ISET Math Camp 12
Tornike Kadeishvili

Linear Algebra

1 Linear Equation

$$a \cdot x = b.$$

Solution:
Case 1. $a \neq 0$, then $x = \frac{b}{a}$ (one solution).
Case 2. $a = 0,\ b \neq 0$, then $x \in \emptyset$ (no solutions).
Case 3. $a = 0,\ b = 0$, then $x \in R$ (infinitely many solutions, moreover, any $x \in R$ is a solution).

1.1 Geometrical Interpretation

Solution is the x-intercept of the graph of the function $y = a \cdot x - b$.
Case 1. Slope $= a \neq 0$ - one intersection.

Case 2. Slope $= a = 0, \quad b \neq 0$ - the graph is parallel to x axes - no intersection.
Case 3. Slope $= a = 0, \quad b = 0$ - the graph coincides with x axes - infinitely many intersections.

2 System of Linear Equations

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

2.1 Solve and Substitute Method

Suppose $b_1 \neq 0$, then from the first equation $y = \frac{c_1 - a_1x}{b_1}$.

Substituting to the second we obtain one variable equation

$$a_2x + b_2 \cdot \frac{c_1 - a_1x}{b_1} = c_2.$$

2.2 Multiply and Add Method

Multiply the first equation by b_2 and the first by $-b_1$.
The summation of obtained equations kills y:

$$\begin{cases} a_1x + b_1y = c_1 & | \quad b_2 \quad | \quad a_1b_2x + b_1b_2y = c_1b_2 \\ a_2x + b_2y = c_2 & | \; -b_1 \; | \; -a_2b_1x - b_1b_2y = -c_2b_1 \end{cases}$$

so we obtain one variable equation

$$(a_1b_2 - a_2b_1)x = c_1b_2 - c_2b_1.$$

Finally we obtain

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}, \quad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

2.3 Determinant Method

Assign to a system

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

three DETERMINANTS

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1,$$

$$\Delta_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = c_1b_2 - c_2b_1, \quad \Delta_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = a_1c_2 - a_2c_1.$$

2.3.1 Cramer's Rule

Case 1. If $\Delta \neq 0$ then

$$x = \frac{\Delta_x}{\Delta}, \quad y = \frac{\Delta_y}{\Delta}.$$

Case 2. If $\Delta = 0$ and either $\Delta_x \neq 0$ or $\Delta_y \neq 0$ then the system has NO SOLUTIONS.
Case 3. If $\Delta = 0$ and $\Delta_x = 0, \quad \Delta_y = 0$ then the system has INFINITELY MANY SOLUTIONS.

2.4 Geometrical Interpretation

Each equation of the system defines linear function

$$y = -\frac{a_1}{b_1}x + \frac{c_1}{b_1}, \quad y = -\frac{a_2}{b_2}x + \frac{c_2}{b_2}.$$

A solution of the system is the intersection point of their graphs.

Case 1. These two graphs have different slopes thus they have one intersection point.

Case 2. These two graphs have equal slopes but different y-intercepts thus they are parallel.

Case 3. These two graphs have equal slopes and the same y-intercepts thus they coincide.

Exercises

Solve these systems using all 3 methods, give the suitable graphical interpretation for

1. $\begin{cases} 2x - y = 5 \\ x + y = 4 \end{cases}$ 2. $\begin{cases} 2x - y = 5 \\ 4x - 2y = 4 \end{cases}$ 3. $\begin{cases} 2x - y = 5 \\ 4x - 2y = 10 \end{cases}$

4. Give examples of systems with (a) one solution, (b) no solutions, (c) infinitely many solutions.

5. Find the values of k and c for which the system $\begin{cases} 2x + ky = 8 \\ 4x + 8y = c \end{cases}$ is (a) *consistent* (i.e. it has solutions), (b) *inconsistent* (i.e. it has no solutions).

