Vectors

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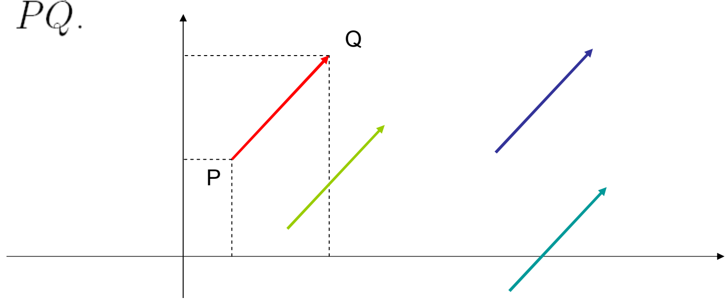
1.1 Euclidian Space R^n

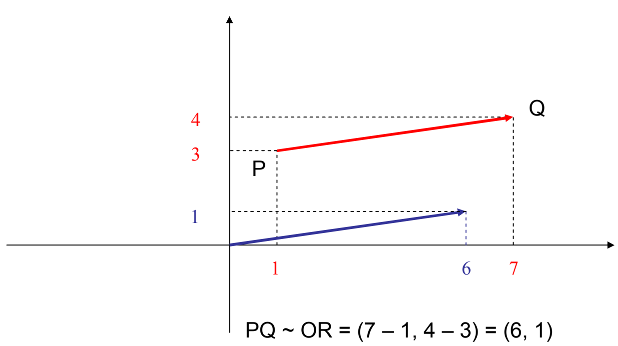
R^1 is the real line.
$R^2 = \{(x_1, x_2),\ x_1, x_2 \in R\}$ is the Euclidian 2-space.
$R^n = \{(x_1, ..., x_n),\ x_i \in R\}$ is the Euclidian n-space which consists of n-tuples of real numbers.

1.1.1 Vectors

A vector is an object which has a *magnitude* (or length) and *direction*. Graphically a vector is represented as an arrow, connecting an *initial point* P with a *terminal point* Q, notation $\overrightarrow{PQ}$.



Two arrows represent the same vector if they have the same magnitude and direction.



A vector is equivalent to the vector of the same magnitude and direction whose initial point is the origin.

Any vector can be identified with its terminal point when as initial point is assumed the origin.

Any two points $P = (p_1, ..., p_n), Q = (q_1, ..., q_n) \in R^n$ determine the vector $\overrightarrow{PQ}$. This vector has *coordinates* and $\overrightarrow{PQ}$ can be written as *row vector*

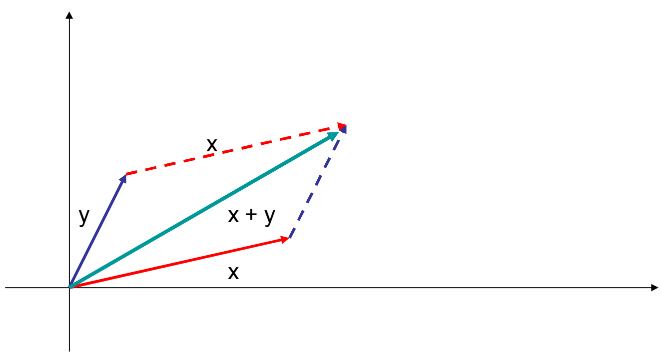
$$(q_1 - p_1, ..., q_n - p_n)$$

or *column vector*

$$\begin{pmatrix} q_1 - p_1 \\ \cdots \\ q_n - p_n \end{pmatrix}.$$

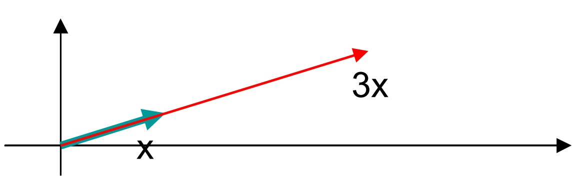
Vector addition: for $x = (x_1, ..., x_n)$, $y = (y_1, ...y_n) \in R^n$ the sum is defined by

$$x + y = (x_1 + y_1, ..., x_n + y_n).$$



Scalar multiplication: for $c \in R$ and $x = (x_1, ..., x_n) \in R^n$

$$c \cdot x = (c \cdot x_1, ..., c \cdot x_n)$$



These operations satisfy the following conditions:

(1) $u+(v+w)=(u+v)+w$,

(2) The zero vector $O=(0,...,0)\in R^n$ is neutral with respect to summation $O+v=v+O=v$,

(3) for each $v=(x_1,...,x_n)\in R^n$ there exists *opposite vector* $(-v)=(-x_1,..,-x_n)\in R^n$ s.t. $v+(-v)=O$,

(4) $v+w=w+v$.

(5) $r\cdot(s\cdot v)=(r\cdot s)\cdot v$, $1\cdot v=v$ for each $r,s\in R$.

(6) $(r+s)\cdot v=r\cdot v+s\cdot v$, $r\cdot(v+w)=r\cdot v+r\cdot w$.

