solution if and only if its determinant $|A - \lambda I|$ (which is called characteristic polynomial of A) is zero, so $|A - \lambda I| = 0$.

So, the eigenvalues can be found from the

characteristic equation $|A - \lambda I| = 0$ that is

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{vmatrix} = 0.$$

Eigenvalues and Eigenvectors

Example. Find the eigenvalues for the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Solution. The characteristic equation looks as

$$\begin{vmatrix} 1 - \lambda & 1 & 1 \\ 1 & 1 - \lambda & 1 \\ 1 & 1 & 1 - \lambda \end{vmatrix} = 0.$$

Calculating this determinant we obtain

$$(1 - \lambda)^3 - 3(1 - \lambda) + 2 = 0, \quad \lambda^3 - 3\lambda^2 = 0,$$

thus $\lambda_1 = 0,\ \lambda_2 = 0, \lambda_3 = 3$.

How to Find Eigenvectors

Eigenvectors corresponding to the eigenvalue λ can be found solving the matrix equation

$$(A - \lambda I)x = 0$$

which is equivalent to the system

$$\begin{cases} (a_{11} - \lambda)x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n = 0 \\ a_{21}x_1 & + & (a_{22} - \lambda)x_2 & + & \ldots & + & a_{2n}x_n = 0 \\ \cdots & \cdots & \cdots & \cdots & & & \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \ldots & + & (a_{nn} - \lambda)x_n = 0 \end{cases}.$$

Since λ is an eigenvalue the determinant of this system is zero. Thus this homogenous system has nonzero solutions.

Example. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}.$$

Solution. The characteristic equation of the matrix A looks as

$$A = \begin{vmatrix} 2 - \lambda & 2 \\ 1 & 3 - \lambda \end{vmatrix} = 0 \quad , \lambda^2 - 5\lambda + 4 = 0.$$

The roots of this equation, that is the eigenvalues are $\lambda_1 = 1, \quad \lambda_2 = 4$.

The eigenvectors can be found solving the system of equations

$$\begin{cases} (2-\lambda)x_1 + & 2x_2 = 0 \\ x_1 + & (3-\lambda)x_2 = 0 \end{cases}$$

For $\lambda = 1$:

$$\left.\begin{cases} (2-1)x_1 + & 2x_2 = 0 \\ x_1 + & (3-1)x_2 = 0 \end{cases}\right|, \quad \left|\begin{matrix} x_1 + & 2x_2 = 0 \\ x_1 + & 2x_2 = 0 \end{matrix}\right|$$

$$x_1 + 2x_2 = 0, x_1 = 2x_2,$$

thus the solution depending on the free parameter x_2 is $(2x_2, x_2)$. Taking, say, $x_2 = 1$ we obtain the eigenvector $v_1 = (2, 1)$.

For $\lambda = 4$:

$$\left.\begin{cases} (2-4)x_1 + & 2x_2 = 0 \\ x_1 + & (3-4)x_2 = 0 \end{cases}\right|, \quad \left|\begin{matrix} -2x_1 + & 2x_2 = 0 \\ x_1 - & x_2 = 0 \end{matrix}\right|$$

$$x_1 - x_2 = 0, \quad x_1 = x_2,$$

thus the solution depending on the free parameter x_2 is (x_2, x_2). Taking, say, $x_2 = 1$ we obtain the eigenvector $v_1 = (1, 1)$.

Exercises

1. Let $\begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix}$.

(a) Check that $\lambda = 2$ is an eigenvalue of A.

(b) Check that $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a corresponding eigenvector of A.

(c) Find all eigenvalues and corresponding eigenvectors of A.

2. Find the eigenvalues and eigenvectors for the matrix $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 5 & 0 \\ 3 & 0 & 2 \end{pmatrix}$.

3. A Markov matrix is a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ if all entries a, b, c, d are non-negative and $a + c = 1,\ b + d = 1$. Show that $\lambda = 1$ is an eigenvector for an arbitrary Markov matrix.

4. Find eigenvalues of an upper-triangular matrix $\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$.

5. Show that a 2×2 symmetric matrix $\begin{pmatrix} a & b \\ b & d \end{pmatrix}$ has real eigenvalues. In which case it has just one eigenvalue?

6. Find eigenvalues and eigenvectors of the following matrices

(a) $\begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix}$. (b) $\begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix}$. (c) $\begin{pmatrix} -1 & 3 \\ -2 & 4 \end{pmatrix}$. (d) $\begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix}$.

7. Find eigenvalues and eigenvectors of the following matrix

(a) $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 5 & 0 \\ 3 & 0 & 2 \end{pmatrix}$. (b) $\begin{pmatrix} 0 & 0 & -2 \\ 0 & 7 & 0 \\ 1 & 0 & -3 \end{pmatrix}$.