General form of m linear equations with n variables

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = c_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = c_2 \\ \dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = c_m. \end{cases}$$

Three ingredients of the system

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} \quad c = \begin{pmatrix} c_1 \\ c_2 \\ \dots \\ c_m \end{pmatrix}$$

A matrix of the system of order $m \times n$, x vector of variables of order $n \times 1$, c vector of constants of order $m \times 1$.

Matrix Algebra

Matrix operations

1. Addition

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{pmatrix}$$

Matrix Algebra

Null matrix

$$O = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

For an arbitrary matrix A one has $O + A = A + O = A$.

Matrix Algebra

2. Scalar multiplication

$$k \cdot \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = \begin{pmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ \dots & \dots & \dots & \dots \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{pmatrix} =$$

3. Multiplication of matrixes

$$\begin{pmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & \dots & b_{1j} & \dots & b_{1k} \\ \dots & \dots & \dots & \dots & \dots \\ b_{i1} & \dots & b_{ij} & \dots & b_{ik} \\ \dots & \dots & \dots & \dots & \dots \\ b_{n1} & \dots & b_{nj} & \dots & b_{nk} \end{pmatrix} =$$

$$\begin{pmatrix} c_{11} & \dots & c_{12} & \dots & c_{1k} \\ \dots & \dots & \dots & \dots & \dots \\ c_{i1} & \dots & c_{ij} & \dots & c_{ik} \\ \dots & \dots & \dots & \dots & \dots \\ c_{m1} & \dots & c_{mj} & \dots & c_{mk} \end{pmatrix}$$

where

$$c_{ij} = a_{i1}b_{1j} + \dots + a_{ik}b_{kj} + \dots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

Transpose of a matrix

Transpose of a $m \times n$ matrix A is the $n \times m$ matrix A^T whose i-th column is is the i-th row of A.

For example the transpose for $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ is $A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$.

Properties of transposes

1. $(A^T)^T = A$;
2. $(A + B)^T = A^T + B^T$;
3. $(A \cdot B)^T = B^T \cdot A^T$.

Matrix Algebra

Multiplication matrix × column vector

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{pmatrix}$$

thus the system can be written in matrix form as

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \dots \\ c_n \end{pmatrix}$$

or

$$A \cdot x = c.$$

Matrix Algebra

Matrix multiplication is not commutative:

$$Let \quad A = \begin{pmatrix} 2 & 0 \\ 3 & 8 \end{pmatrix}, \quad B = \begin{pmatrix} 7 & 2 \\ 6 & 3 \end{pmatrix}.$$

$$Then \quad AB = \begin{pmatrix} 14 & 4 \\ 69 & 30 \end{pmatrix} \neq BA = \begin{pmatrix} 20 & 16 \\ 21 & 24 \end{pmatrix}.$$

Matrix Algebra

We introduce the *identity* or *unit* matrix of dimension n I_n as

$$I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

Note that I_n is always a square $[n \times n]$ matrix (further on the subscript n will be omitted). I_n has the following properties:

a) $AI = IA = A$,

b) $AIB = AB$ for all A, B.

In this sense the identity matrix corresponds to 1 in the case of scalars.

The inverse matrix A^{-1} is defined as $A^{-1}A = AA^{-1} = I$

Matrix Algebra

If a matrix A has inverse A^{-1}, then it solves a system of linear equations $A \cdot x = c$:

multiplying both sides of this equation by A^{-1} from the left we obtain

$$A^{-1} \cdot (A \cdot x) = A^{-1} \cdot c, \quad (A^{-1} \cdot A) \cdot x = A^{-1} \cdot c, \quad I \cdot x = A^{-1} \cdot c, \quad x = A^{-1} \cdot c.$$

But not all matrices have inverse, for example

$$\begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$$

Special Kinds of Matrices

Bellow k denotes the number of rows and n denotes the number of columns.

