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Homotopy types for “gros” toposes

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Introduction

In this thesis we are investigating possibility of assigning homotopical invariants to toposes in an alternative way.

The 2-category of (Grothendieck) toposes, geometric morphisms and their natural transformations (which we will denote by $\mathfrak{Top}$) has been used by many authors to model homotopy types particularly efficiently. Thus for example a topological space X can be represented in $\mathfrak{Top}$ via its category of set-valued sheaves $\mathrm{Shv}(X)$. Moreover a small category $\mathbb{C}$ can be represented, according to convenience, either via the category $\mathrm{Shv}(\mathrm{B}\mathbb{C})$ of sheaves on the geometric realization of its nerve, or just by the category $\mathrm{Set}^{\mathbb{C}^{\mathrm{op}}}$ of set-valued presheaves on $\mathbb{C}$. There are well known ways to read off homotopical and homological invariants, such as the fundamental group or (co)homology with various coefficients, from these representations.

However approximately one half of $\mathfrak{Top}$ stays apparently useless from this point of view. It is well known, for example, that as soon as a small category is filtered or cofiltered, its classifying space is homotopically trivial (see e. g. [29]). This seemingly rules out all toposes of presheaves on, say, categories with finite limits. In fact, there is a problem of interpretation here. Usually one approximates the notion of the set of homotopy classes of maps $[\mathbf{X}, \mathbf{Y}]$ between toposes by the set of connected components of the category $\mathfrak{Top}(\mathbf{X}, \mathbf{Y})$. But how to interpret the situations when this category is not small? And in $\mathfrak{Top}$ this is by no means an exception. There is in fact a distinctive dichotomy between s. c. "petit" and "gros" toposes, pointed out from the very beginning of topos theory. Unfortunately there still does not exist a precise definition of these classes, but there are lots of definite examples of both kinds, and it is generally agreed that $\mathfrak{Top}(\mathbf{X}, \mathbf{Y})$ cannot be expected small for "gros" $\mathbf{Y}$.

A leading example for us will be this: any finitary algebraic theory $\mathbb{T}$ has a *classifying topos* $[\mathbb{T}]$ which represents the 2-functor

$$\mathbb{T}\text{-}\mathbf{mod}(\_) : \mathfrak{Top}^{\mathrm{op}} \to \mathscr{Cat}$$

to (large) categories which sends a topos $\mathbf{X}$ to the category of $\mathbb{T}$-models in $\mathbf{X}$. Thus $\mathfrak{Top}(\mathbf{X}, [\mathbb{T}])$ is not a small category for most $\mathbf{X}$ and $\mathbb{T}$. In fact, $[\mathbb{T}]$ can be taken to be the category of set-valued functors on the category of finitely presented $\mathbb{T}$-models (in sets), so in any case it must be considered "contractible" from the traditional point of view.

On the other hand, there is a growing evidence that one can assign meaningful cohomological invariants to toposes such as $[\mathbb{T}]$. In fact there are concrete classification problems which lead to such cohomology groups. To the author's knowledge, first such problems were explicitly described in [4], where the s. c. Hochschild-Mitchell cohomology was pointed out as the appropriate cohomology theory, and its extension to more general coefficients called *natural systems* was constructed.

To authors of [4], the main interest was in describing various subcategories of the homotopy category. Soon afterwards T. Pirashvili and the author performed some calculations in a number of purely algebraic situations (see [7]) where this approach also gives interesting results. In [15], the starting point of investigation has been a cohomology theory for associative rings introduced by MacLane in late fifties (see [24] and [23]).

As for many other cohomology theories for algebraic systems, one of the main motivations for introducing MacLane cohomology was classification of certain extensions of associative rings. For algebras over a field, such classification problems are successfully dealt with by Hochschild cohomology. In particular, for an algebra A and an A-A-bimodule B, elements of the second Hochschild cohomology group $H^2_{\mathrm{Hoch}}(A;B)$ are in one-to-one correspondence with isomorphism classes of extensions of the form $B \rightarrowtail X \twoheadrightarrow A$ where $X \twoheadrightarrow A$ is a surjective homomorphism of algebras with kernel a square zero ideal, isomorphic to B when equipped with the A-A-bimodule structure induced by the extension. In a sense, success of this classification of extensions by H^2 depends crucially on the fact that the category of vector spaces over the field is semisimple, i. e. every short exact sequence splits. Extending Hochschild cohomology to algebras over more general commutative rings K in a straightforward way yields less satisfactory cohomologies, in the sense that they can only classify these extensions of algebras which split as extensions of K-modules. On the other hand, the appropriate cohomology theory for the classification of non-split extensions of K-modules is provided by Ext groups Ext_K; for any K-modules M and N, elements of the group $\mathrm{Ext}^n_K(M,N)$ correspond bijectively to equivalence classes of exact sequences of K-modules of the form

$$N \rightarrowtail X_1 \to \cdots \to X_n \twoheadrightarrow M,$$

for each $n \geqslant 1$. Thus a more appropriate cohomology theory of K-algebras should somehow combine Hochschild cohomology with Ext_K.

MacLane approached this problem using the so called *cubical construction* Q_* introduced previously by Eilenberg and MacLane in [9] for calculation of stable homologies of Eilenberg-MacLane spaces $K(A,n)$ (topological spaces with a single nontrivial homotopy group A in dimension n). More precisely, Eilenberg and MacLane assign to any abelian group A a chain complex $Q_*(A)$ which satisfies

$$H_m(K(A,n)) \cong H_{m-n}(Q_*(A)) \text{ for any } n \leqslant m < 2n.$$

The construction Q_* is functorial and lax monoidal with respect to tensor products, i. e. there are well-behaved maps of complexes

$$Q_*(A) \otimes Q_*(B) \to Q_*(A \otimes B);$$

in particular, for a bimodule B over a ring A, there is a natural differential graded (DG) ring structure on $Q_*(A)$, and moreover $Q_*(B)$ is a DG-bimodule over it. Moreover there is an augmentation $Q_*(A) \to A$ which is a DG-ring homomorphism when A is a ring considered as a DG ring concentrated in degree 0. Equipping A with a $Q_*(A)$-$Q_*(A)$-bimodule structure via this augmentation, MacLane considers the corresponding two-sided bar construction, i. e. the total complex $\mathbf{B}(A, Q_*(A), A)$ of the bicomplex

$$\mathbf{B}_{m,n}(A, Q_*(A), A) = \left(A \otimes \underbrace{Q_*(A) \otimes ... \otimes Q_*(A)}_{n \text{ times}} \otimes A \right)_m .$$

MacLane cohomology of the ring A with coefficients in the A-A-bimodule B is then defined by

$$H^n_{\mathrm{ML}}(A; B) \cong H^n(\mathrm{Hom}_{A \otimes A^\circ}(\mathbf{B}(A, Q_*(A), A), B)).$$

This yielded satisfactory results for classification of extensions. In particular, MacLane has been able to prove that elements of $H^2_{\mathrm{ML}}(A; B)$ are in a natural one-to-one correspondence with equivalence classes of *arbitrary*, i. e., not necessarily split, extensions $B \rightarrowtail X \twoheadrightarrow A$, given by embedding B into a ring X as a square zero ideal with quotient A and matching bimodule structure. Soon afterwards Shukla in [34] proposed an alternative approach to cohomology of rings, with more clear motivation with respect to existing general principles of approach to cohomology known by that time. Namely the Shukla cohomology $H^*_{\mathrm{Sh}}(A; B)$ of a ring A with coefficients in an A-A-bimodule B is an instance of the cotriple cohomology of Barr and Beck [1] applied to the free ring cotriple induced by the monadic forgetful functor from rings to sets. Moreover the general approach of Quillen from [28] to the construction of cohomology in "sufficiently nice" categories also yields Shukla cohomology when applied to the category of rings. The groups $H^*_{\mathrm{Sh}}(A;B)$ are in general different from the MacLane cohomology groups; however there is an isomorphism up to dimension 2, so that in particular H^2_{Sh} classifies extensions just as well as H^2_{ML}.

There is however one important feature of the Hochschild cohomology which was seemingly lost in both MacLane and Shukla's generalizations: since for algebras over a field one has

$$H^*_{\mathrm{Hoch}}(A;B) \cong \mathrm{Ext}^*_{A\otimes A^\circ}(A,B),$$

Hochschild cohomology groups in this particular case can be characterized by a certain universal property with respect to the second argument. Namely, they form a universal exact connected sequence of functors. It seems impossible to retain this property in the general case, since for algebras over general commutative rings the groups $\mathrm{Ext}^*_{A\otimes A^\circ}(A,B)$ do not seem to be related to algebra extensions in any evident way; in particular they are not isomorphic to either of H^*_{Sh} or H^*_{ML}.

It was T. Pirashvili who first observed that calculations from [7] suggest a possibility to express MacLane cohomology groups using cohomology of categories introduced by Baues and Wirsching in [4] in such a way that they also will form a universal connected sequence of functors. Namely, Pirashvili has been able to construct, for each ring A, a naturally defined abelian category $\mathscr{F}(A)$ containing the category of A-A-bimodules as a full subcategory, and prove that there are natural isomorphisms

$$H^*_{\mathrm{ML}}(A;B) \cong \mathrm{Ext}^*_{\mathscr{F}(A)}(A,B).$$

In fact $\mathscr{F}(A)$ can be taken to be the category $A\text{-}\mathbf{mod}^{\mathbb{M}_A}$ of all functors from the category $\mathbb{M}_A$ of finitely generated free A-modules to the category A-**mod** of all A-modules. One might identify the full subcategory of $\mathscr{F}(A)$ consisting of additive functors with the category of A-A-bimodules, as any such additive functor F is canonically isomorphic to the functor $B\otimes_A {}_-$, where B is $F(A)$, equipped with the evident A-A-bimodule structure. In particular, A itself, considered as an A-A-bimodule, corresponds in this way to the inclusion $\mathbb{M}_A \to A\text{-}\mathbf{mod}$ of finitely generated free modules into all modules.

Later it turned out (see [27]) that this approach yields, in particular, an alternative definition of *topological Hochschild homology* for discrete rings, which is quite useful as the latter homology is quite difficult to calculate in general.

It has been pointed out in [15] that this approach can be also used to provide a natural generalization of MacLane cohomology, retaining all the desirable properties one should ask of a "decent" cohomology theory. First, evidently one might define, for a ring A and any functor $F : \mathbb{M}_A \to A$-**mod**, cohomology groups of A with coefficients in F by the same formula as above,

$$H^*(A; F) := \operatorname{Ext}^*_{\mathscr{F}(A)}(A, F).$$

Thus in these terms one has

$$H^*_{\mathrm{ML}}(A; B) = H^*(A; B \underset{A}{\otimes} \_),$$

for an A-A-bimodule B. A natural question then arises: one knows that elements of $H^2_{\mathrm{ML}}(A; B)$ correspond to ring extensions of A by B; is there a corresponding notion of extension which would be similarly classified by elements of $H^2(A; F)$, for an arbitrary object of $\mathscr{F}(A)$? To answer this question, one is led to a generalization of second kind.

Each ring A gives rise to a certain algebraic theory, namely, the theory $\mathbb{T}_A$ of (say, left) A-modules. Basically, this is just the category which is opposite to the category of finitely generated free A-modules. It has been proved in [15] that to an object F of $\mathscr{F}(A)$ one might assign a certain natural system, in the sense of [4], on $\mathbb{T}_A$, $\tilde{F}$, in such a way that there is an isomorphism

$$\operatorname{Ext}^*_{\mathscr{F}(A)}(A, F) \cong H^*(\mathbb{T}_A; \tilde{F}),$$

the groups on the right being the cohomology groups of the category $\mathbb{T}_A$ studied in [4]. In particular, results of [4] yield an interpretation of elements of $H^2(\mathbb{T}_A; \tilde{F})$ as equivalence classes of *linear extensions*

$$\tilde{F} \longmapsto \mathbb{C} \xrightarrow{p} \mathbb{T}_A,$$

in the sense of [4], of $\mathbb{T}_A$ by $\tilde{F}$. And moreover it turns out that for these particular natural systems $\tilde{F}$, in all linear extensions as above $\mathbb{C}$ is also an algebraic theory, and p is a morphism of theories, up to equivalence of extensions.