Inner Product

The *inner product* of two vectors $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n) \in R^n$ is defined as the number

$$x \cdot y = x_1 \cdot y_1 + ... + x_n \cdot y_n = \sum_{k=1}^{n} x_k \cdot y_k.$$

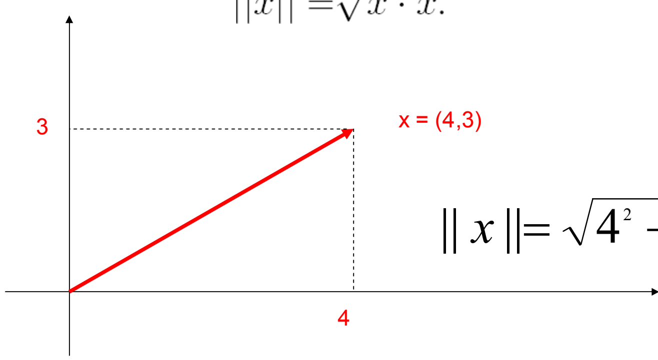
Properties of inner product:

(1) $u \cdot v = v \cdot u$,
(2) $u \cdot (v + w) = u \cdot v + u \cdot w$,
(3) $u \cdot rv = r(u \cdot v) = ru \cdot v$,
(4) $u \cdot u \geq 0$,
(5) $u \cdot u = 0 \ \Rightarrow u = 0$,

Norm of a Vector

The norm (length) of a vector $x = (x_1, ..., x_n) \in R^n$ is given by $||x|| = \sqrt{x_1^2 + ... + x_n^2}$. It fact the norm can be expressed in terms of inner product

$$||x|| = \sqrt{x \cdot x}.$$



$$|| x || = \sqrt{4^2 + 3^2} = 5$$

Angle Between Two Vectors

if the angle between vectors $x, y \in R^n$ is α, then

$$x \cdot y = ||x|| \cdot ||y|| \cdot cos\ \alpha.$$

This formula can be used to find the angle between two vectors:

$$cos\ \alpha = \frac{x \cdot y}{||x|| \cdot ||y||} = \frac{x_1 \cdot y_1 + ... + x_n \cdot y_n}{\sqrt{x_1^2 + ... + x_n^2} \cdot \sqrt{y_1^2 + ... + y_n^2}}.$$

Particularly vectors x an y are orthogonal (perpendicular) if and only if

$$x \cdot y = 0.$$

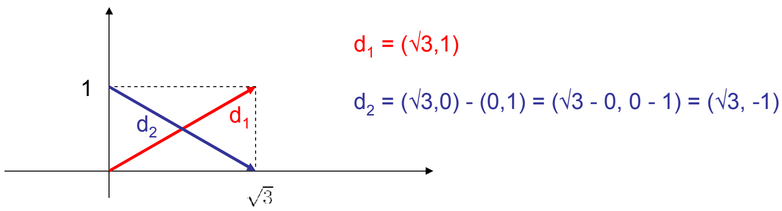
Metric in R^n

Metric (distance) is a function of two arguments $d(x, y)$ which satisfies the following axioms

1. $d(a, b) \geq 0$, $d(a, b) = 0 \Leftrightarrow a = b$;
2. $d(a, b) = d(b, a)$;
3. $d(a, c) + d(c, b) \geq d(a, b)$.

Metric in R^n is given by

$$d(x, y) = ||(x_1, ..., x_n) - (y_1, ..., y_n)|| =$$
$$\sqrt{(x_1 - y_1)^2 + ... + (x_n - y_n)^2}.$$



Example. Consider the rectangle determined by vectors $(\sqrt{3}, 0)$ and $(0, 1)$. Find the angle between the diagonals of this rectangle.

Solution. The diagonals are the vectors $d_1 = (\sqrt{3}, 1)$ and $d_2 = (\sqrt{3} - 0, 0 - 1) = (\sqrt{3}, -1)$, thus

$$cos\ \alpha = \frac{d_1 \cdot d_2}{||d_1|| \cdot ||d_1||} = \frac{3-1}{2 \cdot 2} = \frac{1}{2}.$$

Exercises

1. Find the lengths of the vectors (i) $(3, 4)$, (ii) $(1, 2, 3)$.

2. Find the distances (i) $d((0, 0), (3, 4))$, (ii) $d((5, 2), (1, 2))$.

3. Find the lengths of the vectors (i) $(3, 0, 0, 0)$, (ii) $(1, 1, 1, 1)$.

4. Find the distances
(i) $d((1, 2, 3, 4), (1, 0, -1, 0))$, (ii) $d((1, 2, 1, 2), (2, 1, 2, 1))$.

5. Find the angle between the vectors u and v if (i) $u = (1, 0)$, $v = (-1, 1)$; (ii) $u = (1, 0, 0)$, $v = (0, 0, 1)$.

6. Find the angle between the vectors u and v if (i) $u = (1, 0)$, $v = (2, 2)$; (ii) $u = (\sqrt{3}, 0)$, $v = (0, 1)$.

7. Find a vector of length 1 which points in the same direction as (i) (3,4); (ii) (6,0); (iii) (1,1,1); (iv) (-1,2,-3).

8. Find a vector of length 2 which points in the opposite direction to (i) (3,4); (ii) (6,0); (iii) (1,1,1); (iv) (-1,2,-3).

9. Consider the parallelogram determined by vectors $(1, 0)$ and $(1, 1)$. Find the angle between the diagonals of this parallelogram.