Square matrix. $k = n$. **Example** $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Column matrix. $n = 1$. **Example** $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Row matrix. $k = 1$. **Example** $\begin{bmatrix} 1 & 2 \end{bmatrix}$

Special Kinds of Matrices

A diagonal matrix *is a square matrix whose only non-zero elements appear on the principle (or main) diagonal.*

Example.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

A triangular matrix *is a square matrix which has only zero elements above or below the principle diagonal.*

Example.

$$\begin{pmatrix} 1 & 2 & 5 & 7 \\ 0 & 2 & 3 & 6 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 5 & 2 & 0 & 0 \\ 8 & 6 & 3 & 0 \\ 10 & 9 & 7 & 4 \end{pmatrix}$$

Special Kinds of Matrices

Symmetric matrix. $a_{ij} = a_{ji}$, equivalently $A^T = A$.

Example

$$\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$$

Antisymmetric $\mathbf{A = -A^T}$

Example

$$\begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}$$

Special Kinds of Matrices

Nilpotent matrix. $k = n$ and $A^n = 0$ for some positive integer n.

Example $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Permutation matrix. $k = n$ and each row and each column contains exactly one 1 and all other entries are 0.

Example $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

Orthogonal matrix. $AA^T = I$.

Example $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

Matrix Algebra

Algebra of Square Matrices

The sum, difference, product of square $n \times n$ matrices is $n \times n$ again, besides the $n \times n$ identity matrix I is true multiplicative identity

$$I \cdot A = A \cdot I = A.$$

So the set of all $n \times n$ matrices M_n carries algebraic structure similar to that of real numbers R. But there are some differences:

Matrix Algebra

1. Multiplication in M_n is not commutative: generally $A \cdot B \neq B \cdot A$.

2. M_n has zero divisors: there exist nonzero matrices $A,\ B \in M_n$ such that $A \cdot B = O$.

3. Not all nonzero matrices have inverse.

Example $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

A zero divisor can not have an inverse.

Matrix Algebra

Inverse Matrix

Definition. A matrix $A \in M_n$ is called invertible there exists its inverse, a matrix A^{-1} such that

$$A \cdot A^{-1} = A^{-1} \cdot A = I.$$

Properties of inverse matrix:

1. $(A^{-1})^{-1} = A$;
2. $(cA)^{-1} = 1/c\ A^{-1}$;
3. $(AB)^{-1} = B^{-1}A^{-1}$;
4. $(A^T)^{-1} = (A^{-1})^T$.

Matrix Algebra

Solving Systems Using Inverse

As we know each system of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = c_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = c_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = c_m. \end{cases}$$

can be written in matrix form $A \cdot x = c$.

Theorem. If A is invertible then the system of linear equations $A \cdot x = c$ has the unique solution given by $x = A^{-1} \cdot c$.

Proof.

$$A \cdot x = c \;\Rightarrow\; A^{-1} \cdot (A \cdot x) = A^{-1} \cdot c \;\Rightarrow\; (A^{-1} \cdot A) \cdot x = A^{-1} \cdot c \;\Rightarrow\; I \cdot x = A^{-1} \cdot c \;\Rightarrow\; x = A^{-1} \cdot c.$$

Matrix Algebra

Exercises

Let

$$A=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix},\quad B=\begin{pmatrix} b_{11} & b_{12} & \dots & b_{1q} \\ b_{21} & b_{22} & \dots & b_{2q} \\ \dots & \dots & \dots & \dots \\ b_{p1} & b_{p2} & \dots & b_{pq} \end{pmatrix},\quad x=\begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_r \end{pmatrix}$$

1. Find the values of m, n, p, q, r for which exist the products, find the dimensions of these products when they exist:
(a) $A \cdot B$, (b) $B \cdot A$, (c) $B^T \cdot A^T$, (d) $A^T \cdot B^T$, (e) $A \cdot B^T$,
(f) $A \cdot x$, (g) $A \cdot x^T$, (h) $x \cdot A$, (i) $x^T \cdot A$ (j) $x \cdot x^T$, (k) $x^T \cdot x$, (l) $x \cdot x$, (m) $x^T \cdot x^T$.