It thus becomes apparent that there must be a natural notion of cohomology for algebraic theories, with coefficients such as F above, which yields the MacLane cohomology of a ring A with coefficients in a bimodule B when the theory is $\mathbb{T}_A$ and F is $B \otimes_A \_$. Such cohomology groups $H^*(\mathbb{T}; F)$ have been constructed in [15], where $\mathbb{T}$ is an algebraic theory and F is a contravariant functor from $\mathbb{T}$ to the category $\mathrm{Ab}(\mathbb{T}\text{-}\mathbf{mod})$ of internal abelian groups in the category of models of $\mathbb{T}$. The definition was

$$H^*(\mathbb{T}; F) := \operatorname{Ext}_{\mathrm{Ab}(\mathbb{T}\text{-}\mathbf{mod})^{\mathbb{T}^{\mathrm{op}}}}((\_)_{\mathrm{ab}}, F),$$

where $(\_)_{\mathrm{ab}}$ denotes the composite of the Yoneda embedding (which identifies $\mathbb{T}^{\mathrm{op}}$ with the full subcategory of free finitely generated $\mathbb{T}$-models in $\mathbb{T}$-**mod**) with the abelianization functor, i. e. with the left adjoint to the forgetful functor $\mathrm{Ab}(\mathbb{T}\text{-}\mathbf{mod}) \to \mathbb{T}\text{-}\mathbf{mod}$. This choice of coefficients is natural in view of the fact that for a ring A the category $\mathbb{M}_A$ is equivalent to $\mathbb{T}_A^{\mathrm{op}}$, so that for $\mathbb{T} = \mathbb{T}_A$ for a ring A, the category of these coefficients is equivalent to $\mathscr{F}(A)$ above. Moreover this cohomology turned out to combine nice features of both Shukla and Hochschild cohomology. With respect of the first argument it can be obtained, similarly to the Shukla cohomology, as a Barr-Beck cotriple cohomology, this time for the free theory cotriple (recall that algebraic theories are monadic over the category $\mathrm{Set}^{\mathbb{N}}$ of sequences of sets, the forgetful functor sending a theory $\mathbb{T}$ to the sequence whose n-th set is the set of all n-ary operations of $\mathbb{T}$). Moreover it was pointed out in [15] that the proposed categories of coefficients are probably too restrictive. Indeed for many interesting theories $\mathbb{T}$ – for example, for the theory of rings with unit – cohomology with such coefficients is trivial simply because the category $\mathrm{Ab}(\mathbb{T}\text{-}\mathbf{mod})$ is trivial. It was suggested instead to consider the so called *cartesian natural systems* of abelian groups on $\mathbb{T}$ as coefficients. This conforms with the general approach of Quillen in [28] since the category of such natural systems is equivalent to the category $\mathrm{Ab}(\mathscr{T\!h}/\mathbb{T})$ of internal abelian groups in the slice category of theories over $\mathbb{T}$. In this thesis, we adopt that point of view too.

It has been already apparent in [15] that one is dealing with some kind of homotopy type of an algebraic theory. Recently S. Schwede in [31] gave most clear evidence to this by assigning to an algebraic theory a *stable homotopy* invariant in form of a certain Γ-ring.

The present thesis is aimed at giving some further glimpses of that homotopy type. Using methods of [20] we formulate our results in terms of algebraic theories over a given base topos. This is important since it allows to simplify some aspects of [31] just by choosing the base topos to be the category of simplicial sets.

We will thus work over a base topos $\mathbf{S}$ and will be mostly dealing with the classifying toposes $[\mathbb{T}] \to \mathbf{S}$ of $\mathbf{S}$-algebraic theories. Our proposed model for the homotopy type of such $[\mathbb{T}]$ is the comma category $\mathscr{F}(\mathbb{T}) := \mathrm{End}_{\mathfrak{Top}/\mathbf{S}}([\mathbb{T}])/\mathrm{Id}_{[\mathbb{T}]}$. Thus objects of $\mathscr{F}(\mathbb{T})$ are natural transformations $e \to \mathrm{Id}_{[\mathbb{T}]}$, where $e : [\mathbb{T}] \to [\mathbb{T}]$ is a geometric morphism. In particular, we will show that standard cohomological constructions applied to internal abelian groups in $\mathscr{F}(\mathbb{T})$ yield as a particular case the cohomologies introduced in [15]. Accordingly, results of that paper will be reviewed from this point of view. We also indicate how to construct the analog of the Γ-ring introduced in [31] in terms of $\mathscr{F}(\mathbb{T})$.

Now of course the above choice of $\mathscr{F}(\mathbb{T})$ looks rather ad hoc and obscure. Below we will give some motivation and examples, along with the detailed definition. Using the closely related topos $\widehat{\mathbb{T}}$ of presheaves on $\mathbb{T}$, which classifies *flat* models of the theory $\mathbb{T}$, we introduce a variant of $\mathscr{F}(\mathbb{T})$ which will be denoted $\hat{\mathscr{F}}(\mathbb{T})$. It will be shown that in the particular important case of the theory $\mathbb{T}_A$ of modules over a ring A, our $\hat{\mathscr{F}}(\mathbb{T}_A)$ is equivalent to the category $\mathscr{F}(A)$ introduced in [15]. In particular the cohomology groups of $\mathbb{T}_A$ with coefficients in internal abelian groups of $\hat{\mathscr{F}}(\mathbb{T}_A)$ agree with those from [15]. We will also reproduce from [14] description of low dimensional cohomology groups for more general "abelian" theories $\mathbb{T}$.

One additional feature of the categories like $\mathscr{F}(\mathbb{T})$ or, more generally, $\mathscr{F}(\mathbf{X}) = \mathrm{End}_{\mathfrak{Top}/\mathbf{S}}(\mathbf{X})/\mathrm{Id}_{\mathbf{X}}$ for an $\mathbf{S}$-topos $\mathbf{X}$ is that they have an obvious monoidal structure induced by composition of geometric morphisms. This is useful e. g. for using well-known "bar-construction"-like resolutions for calculating various invariants of homotopical nature in $\mathscr{F}(\mathbf{X})$. The monoidal structure is rather specific in that it is non-symmetric, and its unit is a terminal object. Observe that for a theory $\mathbb{T}$, any object of $\mathscr{F}(\mathbb{T})$ can be viewed as a "natural" endofunctor of the category of $\mathbb{T}$-models. A part of such functors, namely those which have adjoints, sometimes tend to explicit description. It is well known that they correspond to *bimodels* of the theory, i. e. to its models in the category which is opposite to the category of models. In a separate section we will give some exemplary calculations of such categories of bimodels, with the monoidal structure arising from composition of endoadjunctions, which can be interesting in their own right.

Here is the contents of the thesis in some more detail. In the first section, we define the category $\mathscr{F}(\mathbf{X})$ for a general Grothendieck topos $\mathbf{X}$, which we propose to consider as representing the "gros" homotopy type of $\mathbf{X}$. We give several equivalent views of this category when $\mathbf{X}$ is the classifying topos $[\mathbb{T}]$ of an algebraic theory $\mathbb{T}$. One of these views expresses $\mathscr{F}([\mathbb{T}])$ in terms of the Eilenberg-Moore object for the monad on the object classifier in the 2-category of Grothendieck toposes which corresponds to $\mathbb{T}$. In this particular case we also introduce an alternative version of $\mathscr{F}([\mathbb{T}])$ which we denote $\hat{\mathscr{F}}(\mathbb{T})$; it combines the Kleisli and Eilenberg-Moore categories of the aforementioned monad.

In the second section we review some results from [31], notably the construction of the Γ-ring corresponding to an algebraic theory over the topos of simplicial sets. We show how one could simplify some aspects of this construction by employing the classifying properties of the object classifier. The reason of appearance of the object classifier is simply that the category of Γ-objects in a topos $\mathbf{S}$ is equivalent to the slice of its object classifier over the generic object.

In the third section we investigate, for an algebraic theory $\mathbb{T}$, categories of bimodels of $\mathbb{T}$, which, as was mentioned above, are closely related to $\mathscr{F}([\mathbb{T}])$ and $\hat{\mathscr{F}}(\mathbb{T})$ but tend to more explicit description. Along with some well known classical examples corresponding to theories of sets, pointed sets, groups, abelian groups and modules over a ring, we give explicit descriptions of bimodel categories and their monoidal structures for theories generated by constants, and for theories of nilpotent rings, Lie rings and groups of class two.

In the fourth section we introduce "gros" cohomology groups of a topos $\mathbf{X}$, with coefficients in an internal abelian group in $\mathscr{F}(\mathbf{X})$. We show that these cohomology groups, for the classifying topos of an algebraic theory, give as a particular case the groups from [15], and give some examples of their new features. We also present a simplification of coefficients for cohomology and an explicit complex which calculates it. We then give interpretations of elements of low-dimensional cohomology groups.

In the last fifth section we provide more detailed information for the case of classifying toposes of *abelian Maltsev* theories. These are theories related to theories of modules over a ring in the same way as affine spaces relate to vector spaces. We give an explicit presentation of any such theory in terms of defining operations and relations. We show that such theories are in one-to-one correspondence with R-module homomorphisms $M \to R$, for arbitrary ring R and a left R-module M. Moreover we give explicit description of those cartesian natural systems on abelian Maltsev theories giving rise to extensions which are again abelian Maltsev. Also low dimensional cohomology groups are described in a more explicit way than for general theories. Finally we give an alternative description of the category of models of an abelian Maltsev theory, as certain factorizations of the homomorphism $M \to R$ corresponding to it.

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1. Gros homotopy type of a topos

Most of the time we will assume fixed a base topos $\mathbf{S}$, and by an $\mathbf{S}$-topos will be meant a bounded geometric morphism $f : \mathbf{X} \to \mathbf{S}$. Recall (see e. g. [17]) that this means that $\mathbf{X}$ has a generating family as an $\mathbf{S}$-indexed category, or, equivalently, that there exists an internal category $\mathbb{C}$ in $\mathbf{S}$ and a Grothendieck-Lawvere-Tierney topology J on $\mathbb{C}$ such that $\mathbf{X}$ is equivalent over $\mathbf{S}$ to the topos of internal sheaves $\mathrm{Shv}(\mathbb{C}, J)$. With $\mathfrak{Top}/\mathbf{S}$ will be denoted the 2-category of such bounded $\mathbf{S}$-toposes, geometric morphisms over $\mathbf{S}$, and natural transformations (between inverse image parts). Standard references for basic facts about this 2-category are [17] and [25].

Usually for an $\mathbf{S}$-topos $\mathbf{X}$ its homotopy type is represented via $\mathbf{X}$ itself, considered as a category. For example, coefficients for cohomology of $\mathbf{X}$ can be taken to be internal abelian groups of $\mathbf{X}$, defining $H^n(\mathbf{X}; A) = R^n(f_*)(A)$, where $R^n(f_*)$ denotes right derived functors of the direct image functor of f. This means that $\mathbf{X}$ will have trivial cohomology as soon as, for example, f_* has a right adjoint – as is the case with most "gros" toposes. As a substitute we propose to replace $\mathbf{X}$ with the following category $\mathscr{F}(\mathbf{X})$.

1.1. **Definition.** For an $\mathbf{S}$-topos $p : \mathbf{X} \to \mathbf{S}$, let

$$\mathscr{F}(\mathbf{X}) = \mathrm{End}_{\mathfrak{Top}/\mathbf{S}}(\mathbf{X})/\mathrm{Id}_{\mathbf{X}}.$$

Thus objects of $\mathscr{F}(\mathbf{X})$ are geometric morphisms $e : \mathbf{X} \to \mathbf{X}$ with $pe = p$, together with a natural transformation $\varepsilon : e \to \mathrm{Id}_{\mathbf{X}}$ such that $p\varepsilon$ is the identity of p; morphisms $(e, \varepsilon) \to (e', \varepsilon')$ are natural transformations $\varphi : e \to e'$ with $\varepsilon'\varphi = \varepsilon$.

We will write also $\mathscr{F}_{\mathbf{S}}(\mathbf{X})$ if several bases can be suggested from the context.

Motivation for this definition will be apparent after we give some examples. But let us begin with a disappointingly negative example.

1.2. **Proposition.** *If* $\mathbf{X}$ *is the topos of sheaves on a Hausdorff space in* $\mathbf{S} = \mathrm{Set}$ *then* $\mathscr{F}(\mathbf{X})$ *is the trivial category with unique morphism.*

Proof. Simply observe that $\mathfrak{Top}_{\mathbf{S}}(\mathbf{Y}, \mathbf{X})$ is a discrete category for all $\mathbf{Y}$.

Thus most of the "usual" homotopy types are excluded at once! To describe a more reassuring example, let us recall the notion of algebraic theory.