Matrix Algebra

Let

$$A=\begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \end{pmatrix},\quad B=\begin{pmatrix} 0 & 1 & -1 \\ 4 & -1 & 2 \end{pmatrix},\quad C=\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix},$$

$$D=\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},\quad \text{and}\quad E=\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

a) Compute each of the following matrices if it is defined:

$A+B$,	$A-D$,	$3B$,	DC,	B^T,	A^TC^T,
$C+D$,	$B-A$	AB,	CE,	$-D$,	$(CE)^T$,
$B+C$,	$D-C$,	CA,	EC,	$(CA)^T$,	E^TC^T.

b) Verify that $(DA)^T = A^TD^T$.
c) Verify that $CD \neq DC$.

Check that

$$\begin{pmatrix} 2 & 3 & 1 & 4 \\ 0 & -1 & 2 & 1 \\ 5 & 0 & 6 & 0 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}=\begin{pmatrix} 5 & 11 \\ 2 & 3 \\ 10 & 21 \end{pmatrix}.$$

Note that the reverse product is not defined.

Determinant

Bellow w'll study the central question: *which additional conditions must satisfy a quadratic matrix A to be invertible, that is to have A^{-1}?*

Determinant

There is a function which assigns to an $n \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

the real number denoted as

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

or $det\ A$, called **determinant** of A which has the properties described bellow.

Determinant

Properties of Determinant

1.

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{i1}+b_{i1} & a_{i2}+b_{i2} & \dots & a_{in}+b_{in} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} =$$

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ b_{i1} & b_{i2} & \dots & b_{in} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

Determinant

2. If B is obtained from A by multiplying of each entry of row i by a scalar r then $|B| = r \cdot |A|$.

3. If a matrix B is obtained by interchanging two rows of A then $|B| = -|A|$.

4. $|I| = 1$;

5. If two rows of A equal then $|A| = 0$ (prove it using 3).

6. If a matrix A has an all-zero row then $|A| = 0$ (prove it using 2).

Determinant

7. Transform matrix A to matrix B by performing the *elementary row operation* of adding r times row i to row j of A to form row j of B (the other rows remain the same), then $|B| = |A|$ (prove it using 1,2,5).

8. $|A \cdot B| = |A| \cdot |B|$;

9. $|A^{-1}| = |A|^{-1}$ (prove it using 4,7).

10. $|A^T| = |A|$.

Since of the property 10 all the properties remain correct if we replace *row* by *column*.

The formal definition of the determinant is as follows: given $n \times n$ matrix $A = (a_{ij})$,

$$\det(A) = \sum_{(\alpha_1,\dots,\alpha_n)} (-1)^{I(\alpha_1,\dots,\alpha_n)} a_{1\alpha_1} \cdot a_{2\alpha_2} \cdot \ldots \cdot a_{n\alpha_n}$$

where $(\alpha_1, \dots, \alpha_n)$ are all different permutations of $(1, 2, \dots, n)$, and $I(\alpha_1, \dots, \alpha_n)$ is the number of inversions.

Usually we denote the determinant of A as $\det(A)$ or $|A|$.

The *inductive* definition of determinant will be given bellow.

Determinant

Minors and Cofactors

For an $n \times n$ matrix A let A_{ij} be the $(n-1) \times (n-1)$ submatrix obtained by deleting the i-th row and j-th column. The determinant of this matrix $M_{ij} = |A_{ij}|$ is called (i, j)-th *minor* of A and $C_{ij} = (-1)^{i+j} M_{ij}$ is called (i, j)-th

For example for $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ we have

$$A_{21} = \begin{pmatrix} 2 & 3 \\ 8 & 9 \end{pmatrix}, \qquad M_{21} = 2 \cdot 9 - 8 \cdot 3 = -6,$$

$$C_{21} = (-1)^{2+1}(-6) = (-1)^3(-6) = -(-6) = 6.$$

Laplas Expansion - Inductive Definition of Determinant

For a matrix $A = \begin{pmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{pmatrix}$ the determinant $|A|$ can be calculated by i-th row expansion

$$|A| = a_{i1} \cdot C_{i1} + a_{i2} \cdot C_{i2} + \dots + a_{in} \cdot C_{in} = \sum_{k=1}^{n} a_{ik} \cdot C_{ik}$$

or by j-th column expansion

$$|A| = a_{1j} \cdot C_{1j} + a_{2j} \cdot C_{2j} + \dots + a_{nj} \cdot C_{nj} = \sum_{k=1}^{n} a_{kj} \cdot C_{kj}.$$

All row expansions as well as all column expansions give the *same result*, so Laplas expansion can be used as an *inductive* definition of determinant.

Determinant

Determinant of a 3×3 matrix

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$a_{11} \cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \cdot \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \cdot \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{33} \end{vmatrix} =$$

$$\begin{aligned} &(a_{11} \cdot a_{22} \cdot a_{33} - a_{11} \cdot a_{23} \cdot a_{32}) - \\ &(a_{12} \cdot a_{21} \cdot a_{33} - a_{12} \cdot a_{23} \cdot a_{31}) + \\ &(a_{13} \cdot a_{21} \cdot a_{33} - a_{13} \cdot a_{22} \cdot a_{31}) = \\ &(a_{11} \cdot a_{22} \cdot a_{33} + a_{12} \cdot a_{23} \cdot a_{31} + a_{13} \cdot a_{21} \cdot a_{32}) - \\ &(a_{13} \cdot a_{22} \cdot a_{31} + a_{11} \cdot a_{23} \cdot a_{32} + a_{12} \cdot a_{21} \cdot a_{33}). \end{aligned}$$

Inverse Matrix

The inverse A^{-1} exists if and only if A is nonsingular, i.e. $|A| \neq 0$.

The inverse is given by

$$A^{-1} = \begin{pmatrix} \frac{C_{11}}{|A|} & \cdots & \cdots & \frac{C_{n1}}{|A|} & \\ \cdots & \cdots & \cdots & & \\ \frac{C_{1i}}{|A|} & \cdots & \cdots & \frac{C_{ni}}{|A|} & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{C_{1n}}{|A|} & \cdots & \cdots & \frac{C_{nn}}{|A|} & \end{pmatrix}$$

Cramer's Rule

For a system of n linear equations with n variables

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = c_1 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = c_n \end{cases}$$

we define $n+1$ matrixes $A,\ A_1,\ A_2,\ \ldots,,\ A_n$:

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}, \quad A_k = \begin{pmatrix} a_{11} & \ldots & c_1 & \ldots & a_{1n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & \ldots & c_n & \ldots & a_{nn} \end{pmatrix}$$

here A_k is obtained by replacing in A the k-th column by the column of constants c.

Theorem 3 *(Cramer's Rule) Let A be a non-singular matrix i.e. $|A| \neq 0$. Then the system $A \cdot x = c$ has unique solution given by*

$$x_k = \frac{|A_k|}{|A|}, \quad k = 1, 2, \ldots, n \quad .$$

Rank of a Matrix

Definition of rank

The rank of a matrix is *maximum order of nonzero determinant that can be constructed from the rows and columns of that matrix.*

Example.

$$rank\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = 2$$

Rank

How to calculate the rank

By definition the rank of a matrix A is r if there exists nonzero minor of degree r but *all minors of higher degrees* are zero.

In fact there is no need to check *all higher minors*:

Theorem *If in a matrix A there exists nonzero minor M of degree r and all minors bordering it (that is, minors of an order higher by one and containing it) are equal to zero then* ***rank A=r****.*

Rank

Example. Let us calculate *rank A* for $A = \begin{pmatrix} 1 & 4 & 17 & 4 \\ 2 & 12 & 46 & 10 \\ 3 & 18 & 69 & 17 \end{pmatrix}$.

The minor $|a_{11}| = 1$ is nonzero, so the *rank A* is at last 1. Now take the 2×2 minor

$$\begin{vmatrix} 1 & 4 \\ 2 & 12 \end{vmatrix}$$

bordering the previous nonzero minor. It is equal to $1 \cdot 12 - 2 \cdot 4 = 8 \neq 0$, so *rank A* is at last 2.