Let $\mathbb{T}$ be a *finitary algebraic theory* in the sense of Lawvere, i. e. a category with objects X^n, $n \geqslant 0$, and distinguished projection morphisms $x_1, ..., x_n : X^n \to X$ turning each X^n into the n-fold cartesian power of $X^1 = X$. Elements of $\hom_{\mathbb{T}}(X^n, X)$ are called n-ary operations of $\mathbb{T}$. A *model* of $\mathbb{T}$ in a category $\mathbf{X}$ is a product preserving functor $\mathbb{T} \to \mathbf{X}$; usually one identifies a model with its *underlying object* $M \in \mathbf{X}$ – value on X, together with the maps $M^n \to M$ for each n-ary operation, making appropriate diagrams commute. The category of models of $\mathbb{T}$ in $\mathbf{X}$ and their natural transformations will be denoted by $\mathbb{T}\text{-}\mathbf{mod}(\mathbf{X})$. For example, $\mathbb{T}(n) := \hom_{\mathbb{T}}(X^n, \_)$ is a model of $\mathbb{T}$ in Set, the *free model on n generators*; its underlying object is the set of all n-ary operations of $\mathbb{T}$. A $\mathbb{T}$-model in Set is *finitely presentable* if it is isomorphic to a coequalizer of a pair of maps $u, v : \mathbb{T}(m) \to \mathbb{T}(n)$ between free finitely generated models. The category of finitely presentable models of $\mathbb{T}$ is equivalent to the opposite of the *equalizer completion* of $\mathbb{T}$ whose objects are explicit finite presentations as u, v above, morphisms being those of the corresponding quotient models. This category will be denoted $\mathbb{T}^{\text{fp}}$. One then has

1.3. **Definition.** (see [17]). The *classifying topos* of a theory $\mathbb{T}$ is the category

$$[\mathbb{T}] := \mathrm{Set}^{\mathbb{T}^{\text{fp}}}.$$

The *generic model* $U_{\mathbb{T}}$ of $\mathbb{T}$ is the model whose value on X^n is the functor $\mathbb{T}^{\text{fp}} \to \mathrm{Set}$ sending a finitely presentable model to its value on X^n.

Recall from [17] that $[\mathbb{T}]$ represents the 2-functor $\mathfrak{Top}/\mathrm{Set} \to \mathscr{Cat}$ taking $\mathbf{X}$ to $\mathbb{T}\text{-}\mathbf{mod}(\mathbf{X})$. That is, for any Grothendieck topos $\mathbf{X}$, the functor

$$\mathfrak{Top}_{\mathrm{Set}}(\mathbf{X}, [\mathbb{T}]) \to \mathbb{T}\text{-}\mathbf{mod}(\mathbf{X})$$

which takes $f : \mathbf{X} \to [\mathbb{T}]$ to $f^* U_{\mathbb{T}}$, is an equivalence. This readily gives a description of $\mathscr{F}([\mathbb{T}])$, which we will denote simply $\mathscr{F}(\mathbb{T})$.

1.4. **Proposition.** *For an algebraic theory* $\mathbb{T}$ *one has an equivalence*

$$\mathscr{F}(\mathbb{T}) \simeq \mathbb{T}\text{-}\mathbf{mod}([\mathbb{T}])/U_{\mathbb{T}}.$$

This proposition shows the idea behind our definition. One may regard $U_{\mathbb{T}}$ as a "most representative" model of $\mathbb{T}$. On the other hand, as noticed many times, (see especially [28]) homotopical and cohomological invariants of a universal algebra A in some variety of algebras such as $\mathbb{T}$-**mod** "live" in the category $\mathbb{T}\text{-}\mathbf{mod}/A$. Thus it looks plausible that such invariants for the "most typical" such algebra $U_{\mathbb{T}}$ will reflect general homotopical properties of $\mathbb{T}$.

But in fact it is well known that more generally *any* Grothendieck topos $\mathbf{X}$ can be viewed as the classifying topos of an appropriate *infinitary geometric theory* (see [17]), so that $\mathfrak{Top}_{\mathrm{Set}}(\mathbf{Y}, \mathbf{X})$ is equivalent to the category of models of that theory in $\mathbf{Y}$. From this point of view, the identity of $\mathbf{X}$ can be viewed as the *generic model* of the theory. This gives some supporting motivation for our definition 1.1. Below we'll try to give some further evidence for its usefulness.

Note that alternatively, one might view the above description 1.4 as follows:

1.5. **Proposition.** *For an algebraic theory* $\mathbb{T}$ *one has an equivalence*

$$\mathscr{F}(\mathbb{T}) \simeq (\mathbb{T}\text{-}\mathbf{mod}(\mathrm{Set}))^{\mathbb{T}^{\mathrm{fp}}}/I_{\mathbb{T}},$$

where $I_{\mathbb{T}}$ *is the full embedding of finitely presentable models into all models.*

Proof. One has

$$\mathbb{T}\text{-}\mathbf{mod}(\mathbf{E}^{\mathbb{C}}) \simeq (\mathbb{T}\text{-}\mathbf{mod}(\mathbf{E}))^{\mathbb{C}}$$

for any $\mathbf{E}$ with finite limits and any $\mathbb{C}$. Taking $\mathbf{E} = \mathrm{Set}$ and $\mathbb{C} = \mathbb{T}^{\mathrm{fp}}$ gives the proof since obviously $U_{\mathbb{T}}$ corresponds to $I_{\mathbb{T}}$ under the above equivalence in this particular case.

To describe $\mathscr{F}(\mathbf{X})$ for some more general $\mathbf{X}$ we will need to recall some definitions.

1.6. **Definition.** For a small category $\mathbb{C}$, let $\mathbb{C}^{\#}$ denote the category called *twisted arrow category* of $\mathbb{C}$ in [26], and the *category of factorizations* of $\mathbb{C}$ in [4]. Objects of $\mathbb{C}^{\#}$ are morphisms of $\mathbb{C}$, whereas

$\hom_{\mathbb{C}^\#}(\gamma, \gamma')$ consists of pairs (φ_1, φ_2) with $\varphi_1\gamma\varphi_2 = \gamma'$, like this:

$$\begin{array}{ccc} \bullet & \xleftarrow[\varphi_2]{} & \bullet \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\gamma'} \\ \bullet & \xrightarrow{\varphi_1} & \bullet \end{array}$$

A *natural system* on $\mathbb{C}$ with values in some category $\mathbf{S}$ is a functor $D : \mathbb{C}^\# \to \mathbf{S}$. It is thus a collection of $\mathbf{S}$-objects $(D_\gamma)_{\gamma:X_1\to X_2}$ of $\mathbf{S}$, indexed by morphisms of $\mathbb{C}$, together with $\mathbf{S}$-morphisms

$$\gamma_1(\_) : D_\gamma \to D_{\gamma_1\gamma}$$

and

$$(\_)\gamma_2 : D_\gamma \to D_{\gamma\gamma_2},$$

for all composable morphisms γ_1, γ, γ_2 in $\mathbb{C}$, such that certain evident diagrams commute. In other words, one must have

$$\begin{aligned} (\gamma_1\gamma_2)x_3 &= \gamma_1(\gamma_2 x_3), \\ (\gamma_1 x_2)\gamma_3 &= \gamma_1(x_2\gamma_3), \\ (x_1\gamma_2)\gamma_3 &= x_1(\gamma_2\gamma_3) \end{aligned}$$

for any composable $\gamma_1, \gamma_2, \gamma_3$ in $\mathbb{C}$ and any $x_i : X \to D_{\gamma_i}$ in $\mathbf{S}$.

These definitions have immediate extension to the case of internal categories in a topos (in fact, in any category with finite limits). A simplest way to see this is to pass from an internal category to its nerve (simplicial object whose n-simplices are given by the object of composable n-tuples of morphisms), and then perform the *subdivision* of the obtained simplicial object: recall from [32]

1.7. **Definition.** Let $\mathrm{sub} : \Delta \to \Delta$ be the functor from the category of nonempty finite linear orders to itself carrying a monotone map $f : I \to J$ to $f + f^{\mathrm{op}} : I + I^{\mathrm{op}} \to J + J^{\mathrm{op}}$, where $+$ denotes concatenation. For a simplicial object X in any category let $\mathrm{Sub}(X) = X \circ \mathrm{sub}$.

One then sees easily that if X is the nerve $N\mathbb{C}$ of an internal category $\mathbb{C}$ in a category with finite limits $\mathbf{S}$, then $\mathrm{Sub}(X)$ is the nerve of $\mathbb{C}^\#$. One thus can define an *internal natural system* on $\mathbb{C}$ as a discrete fibration (see [17]) over $\mathrm{Sub}(N\mathbb{C})$. With this definition one then has

1.8. **Proposition.** (cf. [14]) *The category of* $\mathbf{S}$*-valued natural systems on an internal category* $\mathbb{C}$ *is equivalent to*

$$\mathbf{S}^{\mathbb{C}^{\mathrm{op}}\times\mathbb{C}}/\hom_{\mathbb{C}}.$$

Proof. For any internal category $\mathbb{D}$ and any internal presheaf P on it $\mathbf{S}^{\mathbb{D}^{\mathrm{op}}}/P$ is well known to be equivalent to $\mathbf{S}^{\mathrm{Tot}(P)^{\mathrm{op}}}$, where $\mathrm{Tot}(P)$ is the total category of the discrete fibration corresponding to P. Now one sees easily that for $\mathbb{D}=\mathbb{C}^{\mathrm{op}}\times\mathbb{C}$, $N\mathrm{Tot}(\hom_{\mathbb{C}})$ is exactly $\mathrm{Sub}(N\mathbb{C})$.

1.9. **Definition.** (cf. [15, 14]) A natural system D on a category $\mathbb{C}$ with finite limits, with values in a category with finite limits $\mathbf{S}$ is called *cartesian* if for any finite limiting cone $(f_i : c \to c_i)_{i\in I}$ and any morphism $f : c' \to c$ in $\mathbb{C}$, the cone $((x \mapsto f_i x) : D_f \to D_{f_i f})_{i\in I}$ is a limiting cone in $\mathbf{S}$.

Once again it is straightforward to internalize this definition – for example, by reducing consideration of all finite limiting cones to only pullbacks and terminal objects (or, say, equalizers, binary products and terminal objects). One thus arrives at the notion of *internal* cartesian natural system on an internal category $\mathbb{C}$ in $\mathbf{S}$.

With this notion one then has

1.10. **Proposition.** *For an internal category* $\mathbb{C}$ *with finite limits in* $\mathbf{S}$*, the category* $\mathscr{F}(\mathbf{S}^{\mathbb{C}^{\mathrm{op}}})$ *is equivalent to the category of* $\mathbf{S}$*-valued cartesian natural systems on* $\mathbb{C}$*.*

Proof. By Diaconescu's theorem (see [17]), $\mathfrak{Top}_{\mathbf{S}}(\mathbf{X},\mathbf{S}^{\mathbb{C}^{\mathrm{op}}})$ is equivalent to the category of finite limit preserving internal functors $\mathbb{C}\to\mathbf{X}$, for any $\mathbf{X}$. Taking $\mathbf{X}=\mathbf{S}^{\mathbb{C}^{\mathrm{op}}}$ we see that $\mathrm{End}_{\mathfrak{Top}/\mathbf{S}}(\mathbf{S}^{\mathbb{C}^{\mathrm{op}}})$ is equivalent to the full subcategory of $\mathbf{S}^{\mathbb{C}^{\mathrm{op}}\times\mathbb{C}}$ whose objects preserve finite limits in the second variable, with the identity endofunctor corresponding to the bifunctor $\hom_{\mathbb{C}}$. One then sees easily that under the equivalence of 1.8, bifunctors preserving finite limits with respect to the second variable correspond precisely to cartesian natural systems.

Returning to the case of theories, let us introduce a relative notion, suitable for working over base toposes other than Set.

1.11. **Definition.** A *geometric monad* is a monad in the 2-category $\mathfrak{Top}$. It is thus a geometric morphism $T : \mathbf{X} \to \mathbf{X}$ together with natural transformations $e : \mathrm{Id}_{\mathbf{X}} \to T^*$ and $m : T^*T^* \to T^*$ turning the inverse image functor T^* into a monad on $\mathbf{X}$.

By the well known observation, going back to Bénabou, a geometric monad is the same as a lax functor from the terminal 2-category to $\mathfrak{Top}$.

Then as usually for 2-categories, one defines

1.12. **Definition.** An *algebra* over a geometric monad $\mathrm{T} = (T, e, m)$ on a topos $\mathbf{X}$ is a geometric morphism $A : \mathbf{Y} \to \mathbf{X}$ together with a natural transformation $a : A^*T^* \to A^*$ satisfying $a \circ A^*e = \mathrm{Id}_{A^*}$ and $a \circ A^*m = a \circ aT^*$. For such an algebra (A, a), its *base change* along a geometric morphism $F : \mathbf{Z} \to \mathbf{Y}$ is (AF, aF).

An *Eilenberg-Moore topos* or *classifying topos* $U_{\mathrm{T}} : [\mathrm{T}] \to \mathbf{X}$ for (T, e, m) is a universal such algebra, i. e. such that for any $\mathbf{Y}$, base change of U_{T} induces an equivalence from $\mathfrak{Top}(\mathbf{Y}, [\mathrm{T}])$ to T-algebras with domain $\mathbf{Y}$. Thus it is the lax limit of the corresponding lax functor.

A *right algebra* over T is a geometric morphism $A : \mathbf{X} \to \mathbf{Y}$ together with a transformation $AT \to T$ with properties dual to the above. The universal such, i. e. the lax colimit of the corresponding lax functor, is called *Kleisli topos* of T, denoted $U^{\mathrm{T}} : \mathbf{X} \to \widehat{\mathrm{T}}$.

Kleisli toposes are much easier to construct than Eilenberg-Moore toposes. In fact, one has

1.13. **Proposition.** *For a geometric monad* $\mathrm{T} = (T, e, m)$ *on a topos* $\mathbf{X}$*, let* $\tilde{m} : T^*T_* \to T_*$ *be the composite natural transformation in*

$$T^*T_* \xrightarrow{\iota T^*T_*} T_*T^*T^*T_* \xrightarrow{T_*mT_*} T_*T^*T_* \xrightarrow{T_*\varepsilon} T_*,$$

where ι *and* ε *are, respectively, the unit and counit of* $T^* \dashv T_*$*. Then,* $\tilde{m}$ *defines a* (T^*, e, m)*-algebra structure on* T_* *(in the* 2*-category of categories); moreover the corresponding functor* $(T_*, \tilde{m}) : \mathbf{X} \to \mathbf{X}^{T^*}$ *has a left exact left adjoint and thus determines a geometric morphism, which yields a Kleisli topos for* T.

Proof. It is well known that whenever a monad such as T^* has a right adjoint, that adjoint acquires a comonad structure such that the corresponding categories of algebras and coalgebras are naturally isomorphic. Moreover it is a standard fact in topos theory that the category of coalgebras over a left exact monad on a topos is also a topos, with the (forgetful)$\dashv$(cofree) adjunction forming a geometric morphism. Now in our case, to a cofree coalgebra with the coaction $T_*X \to T_*T_*X$ corresponds under this isomorphism a T^*-algebra with the adjoint transpose action $T^*T_*X \to T_*X$. Unfolding the comultiplication $T_* \to T_*T_*$ in terms of (T^*, e, m) then shows that this algebra structure is exactly $\tilde{m}X$ above.

Next observe that for a geometric surjection $q : \mathbf{X} \to \mathbf{Y}$ determined by a left exact comonad q^*q_*, and for any geometric morphism $f : \mathbf{X} \to \mathbf{Z}$, factorizations of f through q are in one-to-one correspondence with q^*q_*-coalgebra structures on f^*. In our case, $q^*q_* = T_*$ has a left adjoint T^*, so that T_*-coalgebra structures are the same as T^*-algebra structures. Thus $\mathbf{X}_{T_*} \cong \mathbf{X}^{T^*}$ has the universal property of a Kleisli topos for T.

Remark. Alternatively, one could use the fact that this is a particular case of the Wraith glueing, which gives explicit construction of any lax colimits in $\mathfrak{Top}$.

Also observe that when both the Kleisli topos $U^{\mathrm{T}} : \mathbf{X} \to \widehat{\mathrm{T}}$ and the Eilenberg-Moore topos $U_{\mathrm{T}} : [\mathrm{T}] \to \mathbf{X}$ for a monad T exist, there is a canonical geometric morphism $I(\mathrm{T}) : \widehat{\mathrm{T}} \to [\mathrm{T}]$ from the former to the latter, such that

$$U_{\mathrm{T}} I(\mathrm{T}) U^{\mathrm{T}} = T.$$

Indeed, the T-algebra structure m on T induces a factorization of $T : \mathbf{X} \to \mathbf{X}$ through U_{T}, via, say, $U' : \mathbf{X} \to [\mathrm{T}]$, so that $U_{\mathrm{T}}U' = T$; then there is also a right T-algebra structure on this functor,

$$U_{\mathrm{T}}U'T = TT \xrightarrow{m} T = U_{\mathrm{T}}U';$$

and since this is in fact a homomorphism of (left) T-algebras, by universality of U_{T} it must have form $U_{\mathrm{T}}m'$, for some right T-algebra structure $m' : U'T \to T$. Then by universality of U^{T} this gives $I(\mathrm{T})$ as required.

As for Eilenberg-Moore objects, they have been constructed, generalizing 1.3, in a particularly important case in [17, 20], starting from any topos $\mathbf{S}$ with a natural numbers object. The method is based on the celebrated *object classifier* $\mathscr{R}(\mathbf{S}) \to \mathbf{S}$ of $\mathbf{S}$, which is the topos $\mathbf{S}^{\mathbb{S}^{\mathrm{fp}}}$ of internal functors on the internal full subcategory $\mathbb{S}^{\mathrm{fp}}$ of finite cardinals in $\mathbf{S}$. The object classifier represents the 2-functor $(\mathfrak{Top}/\mathbf{S})^{\mathrm{op}} \to \mathscr{Cat}$ taking $\mathbf{X} \to \mathbf{S}$ to $\mathbf{X}$ considered as a category, and taking geometric morphisms to their inverse image parts. Thus $\mathscr{R}(\mathbf{S})$ contains a *generic object*, $U_{\mathbf{S}}$, namely the inclusion of the full subcategory $\mathbb{S}^{\mathrm{fp}}$ into $\mathbf{S}$, such that any object of any $\mathbf{S}$-topos $\mathbf{X}$ is the inverse image of $U_{\mathbf{S}}$ under some geometric morphism $\mathbf{X} \to \mathscr{R}(\mathbf{S})$ over $\mathbf{S}$, defined up to canonical isomorphism. One thus can say that $\mathscr{R}(\mathbf{S})$ is an *internal topos* in $\mathfrak{Top}/\mathbf{S}$

in an obvious sense that $\mathfrak{Top}_{\mathbf{S}}(\mathbf{X},\mathscr{R}(\mathbf{S}))$ is a topos for any $\mathbf{X}$. One then defines

1.14. **Definition.** ([20]). A *finitary monad* over $\mathbf{S}$ is a geometric monad $\mathrm{T}=(T,e,m)$ on the object classifier $\mathscr{R}(\mathbf{S})\to\mathbf{S}$ in $\mathfrak{Top}/\mathbf{S}$. Its *classifying topos* $U_{\mathrm{T}}:[\mathrm{T}]\to\mathscr{R}(\mathbf{S})$ is the Eilenberg-Moore topos of this monad.

The main point of the definition is that, since $\mathscr{R}(\mathbf{S})$ is the object classifier, one has

- for each $\mathbf{X}\to\mathbf{S}$ in $\mathfrak{Top}/\mathbf{S}$, the functor

$$\mathfrak{Top}_{\mathbf{S}}(\mathbf{X},T):\mathfrak{Top}_{\mathbf{S}}(\mathbf{X},\mathscr{R}(\mathbf{S}))\to\mathfrak{Top}_{\mathbf{S}}(\mathbf{X},\mathscr{R}(\mathbf{S}))$$

  gives rise to a monad on $\mathfrak{Top}_{\mathbf{S}}(\mathbf{X},\mathscr{R}(\mathbf{S}))$, hence a monad $\mathrm{T}_{\mathbf{X}}$ on $\mathbf{X}$ (considered as a mere category);
- for an object $X\in\mathbf{X}$ classified by the geometric morphism $U:\mathbf{X}\to\mathscr{R}(\mathbf{S})$, i. e. such that $U^*(U_{\mathbf{S}})=X$, there is a one-to-one correspondence between T-algebra structures on U and $\mathrm{T}_{\mathbf{X}}$-algebra structures on X.

Thus $[\mathrm{T}]\to\mathscr{R}(\mathbf{S})\to\mathbf{S}$ represents the 2-functor $\mathfrak{Top}/\mathbf{S}\to\mathscr{C}at$ which sends $\mathbf{X}\to\mathbf{S}$ to the category $\mathbf{X}^{\mathrm{T}_{\mathbf{X}}}$ of $\mathrm{T}_{\mathbf{X}}$-algebras in $\mathbf{X}$.

Moreover in this case the Kleisli topos is in fact equivalent to the topos $\widehat{\mathbb{T}}=\mathbf{S}^{\mathbb{T}^{\mathrm{op}}}$ of internal presheaves on an internal category $\mathbb{T}$ in $\mathbf{S}$. Indeed, since $\mathscr{R}(\mathbf{S})=\mathbf{S}^{\mathbb{S}^{\mathrm{fp}}}$, the explicit description of this category $\mathbb{T}$ is available from that of $\mathbb{S}^{\mathrm{fp}}$. The object of objects of $\mathbb{T}$ can be identified with the object of natural numbers N, while morphisms, in terms of generalized elements, are given by $\hom_{\mathbb{T}}(n_1,n_2)=T_{\mathbf{S}}(n_1)^{n_2}$. Here the functor $T_{\mathbf{S}}:\mathbf{S}\to\mathbf{S}$ corresponds to $\mathfrak{Top}_{\mathbf{S}}(\mathbf{S},T):\mathfrak{Top}_{\mathbf{S}}(\mathbf{S},\mathscr{R}(\mathbf{S}))\to\mathfrak{Top}_{\mathbf{S}}(\mathbf{S},\mathscr{R}(\mathbf{S}))$ under the identification of $\mathfrak{Top}_{\mathbf{S}}(\mathbf{S},\mathscr{R}(\mathbf{S}))$ with $\mathbf{S}$ (considered as a mere category). It follows that in $\mathbb{T}$, the object n is an n-fold cartesian power of 1, in the appropriate internal sense. Thus such $\mathbb{T}$ can be viewed as analogs of finitary algebraic theories and indeed coincide with them when $\mathbf{S}=\mathrm{Set}$. Indeed, as shown in [20, Lemmas 5.18, 5.19], T-algebras in $\mathbf{X}$ for any $f:\mathbf{X}\to\mathbf{S}$ can be identified with internal finite product preserving functors from $f^*(\mathbb{T})$ to $\mathbf{X}$, i. e. with $\mathbb{T}$-models in $\mathbf{X}$, and moreover one can construct the internal category $\mathbb{T}^{\mathrm{fp}}$ of finitely presented $\mathbb{T}$-models such that $[\mathrm{T}]$ is equivalent to $[\mathbb{T}]=\mathbf{S}^{\mathbb{T}^{\mathrm{fp}}}$.

In particular, one has obvious generalizations of 1.4 and 1.5, with Set replaced by any topos $\mathbf{S}$ with natural numbers.

Besides being the Kleisli topos for T, the topos $\widehat{\mathbb{T}}$ also has another universal property: by [20, 5.22], for any $\mathbf{X} \to \mathbf{S}$, the category $\mathfrak{Top}_{\mathbf{S}}(\mathbf{X}, \widehat{\mathbb{T}})$ is equivalent to the category of *flat* $\mathbb{T}$-models in $\mathbf{X}$, i. e. those expressible as colimits of free $\mathbb{T}$-models in $\mathbf{X}^{\mathbb{C}}$, for some filtered internal category $\mathbb{C}$ in $\mathbf{S}$. Here a $\mathbb{T}$-model is free if the corresponding T-algebra is. One then has

1.15. **Proposition.** *For an algebraic theory* $\mathbb{T}$ *over a topos* $\mathbf{S}$*,*

$$\mathfrak{Top}_{\mathbf{S}}(\widehat{\mathbb{T}}, [\mathbb{T}]) \simeq (\mathbb{T}\text{-}\mathbf{mod}(\mathbf{S}))^{\mathbb{T}^{\mathrm{op}}}.$$

Proof. This is just a particular case of the obvious equivalence

$$\mathfrak{Top}_{\mathbf{S}}(\mathbf{S}^{\mathbb{C}^{\mathrm{op}}}, [\mathbb{T}]) \simeq \mathbb{T}\text{-}\mathbf{mod}(\mathbf{S}^{\mathbb{C}^{\mathrm{op}}}) \simeq (\mathbb{T}\text{-}\mathbf{mod}(\mathbf{S}))^{\mathbb{C}^{\mathrm{op}}}.$$

There is an obvious geometric morphism $F_{\mathbb{T}} : \widehat{\mathbb{T}} \to [\mathbb{T}]$ classifying the generic flat model viewed just as a $\mathbb{T}$-model. It is induced by the functor $\mathbb{T}^{\mathrm{op}} \to \mathbb{T}^{\mathrm{fp}}$ which embeds free $\mathbb{T}$-models into finitely presented ones. It is easy to see that $F_{\mathbb{T}}$ is canonically isomorphic to the geometric morphism $I(\mathbb{T})$ defined above. We then define

1.16. **Definition.** $\hat{\mathscr{F}}(\mathbb{T}) = \mathfrak{Top}_{\mathbf{S}}(\widehat{\mathbb{T}}, [\mathbb{T}])/F_{\mathbb{T}}$.