Rank

Next we take the 3×3 minor

$$\begin{pmatrix} 1 & 4 & 17 \\ 2 & 12 & 46 \\ 3 & 18 & 69 \end{pmatrix}.$$

bordering the previous one. Calculation shows that it is zero, so this is bad choice. Let us try another 3×3 minor bordering previous nonzero 2×2 minor

$$\begin{vmatrix} 1 & 4 & 4 \\ 2 & 12 & 410 \\ 3 & 18 & 17 \end{vmatrix}.$$

Calculation shows that this minor is equal to 8.There are no larger minors in A, so this is a basic minor and $rank\ A = 3$.

Rank

Criterion of Consistence (Cronecer-Capelly Theorem)

Theorem *A linear system $A \cdot X = c$ is consistent if and only if the rank of the matrix A equals to the rank of augmented matrix $A|c$:*

$$rank \begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \dots & \dots & \dots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} = rank \begin{pmatrix} a_{11} & \dots & a_{1n}|c_1 \\ a_{21} & \dots & a_{2n}|c_2 \\ \dots & \dots & \dots \\ a_{m1} & \dots & a_{mn}|c_m \end{pmatrix}.$$

Solution of Consistent Systems

Suppose $rank(A) = rank(A|c) = r$. We can assume that the nonzero minor of degree r (*the basic minor*) is $M_{(1,2,\ldots,r);(1,2,\ldots,r)}$.

In this case the $(r+1)$-th, $(r+2)$-th, ... , m-th equations are linear combinations of first r equations, so they can be ignored.

The first r equations we write in the form

$$\begin{cases} a_{11}x_1 + \ldots + a_{1r}x_r = c_1 - (a_{1r+1}x_{r+1} + \ldots + a_{1n}x_n) \\ a_{21}x_1 + \ldots + a_{2r}x_r = c_1 - (a_{2r+1}x_{r+1} + \ldots + a_{2n}x_n) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{r1}x_1 + \ldots + a_{rr}x_r = c_1 - (a_{r+1}x_{r+1} + \ldots + a_nx_n). \end{cases}$$

The determinant of this system $M_{(1,2,\ldots,r);(1,2,\ldots,r)}$ is **nonzero**, thus for each values of *free* variables $x_{r+1}, x_{r+2}, \ldots, x_n$ we can find by Cramer's rule unique *basic* variables $x_1, x_2, \ldots, x_n$.

Then $x_1, x_2, \ldots, x_n,\ x_{r+1}, x_{r+2}, \ldots, x_n$ is a solution of our system.

Example. We want to solve the system

$$\begin{cases} x + 4y + 17z + 4t = 38 \\ 2x + 12y + 46z + 10t = 98 \\ 3x + 18y + 69z + 17t = 153 \end{cases}.$$

Write the augmented matrix $(A|c)$ of this system

$$\left(\begin{array}{cccc|c} 1 & 4 & 17 & 4 & 38 \\ 2 & 12 & 46 & 10 & 98 \\ 3 & 18 & 69 & 17 & 153 \end{array}\right).$$

We already know that rank A for $A = \begin{pmatrix} 1 & 4 & 17 & 4 \\ 2 & 12 & 46 & 10 \\ 3 & 18 & 69 & 17 \end{pmatrix}$ is 3.

Augmentation of A by c can not increase the rank, so the rank of $(A|c)$ is also 3, thus the system is consistent.

So we have one free variable z and 3 basic variables x, y, t.

Next we rewrite the system so that the *basic minor* becomes the determinant of system

$$\begin{cases} x + 4y + 4t = 38 - 17z \\ 2x + 12y + 10t = 98 - 46z \\ 3x + 18y + 17t = 153 - 69z \end{cases}$$

and solve it by Cramer's rule:

Next we rewrite the system so that the *basic minor* becomes the determinant of system

$$\begin{cases} x + 4y + 4t = 38 - 17z \\ 2x + 12y + 10t = 98 - 46z \\ 3x + 18y + 17t = 153 - 69z \end{cases}$$

and solve it by Cramer's rule:

Rank

$$x = \frac{\Delta_x}{\Delta} = \frac{\begin{vmatrix} 38-17z & 4 & 4 \\ 98-46z & 12 & 410 \\ 153-69z & 18 & 17 \end{vmatrix}}{\begin{vmatrix} 1 & 4 & 4 \\ 2 & 12 & 10 \\ 3 & 18 & 17 \end{vmatrix}} = \frac{80-40z}{8} = 10-5z,$$

$$y = \frac{\Delta_x}{\Delta} = \frac{\begin{vmatrix} 1 & 38-17z & 4 \\ 2 & 98-46z & 410 \\ 3 & 153-69z & 17 \end{vmatrix}}{\begin{vmatrix} 1 & 4 & 4 \\ 2 & 12 & 10 \\ 3 & 18 & 17 \end{vmatrix}} = \frac{32-24z}{8} = 4-3z,$$

$$t = \frac{\Delta_x}{\Delta} = \frac{\begin{vmatrix} 1 & 4 & 38-17z \\ 2 & 12 & 98-46z \\ 3 & 18 & 153-69z \end{vmatrix}}{\begin{vmatrix} 1 & 4 & 4 \\ 2 & 12 & 10 \\ 3 & 18 & 17 \end{vmatrix}} = \frac{24}{8} = 3.$$

So the solution is

$$x = 15 - 5z, \quad y = 4 - 3z, \quad z, \quad t = 3.$$

Examples

1. The system $\begin{cases} x+y=5 \\ x-y=1 \\ 2x-2y=2 \end{cases}$ has unique solution.

2. The system $\begin{cases} x+y=5 \\ x-y=1 \\ 2x-2y=3 \end{cases}$ has no solutions.

3. The system $\begin{cases} x+y=5 \\ 2x+2y=10 \\ 3x+3y=15 \end{cases}$ has infinitely many solutions.

Explain why?

Exercises

1. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & ... & b_{1q} \\ b_{21} & b_{22} & ... & b_{2q} \\ ... & ... & ... & ... \\ b_{p1} & b_{p2} & ... & b_{pq} \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_r \end{pmatrix}$$

Find the values of m, n, p, q, r for which exist the following products, and find the dimensions of these products when they exist:
(a) $A \cdot B$, (b) $B \cdot A$, (c) $B^T \cdot A^T$, (d) $A^T \cdot B^T$, (e) $A \cdot B^T$,
(f) $A \cdot x$, (g) $A \cdot x^T$, (h) $x \cdot A$, (i) $x^T \cdot A$ (j) $x \cdot x^T$, (k) $x^T \cdot x$, (l) $x \cdot x$, (m) $x^T \cdot x^T$.

2. Is the product of two symmetric matrices symmetric ?

3. (a) There are only two 2×2 permutation matrices and both are symmetric. Is it true that any 3×3 permutation matrix is also symmetric?

4. Evaluate the following determinants

$$(a) \begin{pmatrix} 8 & 1 & 3 \\ 4 & 0 & 1 \\ 6 & 0 & 3 \end{pmatrix}. \quad (b) \begin{pmatrix} 1 & 2 & 3 \\ 4 & 7 & 5 \\ 3 & 6 & 9 \end{pmatrix}. \quad (c) \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}.$$

$$(d) \begin{pmatrix} 1 & 2 & 0 & 9 \\ 2 & 3 & 4 & 6 \\ 1 & 6 & 0 & -1 \\ 0 & -5 & 0 & 8 \end{pmatrix}. \quad (e) \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 3 & 6 & -5 \\ 0 & 4 & 0 & 0 \\ 9 & 6 & -1 & 8 \end{pmatrix}.$$

5. Calculate the determinant of lower-triangular 4×4 matrix

$$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}.$$

7. Suppose $|A| = a$. Find $| - A|$.

9. What can you say about the determinant of a permutation matrix?

10. Calculate the determinant of upper-triangular 4×4 matrix.

11. Check that $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$.

12. Find A^{-1} for (a) $A = \begin{pmatrix} 4 & 5 \\ 4 & 2 \end{pmatrix}$. $(b) A = \begin{pmatrix} 2 & 4 & 0 \\ 4 & 6 & 3 \\ -6 & -10 & 0 \end{pmatrix}$.