Note that, similarly to 1.4, one has

1.17. **Proposition.** *For any theory* $\mathbb{T}$ *there is an equivalence*

$$\hat{\mathscr{F}}(\mathbb{T}) \simeq \mathbb{T}\text{-}\mathbf{mod}(\widehat{\mathbb{T}})/U^{\mathbb{T}}.$$

There is another naturally defined category for $\mathbb{T}$ which embeds fully both in $\mathscr{F}(\mathbb{T})$ and $\hat{\mathscr{F}}(\mathbb{T})$. Namely, one has

1.18. **Proposition.** *The following categories are equivalent:*

- *the category of endofunctors* $\mathbb{T}$-$\mathbf{mod} \to \mathbb{T}$-$\mathbf{mod}$ *which preserve colimits;*
- *the category of functors* $\mathbb{T}^{\mathrm{fp}} \to \mathbb{T}$-$\mathbf{mod}$ *which preserve finite colimits;*
- *the category of functors* $\mathbb{T}^{\mathrm{op}} \to \mathbb{T}$-$\mathbf{mod}$ *which preserve finite coproducts;*
- *the category of* $\mathbb{T}$-bimodels *in* $\mathbf{S}$*, i. e. opposite of the category of* $\mathbb{T}$*-models in* $(\mathbb{T}\text{-}\mathbf{mod})^{\mathrm{op}}$.

These are well known standard facts for $\mathbf{S} = \mathrm{Set}$. Generalizing to arbitrary $\mathbf{S}$ with natural numbers is trivial except for the first item; it can be easily handled invoking indexed category theory to view $\mathbb{T}$-**mod** as an $\mathbf{S}$-indexed category. It is straightforward to generalize to the indexed setting the fact that colimit preserving indexed functors on $\mathbb{T}$-**mod** are uniquely determined by their values on the subcategory $\mathbb{T}^{\mathrm{fp}}$.

Denoting the category of $\mathbb{T}$-bimodels by $\mathscr{E}(\mathbb{T})$, we thus see immediately from 1.18 that it is a monoidal category, and that there are full embeddings

$$\mathscr{E}(\mathbb{T}) \to \mathfrak{Top}_{\mathbf{S}}(\widehat{\mathbb{T}}, [\mathbb{T}]),\ \mathscr{E}(\mathbb{T}) \to \mathrm{End}_{\mathfrak{Top}/\mathbf{S}}([\mathbb{T}])$$

which carries the unit I of this monoidal structure to respectively, the geometric morphism $F_{\mathbb{T}} = I(\mathbb{T})$ and the identity of $[\mathbb{T}]$. So trivially there are full embeddings

$$\mathscr{E}(\mathbb{T})/I \to \hat{\mathscr{F}}(\mathbb{T}),\ \mathscr{E}(\mathbb{T})/I \to \mathscr{F}(\mathbb{T}).$$

2. Stable homotopy

In [31], a certain ring spectrum $\mathbb{T}^s$ is assigned to any algebraic theory $\mathbb{T}$ over the topos $\mathrm{Set}^{\Delta^{\mathrm{op}}}$ of simplicial sets. Among other things, it allows to extend homological invariants of $\mathbb{T}$ previously introduced in [15] only for *discrete* $\mathbb{T}$, i. e. those coming from Set. After recalling relevant notions from [31] we will show how to construct an analog of $\mathbb{T}^s$ for an algebraic theory over a general topos $\mathbf{S}$.

Recall from [5] that a *spectrum* is a sequence

$$X = (S^1 \wedge X_n \to X_{n+1})_{n\geqslant 0}$$

of maps between pointed simplicial sets, S^1 being the standard simplicial circle. To a pointed simplicial set X_0 corresponds its *suspension spectrum* with $X_{n+1} = S^1 \wedge X_n$ and maps identities. In particular, the *sphere spectrum* $\mathbb{S}^0$ with $(S^1 \wedge S^n \to S^{n+1})_n$ is the suspension spectrum of S^0, discrete simplicial set with two vertices. A morphism of spectra $f : X \to Y$ is a sequence of maps $f_n : X_n \to Y_n$ compatible with the structure in the obvious sense. *Homotopy groups* $\pi_n(X)$, $n \in \mathbb{Z}$, of a spectrum X are defined by

$$\pi_n(X) = \lim_{i\to\infty}(... \to [S^{n+i}, X_i] \to [S^1 \wedge S^{n+i}, S^1 \wedge X_i] \to [S^{n+i+1}, X_{i+1}] \to ...),$$

where $[\_,\_]$ denotes sets of homotopy classes of pointed maps between geometric realizations. A spectrum X is *connective* if its negative homotopy groups are trivial. Homotopy groups are functorial, and a morphism of spectra is called a *stable equivalence* if it induces isomorphisms of all homotopy groups. The *stable homotopy category* is the localization of the category of spectra with respect to stable equivalences.

Connective spectra can be accessed via the category $\mathbf{\Gamma S}$ of Γ-*spaces* of Segal (see [33]). This is just the category of functors from the category Γ^{op} of finite pointed sets to simplicial sets which preserve the terminal object. To assign a spectrum to a Γ-space S one uses left Kan extension along the embedding of finite pointed sets into all pointed sets, thus obtaining a functor from pointed sets to pointed simplicial sets; next, composing this functor with functors from Δ^{op} to pointed sets yields a functor from pointed simplicial sets to bisimplicial sets; and finally taking diagonals of bisimplicial sets yields an endofunctor F_S of the category of pointed simplicial sets. One then shows that this endofunctor has *strength* with respect to the smash product, i. e. can be equipped with maps $\sigma_{K,L} : K \wedge F_S(L) \to F_S(K \wedge L)$ satisfying certain naturality conditions (see e. g. [21]). This finally gives a spectrum X with $X_n = F_S(S^n)$ and maps $\sigma_{S^1,S^n} : S^1 \wedge F_S(S^n) \to F_S(S^{n+1})$. As explained in [5], Γ-spaces can be viewed as connective reduced homology theories corresponding to connective spectra. A typical example of a Γ-space is produced from an arbitrary abelian group A and is given by

$$\tilde{A}(n_+) = A[n] := \underbrace{A \oplus ... \oplus A}_{n\text{times}}.$$

The corresponding endofunctor of pointed simplicial sets assigns to a pointed simplicial set K the simplicial abelian group such that the chain complex $\tilde{C}_*(K; A)$ corresponding to it under the Dold-Kan correspondence is none other than the reduced chain complex of K with coefficients in A.

There is in fact a more straightforward way to define $F_S(K)$ for a Γ-space S and a simplicial set K. For this one notes that

2.1. **Proposition.** *There is a full embedding*

$$\mathbf{\Gamma S} \hookrightarrow \mathscr{F}_{\mathrm{Set}^{\Delta^{\mathrm{op}}}}(\mathscr{R}(\mathrm{Set}^{\Delta^{\mathrm{op}}})).$$

Proof. Clearly $\mathbf{\Gamma S}$ is a full subcategory of the category of pointed functors from pointed finite cardinals to simplicial sets. On the other hand one has for any topos $\mathbf{S}$

$$\begin{aligned} \mathscr{F}_{\mathbf{S}}(\mathscr{R}(\mathbf{S})) &= \mathfrak{Top}/\mathbf{S}(\mathscr{R}(\mathbf{S}), \mathscr{R}(\mathbf{S}))/\mathrm{Id}_{\mathscr{R}(\mathbf{S})} \\ &\simeq \mathscr{R}(\mathbf{S})/U_{\mathbf{S}} \\ &= \mathbf{S}^{\mathbb{S}^{\mathrm{fp}}}/I \\ &\simeq \mathbf{S}^{\mathrm{Tot}(I)}, \end{aligned}$$

where I is the internal full embedding of finite cardinals into $\mathbf{S}$ and "Tot" has the same meaning as in the proof of 1.8 above. Thus $\mathrm{Tot}(I)$ is equivalent to the internal category of pointed finite cardinals of $\mathbf{S}$, which for $\mathbf{S} = \mathrm{Set}^{\Delta^{\mathrm{op}}}$ is the same as the category of ordinary pointed finite cardinals, considered as discrete simplicial sets.

We thus see that each Γ-space S gives rise to a geometric morphism F_S from $\mathscr{R}(\mathrm{Set}^{\Delta^{\mathrm{op}}})$ to itself. On the other hand for any topos $\mathbf{S}$ whatsoever every such geometric morphism $F : \mathscr{R}(\mathbf{S}) \to \mathscr{R}(\mathbf{S})$ induces a natural transformation from the corresponding representable 2-functor to itself. The latter functor, as we know, sends an $\mathbf{S}$-topos $\mathbf{X} \to \mathbf{S}$ to $\mathbf{X}$; so F gives rise to functors $F_{\mathbf{X}} : \mathbf{X} \to \mathbf{X}$. More explicitly, these functors can be described as follows: since $\mathfrak{Top}_{\mathbf{S}}(\mathscr{R}(\mathbf{S}), \mathscr{R}(\mathbf{S})) \simeq \mathscr{R}(\mathbf{S})$, to specify F as above is the same as to specify an object $E_0 = E^*(U_{\mathbf{S}})$ of $\mathscr{R}(\mathbf{S})$. Similarly for any $\mathbf{X} \to \mathbf{S}$ objects geometric morphisms $X : \mathbf{X} \to \mathscr{R}(\mathbf{S})$ over $\mathbf{S}$ correspond to objects $X_0 = X^*(U_{\mathbf{S}})$ of $\mathbf{X}$. Under this correspondence one then has $F_{\mathbf{X}}(X_0) = X^*(E_0)$. We thus can give

2.2. **Definition.** For a topos $\mathbf{S}$ and an object F of $\mathscr{R}(\mathbf{S})$, we call the *value of* F at an object X_0 of an $\mathbf{S}$-topos $\mathbf{X} \to \mathbf{S}$, the object $X^*(F)$, defined up to isomorphism, where $X : \mathbf{X} \to \mathscr{R}(\mathbf{S})$ classifies X_0, i. e. it is such that $X_0 = X^*(U_{\mathbf{S}})$. Notation will be simply $F(X_0)$.

With this notation we then have

2.3. **Theorem.** *For a Γ-space S, the endofunctor F_S of the category of pointed simplicial sets corresponding to S by* [5] *carries a simplicial set K to $\bar{S}(K)$, where $\bar{S}$ is the object of $\mathscr{R}(\mathrm{Set}^{\Delta^{\mathrm{op}}})$ classified by the geometric morphism from $\mathscr{R}(\mathrm{Set}^{\Delta^{\mathrm{op}}})$ to itself corresponding to S by 2.1.*

Proof. For any object X_0 of any $\mathbf{S}$-topos $f : \mathbf{X} \to \mathbf{S}$, in [20] is given an explicit formula for the corresponding geometric morphism $\mathbf{X} \to \mathscr{R}(\mathbf{S})$. In particular, its direct image is given by

$$X_1 \mapsto f_*(X_1^{X_0^n}).$$

The induced functor $\mathfrak{Top}/\mathbf{S}(\mathbf{X}, \mathscr{R}(\mathbf{S})/U_{\mathbf{S}}) \to \mathfrak{Top}/\mathbf{S}(\mathbf{X}, \mathscr{R}(\mathbf{S})/U_{\mathbf{S}})$ is then given by a similar expression except that X_0 and X_1 are now pointed and $X_1^{X_0^n}$ is replaced by the subobject of pointed maps. Calculating then value of the left adjoint of this functor on a pointed simplicial set K gives precisely the coend formula from [31, page 4], namely

$$\int^{n_+ \in \Gamma^{\mathrm{op}}} S(n_+) \wedge K^n$$

(note that there is a misprint in [31] at that place).

It is clear that there is a monoidal structure on Γ-spaces corresponding to the composition of associated endofunctors of the category of pointed simplicial sets. Explicitly, it is given by

$$(X \circ Y)(n_+) = X(Y(n_+)) = \int^{m_+ \in \Gamma^{\mathrm{op}}} X(m_+) \wedge Y(n_+)^m,$$

and the unit object, corresponding to the identity endofunctor, is the Γ-space $\mathbb{S}$ given by $\mathbb{S}(n_+) = n_+$. There is another, symmetric monoidal structure, recently exploited by Lydakis in [22], which is obtained by convolution of the smash product on Γ^{op}. Explicitly, one has

$$(X \wedge Y)(n_+) = \lim_{x_+ \wedge y_+ \to n_+} X(x_+) \wedge Y(y_+).$$

The unit object is, remarkably, the same $\mathbb{S}$; moreover this symmetric monoidal structure is closed, with the internal hom given by

$$\mathrm{Hom}(X, Y)(n_+) = \mathrm{hom}(X, Y(n_+ \wedge \_)).$$

The two structures are related by the s. c. *assembly maps* $X \wedge Y \to X \circ Y$, providing a structure of a lax monoidal functor from $\wedge$ to $\circ$ on the identity functor of $\Gamma\mathbf{S}$. It is obtained from the canonical strength transformation maps

$$X(x_+) \wedge Y(y_+) \to X(x_+ \wedge Y(y_+)) \to X(Y(x_+ \wedge y_+))$$

using the convolution universality for $\wedge$. In [22], a remarkable fact is proved: the assembly map is a stable equivalence whenever any of the X or Y is cofibrant in the sense of the model structure on $\Gamma\mathbf{S}$ constructed in [5].

Now monoids with respect to $\circ$ correspond to arbitrary finitary monads on simplicial sets, and the latter property of the assembly map suggests that such monads can be, up to stable homotopy, profitably studied using monoids with respect to $\wedge$. The latter are called Γ-rings in [31]:

2.4. **Definition.** A Γ-*ring* is a Γ-space R equipped with maps $e : \mathbb{S} \to R$ and $m : R \wedge R \to R$ turning it into a monoid with respect to the monoidal structure $\wedge$, i. e. making appropriate unitality and associativity diagrams commute in $\Gamma\mathbf{S}$. A *module* over a Γ-ring R is a Γ-space equipped with an action $R \wedge M \to M$ of the monoid R.

Importance of Γ-rings lies in the fact that under the correspondence between Γ-spaces and homology theories outlined above, Γ-rings correspond to *multiplicative* homology theories. For example, the Γ space $\tilde{A}$ above which corresponds to an abelian group A has a Γ-ring structure whenever A has a ring structure in the usual sense. More general important examples are provided by various connective K-homology theories and bordisms.

We then can, following [31], assign a Γ-ring T^s to any finitary monad T over $\mathrm{Set}^{\Delta^{\mathrm{op}}}$, in the sense of 1.14. Namely, T^s is defined to be the free T-algebra functor restricted to finite pointed sets. The Γ-ring structure is defined via the assembly map

$$\mathrm{T}^s \wedge \mathrm{T}^s \to \mathrm{T}^s \circ \mathrm{T}^s \to \mathrm{T}^s.$$

Homotopy groups of the corresponding spectrum are then called *stable homotopy groups* of T.

In [31], stable homotopy groups of T are interpreted as groups of stable homotopy operations between T-algebras. A *homotopy operation* of T-algebras is a natural transformation $\pi_n \to \pi_m$ for some n, m, where

$$\pi_n, \pi_m : \mathrm{T}\text{-}\mathbf{mod} \to \mathrm{Set}$$

are given by homotopy groups of geometric realizations of simplicial sets underlying T-models. Since these functors are represented, in the homotopy category, by free T-models on spheres,

$$\pi_n(M) = [F^{\mathrm{T}}(S^n), M],$$

the set of homotopy operations from π_n to π_m is bijective to the set $[F^{\mathrm{T}}(S^m), F^{\mathrm{T}}(S^n)]$ of homotopy classes of maps between geometric realizations of underlying simplicial sets of free T-models. One then defines suspension Σ of homotopy operations carrying an operation $\tau : \pi_n \to \pi_m$ to $\Sigma\tau : \pi_{n+1} \to \pi_{m+1}$, via the corresponding suspension map

$$\Sigma : [F^{\mathrm{T}}(S^m), F^{\mathrm{T}}(S^n)] \to [F^{\mathrm{T}}(S^{m+1}), F^{\mathrm{T}}(S^{n+1})].$$

A *stable* homotopy operation of degree n is then a sequence (τ_i) of homotopy operations $\tau_i : \pi_i \to \pi_{n+i}$ with $\tau_{i+1} = \Sigma\tau_i$, two such said to be equivalent if there is an i_0 such that they have the same components for $i > i_0$. As proved in [31], for each n the set of equivalence classes of stable homotopy operations on T is bijective to the n-th stable homotopy group of T^s. In fact this follows easily from the "Yoneda-like" considerations, as equivalence classes of stable homotopy operations of degree n on T are clearly given by

$$\lim_{i\to\infty} [F^{\mathrm{T}}(S^{n+i}), F^{\mathrm{T}}(S^i)],$$

whereas the n-th stable homotopy group of T^s is by definition given by

$$\lim_{i\to\infty} \pi_{n+i} F^{\mathrm{T}}(S^i).$$

According to the explication after 1.14, in our terms T^s is nothing but the generic model $U_{\mathbb{T}}$ of the corresponding theory, situated in the classifying topos [T] equipped with the canonical functor $[\mathrm{T}] \to \mathscr{R}(\mathrm{Set}^{\Delta^{\mathrm{op}}})$. Moreover as explained in [20], a monad T on $\mathscr{R}(\mathbf{S})$, for any topos $\mathbf{S}$, gives rise, in a compatible way, to monads $\mathrm{T}_{\mathbf{X}}$ on all $\mathbf{S}$-toposes $\mathbf{X} \to \mathbf{S}$, such that for any X_0 in any $\mathbf{X}$, one has $\mathrm{T}_{\mathbf{X}}(X_0) = U_{\mathrm{T}}(X_0)$, i. e. the value on X_0, in the sense of 2.2, of U_{T}, considered as the object of $\mathscr{R}(\mathbf{S})$ classified by $T : \mathscr{R}(\mathbf{S}) \to \mathscr{R}(\mathbf{S})$. Putting this together with 2.3 thus gives

2.5. **Corollary.** *For a finitary monad* T *on* $\mathrm{Set}^{\Delta^{\mathrm{op}}}$, *the spectrum* X *corresponding to the* Γ*-space* T^s *by* [31] *has* $X_n = \mathrm{T}_{\mathrm{Set}^{\Delta^{\mathrm{op}}}}(S^n)$.

This moreover enables us to read off geometric realizations of constituent simplicial sets X_n of the spectrum X. Indeed there are several toposes containing as full subcategories sufficiently large parts of the category of topological spaces. See e. g. the *topological topos* $\mathscr{T}$ from [18] which contains the category of sequential spaces. It is known that the topos $\mathrm{Set}^{\Delta^{\mathrm{op}}}$ classifies linear orders with endpoints, thus the standard unit interval equips $\mathscr{T}$ with a geometric morphism $I : \mathscr{T} \to \mathrm{Set}^{\Delta^{\mathrm{op}}}$. Then similarly to 2.5 one sees that for a finitary monad T on $\mathrm{Set}^{\Delta^{\mathrm{op}}}$, the realization of the n-th space of the spectrum corresponding to T^s is $\mathrm{T}_{\mathscr{T}}(S^n)$.

Finally, let us briefly mention another possibility of assigning Γ-spaces to general Grothendieck toposes. It is shown in [19] that for an object G in a topos $\mathbf{X}$ over $\mathbf{S}$, the following conditions are equivalent:

- there is a generating family for $\mathbf{X}$ over $\mathbf{S}$ consisting of subobjects of finite cartesian powers of G;
- the geometric morphism $\ulcorner G \urcorner : \mathbf{X} \to \mathscr{R}(\mathbf{S})$ which classifies G is *localic*, so that $\mathbf{X}$ is equivalent, over $\mathscr{R}(\mathbf{S})$, to the category of sheaves on an internal complete Heyting algebra H_G in $\mathscr{R}(\mathbf{S})$.

Thus each bounded $\mathbf{S}$-topos $\mathbf{X}$ can be represented as a localic topos over $\mathscr{R}(\mathbf{S})$; conversely, internal locales in $\mathscr{R}(\mathbf{S})$ capture all possible bounded $\mathbf{S}$-toposes. In fact, that complete Heyting algebra can be described explicitly – one has $H_G = \ulcorner G \urcorner_* (\Omega_{\mathbf{X}})$, where $\Omega_{\mathbf{X}}$ is the *subobject classifier* of $\mathbf{X}$. On the other hand, recall from 1 that $\mathscr{R}(\mathbf{S})$ is $\mathbf{S}^{\mathbb{S}^{\mathrm{fp}}}$ and for any object G in $\mathbf{X}$ direct image of the corresponding classifying morphism $\ulcorner G \urcorner : \mathbf{X} \to \mathscr{R}(\mathbf{S})$ is given by

$$\ulcorner G \urcorner_* (X)(n) = \hom(G^n, X);$$

thus for H_G this means that $H_G(n)$ is equal to the Heyting algebra of subobjects of G^n.

This means that once H_G, considered as an internal locale, has enough points, it will be the lattice of open sets of an internal topological space in $\mathbf{S}^{\mathbb{S}^{\mathrm{fp}}}$. Moreover choices of a basepoint $* : 1 \to G$ for the generator G are in one-to-one correspondence with liftings of the classifying geometric morphism $\ulcorner G \urcorner : \mathbf{X} \to \mathscr{R}(\mathbf{S})$ to $\mathscr{R}(\mathbf{S})/U_{\mathbf{S}}$, i. e. to the topos of Γ-sets. Thus such a basepointed generator will assign to $\mathbf{X}$ a Γ-space in this case. But also in general, one sees that bounded **S**-toposes with pointed generators are essentially the same thing as "Γ-locales" over **S**. It is easy to describe explicitly the corresponding locale; the result can be summarized as follows:

2.6. **Proposition.** *For any bounded Grothendieck topos* $\mathbf{X}$, *there is a* Γ*-locale* H, *i. e. a functor from finite pointed sets to locales, which determines* $\mathbf{X}$ *up to equivalence. Explicitly, given any pointed generator* $* : 1 \to G$ *of* $\mathbf{X}$, *the lattice of open sets of* $H(n_+)$ *is the lattice of subobjects of* G^n. *Here* $n_+ = \{0, 1, ..., n\}$. *The topos* $\mathbf{X}$ *can be then recovered as the topos of internal* H*-sheaves in* Γ*-sets.*

3. Examples of bimodel monoidal structures

Note that for any topos $\mathbf{X}$, the category $\mathscr{F}(\mathbf{X})$ has a monoidal structure given by composition. We saw in 2 that for $\mathbf{X} = \mathscr{R}(\mathbf{S})$, another monoidal structure becomes important – that of smash product $\wedge$. In fact it has been proved by Lydakis in [22] that there is a s. c. *assembly map* from smash product to composition product of Γ-spaces which induces a stable equivalence of corresponding spectra. Thus one might say that composition of endofunctors of simplicial sets corresponding to Γ-spaces is "commutative up to stable equivalence".

Although we do not know about such commutativity phenomena for general $\mathscr{F}(\mathbf{X})$, we would like to give examples of closely related monoidal structures arising from full subcategories $\mathscr{E}(\mathbb{T})/I(\mathbb{T})$ of $\hat{\mathscr{F}}(\mathbb{T})$ introduced in the end of 1. These have been investigated in [13].

In fact monoidal categories of bimodels have been studied by many authors (for one of the earliest investigations see e. g. [11]).

3.1. **Examples.**

3.1.1. A bimodel of the theory of sets $\mathbb{S}$ is, clearly, the same as just a set, and $\mathscr{E}(\mathbb{S})$ is equivalent to the category of sets with monoidal structure given by cartesian product.

3.1.2. For the theory $\mathbb{A}$ of abelian groups $\mathscr{E}(\mathbb{A})$ is equivalent to the category of abelian groups Ab with tensor product as the monoidal operation.

3.1.3. In fact, more generally, for any ring R, the category $\mathscr{E}(\mathbb{T}_R)$ for the theory $\mathbb{T}_R$ of (left) R-modules is well known to be equivalent to the category of R-R-bimodules, with unit R and the monoidal structure given by $\_\otimes_R\_$.

3.1.4. Another widely known fact is that for the theory $\mathbb{G}$ of groups one has

$$\mathscr{E}(\mathbb{G}) \simeq \mathrm{Set},$$

with the cartesian product monoidal structure. Indeed, it is proved in [8] that any comultiplication

$$G \to G * G$$

on a group G which has a two-sided counit, is isomorphic to

$$F(S) \xrightarrow{F(\text{diagonal})} F(S \times S) \subset F(S) * F(S),$$

where the rightmost inclusion corresponds to the kernel of the canonical homomorphism $F(S) * F(S) \to F(S) \times F(S)$. In other words, for any adjunction $L \dashv R$ in $\mathscr{E}(\mathbb{G})$, the right adjoint is isomorphic to the functor $G \mapsto G^S$ for some set S.