13. Invert the coefficient matrix to solve the following systems

$$(a) \begin{cases} 2x_1 + x_2 = & 5 \\ x_1 + x_2 = & 3 \end{cases} \qquad (b) \begin{cases} 2x_1 + & 4x_2 & & = 2 \\ 4x_1 + & 6x_2 + & 3x_3 & = 1 \\ -6x_1 - & 10x_2 & & = 60 \end{cases}$$

14. Solve the system

$$\begin{pmatrix} 2x & + & 3y & + & 3z & = & 2 \\ 2x & + & 2y & + & z & = & 5 \\ x & + & y & + & z & = & 14 \end{pmatrix}$$

inverting the coefficient matrix.

15. What is the inverse of the 3×3 diagonal matrix $\begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}$.

16. Show that the inverse of 2×2 upper-triangular matrix is upper-triangular.

17. Show that the inverse of 3×3 lower-triangular matrix is lower-triangular.

18. Show that the inverse of 2×2 symmetric matrix is symmetric.

19. Find numbers a and b that make A the inverse of B when

$$A = \begin{pmatrix} 2 & -1 & -1 \\ a & \frac{1}{4} & b \\ \frac{1}{8} & \frac{1}{8} & \frac{-1}{8} \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 6 \\ 1 & 3 & 2 \end{pmatrix}.$$

20. Prove that if all entries of A are all integers and $det\ A = \pm 1$ then the entries of A^{-1} are also integers.

21. Calculate the rank of each of the following matrixes

$$(a) \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}. \quad (b) \begin{pmatrix} 2 & -4 & 2 \\ -1 & 2 & 1 \end{pmatrix}. \quad (c) \begin{pmatrix} 1 & 6 & -7 & 3 \\ 1 & 9 & -6 & 4 \\ 1 & 3 & -8 & 4 \end{pmatrix}.$$

$$(d) \begin{pmatrix} 1 & 6 & -7 & 3 & 5 \\ 1 & 9 & -6 & 4 & 9 \\ 1 & 3 & -8 & 4 & 2 \\ 2 & 15 & -13 & 11 & 16 \end{pmatrix}. \quad (e) \begin{pmatrix} 1 & 6 & -7 & 3 & 1 \\ 1 & 9 & -6 & 4 & 2 \\ 1 & 3 & -8 & 4 & 5 \end{pmatrix}.$$

22. Solve the system whose coefficient matrix is $\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$ and augmented matrix is $\begin{pmatrix} 2 & -4 & 2 \\ -1 & 2 & 1 \end{pmatrix}$.

23. Solve the system whose coefficient matrix is $\begin{pmatrix} 1 & 6 & -7 & 3 \\ 1 & 9 & -6 & 4 \\ 1 & 3 & -8 & 4 \end{pmatrix}$ and augmented matrix is $\begin{pmatrix} 1 & 6 & -7 & 3 & 1 \\ 1 & 9 & -6 & 4 & 2 \\ 1 & 3 & -8 & 4 & 5 \end{pmatrix}$.

24. For the system

$$\begin{cases} x+ & 2y+ & z- & w = 3 & 1 \\ 3x+ & 6y- & z- & 3w = & 2 \end{cases}$$

(a) determine how many variables can be endogenous, (b) determine a successful separation into exogenous and endogenous variables, (c) find an explicit formula for the endogenous variables in terms of exogenous variables.

25. For

$$\begin{pmatrix} w - x + 3y - z = 0 \\ w + 4x - y + z = 3 \\ 3w + 7x + y + z = 6 \\ 3w + 2x + 5y - z = 3 \end{pmatrix}$$

(a) Check the consistence;
(b) Separate free and basic variables;
(c) Solve the system.

26. Compose a system with 3 variables and 4 equations with
(a) No solution;
(b) One solution;
(c) Infinitely many solutions depending on one free variable;
(d) Infinitely many solutions depending on two free variables.