Closer to our theme is the case when $\mathbb{T}$ is the theory generated by a single constant, i. e. such that $\mathbb{T}$-**mod** is the category of pointed sets. It is equally well known that the result is again the category of pointed sets, with the smash product as monoidal structure. More generally, for a set I, let $\mathbb{T}_I$ be the theory generated by the set of constants I. Thus $\hom_{\mathbb{T}_I}(X^n, X) = \{x_1, ..., x_n\} \sqcup I$, and $\mathbb{T}$-**mod** is equivalent to the coslice I/Set. One then has

3.2. **Theorem.** *For a set I, the monoidal category $\mathscr{E}(\mathbb{T}_I)$ is equivalent to the category of factorizations of the map*

$$\mathrm{const} : I \to I^I$$

assigning to $i \in I$ the constant self-map with value i. In detail, the objects of the latter category are pairs

$$I \xrightarrow{f} X \xrightarrow{e} I^I$$

with $e(f(i))(i') = i$ for all $i, i' \in I$. Morphisms from (e, f) to (e', f') are maps x with $xf = f'$, $e'x = e$. The monoidal operation corresponding to composition is given by

$$\left(I \xrightarrow{f_X} X \xrightarrow{e_X} I^I\right) \circ \left(I \xrightarrow{f_Y} Y \xrightarrow{e_Y} I^I\right) = \left(I \xrightarrow{f_\wedge} X \wedge_I Y \xrightarrow{e_\wedge} I^I\right),$$

where

$$X \wedge_I Y = (X \times Y)/ \langle x, f_Y(i)\rangle \sim \langle f_X(e_X(x)i), y\rangle$$

for $x \in X$, $i \in I$, $y \in Y$; the maps are given by

$$e_\wedge(x \wedge y) = e_X(x)e_Y(y) \text{ (composition)},$$
$$f_\wedge(i) = f_X(i) \wedge f_Y(i),$$

where $x \wedge y$ stands for the equivalence class of $\langle x, y\rangle$. Finally, the unit for the monoidal structure is

$$I_I := \left(I \xrightarrow{\mathrm{const}} \mathrm{const}(I) \cup \{\mathrm{Id}_I\} \xrightarrow{\subset} I^I\right),$$

where const *is the map assigning to* $i \in I$ *the constant map with value* i *while* Id_I *denotes the identity map of* I*.*

Proof. One assigns to an object (e, f) as above the functor

$$R_{(e,f)} : I/\mathrm{Set} \to I/\mathrm{Set}$$

carrying $s : I \to S$ to $R_{(e,f)}(s) : I \to \{ u \mid uf = s \}$ with

$$R_{(e,f)}(s)(i)(x) = s(e(x)i).$$

Then $R_{(e,f)}$ has a left adjoint $L_{(e,f)}$ given by

$$L_{(e,f)}(f_Y : I \to Y) = (f_\wedge : I \to X \wedge_I Y)$$

as above. Checking that composition of such functors corresponds to the described monoidal structure is straightforward.

This gives immediately also a description of the slice of $\mathscr{E}(\mathbb{T}_I)$ over the unit:

3.3. **Corollary.** *For a set* I*, the monoidal category* $\mathscr{E}(\mathbb{T}_I)/I_I$ *has, up to equivalence, the following description: objects are sets* X *equipped with a disjoint sum decomposition*

$$X = X_0 \sqcup \coprod_{i \in I} X_i$$

and basepoints $*_i \in X_i$*, for* $i \in I$*. Morphisms are maps preserving decomposition and basepoints. Unit of the monoidal structure is the terminal, and the operation* $(X, Y) \mapsto X \wedge Y$ *is given by*

$$(X \wedge Y)_0 = X_0 \times Y_0; \quad (X \wedge Y)_i = (X_0 \times Y_i \sqcup X_i \times Y) / \langle x_0, *_i \rangle \sim \langle *_i, y \rangle$$

for any $x_0 \in X_0$*,* $y \in Y$*.*

3.4. **Definition.** Let $\mathrm{n}^2\mathbb{A}$ be the theory of class two nilpotent rings without unit, i. e. rings with the identities

$$(xy)z = x(yz) = 0.$$

3.5. **Theorem.** *The category* $\mathscr{E}(\mathrm{n}^2\mathbb{A})$ *is equivalent to the category whose objects are triples* (A, m_0, m_1) *where* A *is an abelian group and* $m_0, m_1 : A \to A \otimes A$ *are homomorphisms. A morphism from* (A, m_0, m_1) *to* (A', m_0', m_1') *is a homomorphism* $f : A \to A'$ *satisfying*

$m_i'f = f(m_i \otimes m_i)$, $i = 0, 1$. *The monoidal structure corresponding to the composition in* $\mathscr{E}(\mathrm{n}^2\mathbb{A})$ *is given by*

$$\begin{aligned}&(A, m_0, m_1) \circ (B, n_0, n_1)\\ &\quad = (A \otimes B, \tau_{23}(m_0 \otimes n_0 + (\tau m_1) \otimes n_1), \tau_{23}((\tau m_0) \otimes n_1 + m_1 \otimes n_0)),\end{aligned}$$

where τ *is the symmetry isomorphism and* $\tau_{23} = \mathrm{Id} \otimes \tau \otimes \mathrm{Id}$. *The unit object is* $(\mathbb{Z}, 1, 0)$.

Proof. For any model X of $\mathrm{n}^2\mathbb{A}$, let $X^2 \subseteq X$ be the image of the multiplication operation, and let $X_{\mathrm{ab}} = X/X^2$. Thus up to isomorphism the additive structure of X is determined by a symmetric 2-cocycle of X_{ab} with values in X^2: given such a cocycle χ, one obtains an isomorphism of X, as an abelian group, to the set $X^2 \times X_{\mathrm{ab}}$ with addition

$$(\xi, x) + (\eta, y) = (\xi + \eta + \chi(x, y), x + y).$$

Multiplicative structure of X is determined by any surjective homomorphism $X_{\mathrm{ab}} \otimes X_{\mathrm{ab}} \twoheadrightarrow X^2$.

Now suppose X carries structure of an internal abelian group in $(\mathrm{n}^2\mathbb{A}\text{-}\mathbf{mod})^{\mathrm{op}}$, i. e. X has coaddition

$$\Delta : X \to X * X$$

and cozero

$$0 : X \to 0,$$

where $*$ denotes coproduct in $\mathrm{n}^2\mathbb{A}$-**mod**. For any X, Y in $\mathrm{n}^2\mathbb{A}$-**mod** this coproduct fits in a short exact sequence

$$0 \to X_{\mathrm{ab}} \otimes Y_{\mathrm{ab}} \oplus Y_{\mathrm{ab}} \otimes X_{\mathrm{ab}} \xrightarrow{\iota} X * Y \to X \times Y \to 0$$

with $\iota(\bar{x} \otimes \bar{y}) = \iota_X(x)\iota_Y(y)$ and $\iota(\bar{y} \otimes \bar{x}) = \iota_Y(y)\iota_X(x)$, where ι_X and ι_Y are the coproduct embeddings. Since the coaddition has a two-sided cozero, one sees that $\Delta(x)$ for $x \in X$ may be uniquely written in the form

$$\Delta(x) = \iota_\ell(x) + \lambda(x) + \rho(x) + \iota_r(x)$$

where $\iota_\ell, \iota_r : X \to X * X$ are the coproduct embeddings and $\lambda(x) \in X_{\mathrm{ab}} \otimes X_{\mathrm{ab}} \cong \iota_\ell(X)\iota_r(X)$, $\rho(x) \in X_{\mathrm{ab}} \otimes X_{\mathrm{ab}} \cong \iota_r(X)\iota_\ell(X)$. Furthermore Δ is a multiplicative homomorphism which implies

$$\lambda(xy) + \rho(xy) = \iota_\ell(x)\iota_r(y) + \iota_r(x)\iota_\ell(y) \in X_{\mathrm{ab}} \otimes X_{\mathrm{ab}} \;\oplus\; X_{\mathrm{ab}} \otimes X_{\mathrm{ab}},$$

which means that the diagram

$$
(D)\qquad \begin{array}{ccc}
X_{\mathrm{ab}}\otimes X_{\mathrm{ab}} & \xrightarrow{\text{diagonal}} & X_{\mathrm{ab}}\otimes X_{\mathrm{ab}} \ \oplus \ X_{\mathrm{ab}}\otimes X_{\mathrm{ab}} \\
\Big\downarrow \mu & & \Big\downarrow \iota \\
X & \xrightarrow{\lambda+\rho} & X * X
\end{array}
$$

commutes, where μ is the multiplication map. Since the diagonal and ι are monos, this yields a short exact sequence

$$(S)\qquad 0 \to X_{\mathrm{ab}}\otimes X_{\mathrm{ab}} \xrightarrow{\mu} X \to X_{\mathrm{ab}} \to 0,$$

i. e. $X^2 \cong X_{\mathrm{ab}}\otimes X_{\mathrm{ab}}$; let's identify these groups from now on. Now additivity of Δ implies that both λ and ρ are additive, and composing the diagonal in (D) with either of the projections

$$\pi_\ell, \pi_r : X_{\mathrm{ab}}\otimes X_{\mathrm{ab}} \ \oplus \ X_{\mathrm{ab}}\otimes X_{\mathrm{ab}} \to X_{\mathrm{ab}}\otimes X_{\mathrm{ab}}$$

shows that both λ and ρ provide retractions for μ in (S). This means that X is isomorphic to the "truncated tensor algebra" of X_{ab},

$$T(X_{\mathrm{ab}}) = X_{\mathrm{ab}} \ \oplus \ X_{\mathrm{ab}}\otimes X_{\mathrm{ab}}.$$

So up to isomorphism an adjoint pair between $\mathrm{n}^2\mathbb{A}$-**mod** and abelian groups is determined by a single abelian group.

Now let us add into consideration the comultiplication; suppose an X as above, determined by $X_{\mathrm{ab}} = A$, is equipped with a morphism

$$T(A) \to T(A) * T(A),$$

or equivalently

$$A \to U(T(A) * T(A)),$$

where U is the forgetful functor right adjoint to T. If this has to define a coring structure on X, comultiplying from either side by cozero must give cozero, i. e. the above map lands in

$$\mathrm{Ker}(\ T(A) * T(A) \to T(A) \times T(A)),$$

i. e. is determined by $(m_0, m_1) : A \to A\otimes A \ \oplus \ A\otimes A$. For a class two nilpotent ring Z, multiplication on $\mathrm{Hom}(A, U(Z))$ determined by these data is given by

$$(f\cdot g)a = (fg)(m_0a) + (gf)(m_1a),$$

in other words, for $f, g \in \mathrm{Hom}(A, U(Z))$, their product is the composite

$$A \xrightarrow{(m_0,m_1)} A\otimes A \oplus A\otimes A \xrightarrow{f\otimes g+g\otimes f} Z\otimes Z \oplus Z\otimes Z \xrightarrow{\mu\oplus\mu} Z\oplus Z \xrightarrow{+} Z.$$

It remains to calculate the effect of composing the adjunctions on the corresponding corings, which is straightforward.

For the next example, let $\mathrm{n}^2\mathbb{L}$ be the theory of class two nilpotent square zero rings, i. e. rings in which the identities

$$x^2 = xyz = 0$$

hold. For these one has

3.6. **Theorem.** *The category $\mathscr{E}(\mathrm{n}^2\mathbb{L})$ is equivalent to the category with objects looking like*

$$(\delta : C \to C, \mu : C \to C \otimes C)$$

where C is an abelian group, δ is an additive endomorphism with $2\delta = 0$, and μ is a cocommutative comultiplication, i. e. $\tau\mu = \mu$, where τ is the symmetry. Morphisms from (δ, μ) to (δ', μ') are homomorphisms f satisfying $\delta' f = f\delta$, $\mu' f = (f \otimes f)\mu$. The monoidal structure corresponding to the composition in $\mathscr{E}(\mathrm{n}^2\mathbb{L})$ is

$$(\delta_C, \mu_C) \circ (\delta_D, \mu_D) = (\delta_{C \otimes D}, \mu_{C \otimes D})$$

with

$$\delta_{C \otimes D}(c \otimes d) = c \otimes \delta_D(d) + \delta_C(c) \otimes \overline{\mu_D(d)},$$

where $\overline{\mu_D(d)}$ denotes $\mu_D(d)$ modulo elements of the form $d_1 \otimes d_2 + d_2 \otimes d_1$, and

$$\mu_{C \otimes D} = (\mathrm{Id}_C \otimes \tau \otimes \mathrm{Id}_D)(\mu_C \otimes \mu_D);$$

the unit object is $(0 : \mathbb{Z} \to \mathbb{Z}, 1 \otimes 1 : \mathbb{Z} \cong \mathbb{Z} \otimes \mathbb{Z})$.

Proof. Let X be any model of $\mathrm{n}^2\mathbb{L}$. Up to isomorphism it is determined by the exqact sequence of abelian groups

$$\Lambda^2 C \xrightarrow{m} X \to C \to 0,$$

where $C = X/X^2$ and $m(\bar{x} \wedge \bar{x}') = xx'$. Now suppose X carries a coabelian cogroup structure in $\mathrm{n}^2\mathbb{L}$-**mod**, with the coaddition

$$\Delta : X \to X \sqcup X.$$

Since $X \to 0$ must be a two-sided cozero, one has

$$\Delta(x) = \iota_\ell(X) + \Delta_0(x) + \iota_r(x),$$

with $\iota_\ell, \iota_r : X \to X \sqcup X$ the coproduct embeddings, and $\mathrm{im}(\Delta_0) \subseteq \iota_\ell(X)\iota_r(X) \cong C \otimes C$. But Δ must be a multiplicative homomorphism, which implies

$$\Delta_0 m(c \wedge c') = c \otimes c' - c' \otimes c.$$

Indeed one has

$$\begin{aligned}\Delta(x)\Delta(x') &= (\iota_\ell(x)+\Delta_0(x)+\iota_r(x))(\iota_\ell(x')+\Delta_0(x')+\iota_r(x'))\\ &= \iota_\ell(xx')+\iota_\ell(x)\iota_r(x')+\iota_r(x)\iota_\ell(x')+\iota_r(xx')\\ &= \iota_\ell(xx')+\iota_\ell(x)\iota_r(x')+\iota_\ell(x')\iota_r(x)+\iota_r(xx').\end{aligned}$$

Hence in particular we see that m is a monomorphism, since $\Delta_0 m$ is. Furthermore mutativity (=cocommutativity) of Δ implies that Δ_0 takes values in the subgroup $A^2C \subseteq C\otimes C$ of antisymmetric tensors, and hence induces

$$\delta : C = X/\Lambda^2C \to A^2C/\Lambda^2C \cong {}_2C = \{x\in C | 2x=0\}$$

where the latter isomorphism may be deduced from the short exact sequence

$$0 \to A^2C/\Lambda^2C \to (C\otimes C)/\Lambda^2C \to (C\otimes C)/A^2C \to 0,$$

as $(C\otimes C)/\Lambda^2C \cong S^2C$ is the symmetric square and

$$(C\otimes C)/A^2C = (C\otimes C)/\mathrm{Ker}(\mathrm{Id}_{C\otimes C}+\tau) \cong \mathrm{Im}(\mathrm{Id}_{C\otimes C}+\tau).$$

It follows that there is a commutative square

$$\begin{array}{ccc} X & \xrightarrow{\Delta_0} & A^2C \\ \downarrow & & \downarrow \\ C & \xrightarrow{\delta} & {}_2C \end{array}$$

and since the induced homomorphism on kernels of vertical surjections is an isomorphism, a standard diagram-chasing shows that the square is a pullback. Hence the whole coabelian cogroup X is up to isomorphism determined by $\delta : C \to {}_2C$. Moreover any morphism $f : X \to Y$ between coabelian cogroups which preserves coaddition is determined by the induced map $X/X^2 \to Y/Y^2$ since the upper left corner map in a morphism between pullback squares is determined by the remaining maps.

Now let us turn to the comultiplication, which is another morphism of $\mathrm{n}^2\mathbb{L}$-models

$$M : X \to X \sqcup X.$$

Since comultiplying by cozero must be cozero, it follows that

$$\mathrm{Im}(M) \subseteq \mathrm{Ker}(X\sqcup X \to X\times X) = \iota_\ell(X)\iota_r(X),$$

hence $X^2 \subseteq \mathrm{Ker}(M)$. So M factors as

$$X \twoheadrightarrow C \xrightarrow{\mu} C\otimes C \rightarrowtail X\sqcup X$$

where μ may be any homomorphism. Moreover the comultiplication must be coexterior, i. e. satisfy the coidentity dual to $x^2 = 0$, which means

$$\mathrm{Im}(\mu) \subseteq \mathrm{Ker}(C \otimes C \rightarrowtail X \sqcup X \xrightarrow{\nabla} X)$$

where ∇ is the codiagonal. But the latter composition is easily seen to coincide with

$$C \otimes C \twoheadrightarrow \Lambda^2 C \xrightarrow{m} X$$

so that $\mathrm{Im}(\mu) \subseteq \mathrm{Ker}(1 - \tau)$ which means comultiplication is cocommutative.

It remains to check the effect of composition in $\mathscr{E}(\mathrm{n}^2\mathbb{L})$ on the corresponding coexterior rings. This is straightforward and gives the result claimed.

Our final example of "exotic" monoidal structures is given by the category $\mathscr{E}(\mathrm{n}^2\mathbb{G})$ for the theory of class two nilpotent groups. For this let us recall the notion of the universal degree 2 map.

3.7. **Definition.** For an abelian group A, let $P^2(A)$ be the abelian group generated by symbols $p_2(a)$ for $a \in A$, subject to the relations

$$p_2(x+y+z) = p_2(x+y) + p_2(x+z) + p_2(y+z) - p_2(x) - p_2(y) - p_2(z).$$

Thus the map $p_2 : A \to P^2(A)$ is universal among maps with domain A which satisfy the above relations. There is a natural short exact sequence

$$0 \to S^2(A) \xrightarrow{\iota_A} P^2(A) \xrightarrow{\pi_A} A \to 0$$

with $\iota_A(xy) = p_2(x+y) - p_2(x) - p_2(y)$, $xy \in S^2(A)$, and $\pi_A p_2(a) = a$.

3.8. **Theorem.** *The category $\mathscr{E}(\mathrm{n}^2\mathbb{G})$ has, up to equivalence, the following description: objects are homomorphisms*

$$\sigma : A \to P^2(A)$$

for abelian groups A, satisfying

$$\pi_A \circ \sigma = \mathrm{Id}_A,$$

and morphisms from $\sigma : A \to P^2(A)$ to $\sigma' : A' \to P^2(A')$ are homomorphisms $f : A \to A'$ with $\sigma' f = (P^2(f))\sigma$. The monoidal structure is given by

$$(A \xrightarrow{\sigma} P^2 A) \circ (B \xrightarrow{\tau} P^2 B)$$
$$= (A \otimes B \xrightarrow{\sigma \otimes \tau} (P^2 A) \otimes (P^2 B) \xrightarrow{p_{A,B}} P^2(A \otimes B)),$$

where the natural transformation p is given by

$$p_{A,B}(p_2(a) \otimes p_2(b)) = p_2(a \otimes b).$$

The unit is $(\iota : \mathbb{Z} \to P^2\mathbb{Z})$ with $\iota(n) = np_2(1)$.

Proof. Let $\Delta : G \to G *_2 G$ be a comultiplication homomorphism on a class 2 nilpotent group G, where $*_2$ denotes coproduct in $\mathrm{n}^2\mathbb{G}$-**mod**. If Δ has a two-sided counit, then for all $g \in G$

$$\Delta(g) = \iota_\ell(g)\Delta_0(g)\iota_r(g)$$

where $\iota_\ell, \iota_r : G \to G *_2 G$ are the coproduct embeddings and

$$\mathrm{Im}(\Delta_0) \subseteq \mathrm{Ker}(G *_2 G \to G \times G) \cong A \otimes A$$

where $A = G_{\mathrm{ab}} = G/[G, G]$. The latter isomorphism relates $\bar{g} \otimes \bar{g}'$ to $[\iota_\ell g, \iota_r g']$: more generally $X *_2 Y$ for any X, Y fits in a central extension of groups

$$X_{\mathrm{ab}} \otimes Y_{\mathrm{ab}} \rightarrowtail X *_2 Y \twoheadrightarrow X \times Y.$$

Now $\Delta(xy) = \Delta(x)\Delta(y)$ and $\Delta([x, y]) = [\Delta(x), \Delta(y)]$ imply

$$\Delta_0(xy) = [\iota_\ell x, \iota_r y]\Delta_0(x)\Delta_0(y)$$
$$\Delta_0([x, y]) = [\iota_\ell x, \iota_r y][\iota_\ell y, \iota_r x]^{-1}.$$

This means that Δ_0 factors via $\delta : A \to S^2A$ to fill in the diagram

$$\begin{array}{ccccc} G & \xrightarrow{\Delta_0} & [\iota_\ell(G), \iota_r(G)] & \overset{\sim}{=\!=} & A \otimes A \\ \downarrow & & & & \downarrow \pi \\ A & & \xrightarrow{\qquad\qquad\delta\qquad\qquad} & & S^2A \end{array}$$

where π is the canonical map; moreover the composition

$$\Lambda^2 A \xrightarrow{[-,-]} G \xrightarrow{\Delta_0} [\iota_\ell(G), \iota_r(G)] \cong A \otimes A$$

is the canonical embedding given by

$$a \wedge a' \mapsto a \otimes a' - a' \otimes a.$$

It follows that the diagram is a pullback of sets and G is up to isomorphism the set $\{(x, a) \in A \otimes A | \pi x = \delta a\}$ with multiplication

$$(x, a)(x', a') = (x + x' + a \otimes a', a + a'),$$

whilst δ must satisfy

$$\delta(x + y) - \delta(x) - \delta(y) = xy,$$

$x, y \in A$. This means that δ factors through p_2 to give a homomorphism

$$\delta' : P^2(A) \to S^2(A)$$

satisfying

$$\delta' \circ \iota_A = \mathrm{Id},$$

and such homomorphisms are clearly in one-to-one correspondence with sections of π_A. Moreover given a homomorphism between such objects which preserves comultiplication, it will determine a transformation of corresponding pullback squares, and the upper left corner map of this transformation will be determined by the remaining maps. Finally one calculates the effect of the endofunctor composition on the representing objects, which is tedious but straightforward.

4. Cohomology

We now turn to cohomology groups of toposes corresponding to our notion of homotopy type. Thus coefficients for cohomology of a topos $\mathbf{X}$ will be internal abelian groups in $\mathscr{F}(\mathbf{X})$. (Note that the latter category does not necessarily have any products; still observe that it has small homs, so by an internal abelian group we can understand an object A together with a lifting of the functor $\hom_{\mathscr{F}(\mathbf{X})}(\_, A) : \mathscr{F}(\mathbf{X}) \to \mathrm{Set}$ to abelian groups, along the forgetful functor $\mathrm{Ab} \to \mathrm{Set}$.)

When $\mathbf{X}$ is a classifying topos for an algebraic theory $\mathbb{T}$, we will produce explicit complexes for calculating these cohomologies, and will give interpretations of elements in lower dimensions by certain extensions. Exposition is mainly based on [15], with some generalizations corresponding to switching from cohomology with coefficients in objects of $\mathrm{Ab}(\mathbb{T}\text{-}\mathbf{mod})^{\mathbb{T}^{\mathrm{op}}}$ to our slightly more general coefficients in objects of $\mathrm{Ab}(\mathscr{F}(\mathbb{T}))$.

In particular, there is the global sections functor $\Gamma : \mathrm{Ab}(\mathscr{F}(\mathbf{X})) \to \mathrm{Ab}$, $\Gamma(A) = \hom_{\mathscr{F}(\mathbf{X})}(\mathrm{Id}_{\mathbf{X}}, A)$. We then define

4.1. **Definition.** For a Grothendieck topos $\mathbf{X}$ and an internal abelian group A in $\mathscr{F}(\mathbf{X})$, the *cohomology groups* $H^n(\mathbf{X}; A)$ of $\mathbf{X}$ with coefficients in A, $n \geqslant 0$, are values on A of an universal exact connected sequence of functors $H^n(\mathbf{X}; \_) : \mathrm{Ab}(\mathscr{F}(\mathbf{X})) \to \mathrm{Ab}$ with $H^0(\mathbf{X}; \_) = \Gamma$.

Thus for a general Grothendieck topos it is not clear whether these cohomology groups exist. Situation is better when $\mathbf{X}$ is a classifying topos for an algebraic theory, because of the following:

4.2. **Proposition.** *For an algebraic theory $\mathbb{T}$ over a base topos $\mathbf{S}$ with a natural numbers object and a model M of $\mathbb{T}$ in $\mathbf{S}$, there exists an internal ringoid $\mathscr{U}(M)$ in $\mathbf{S}$ (the enveloping ringoid of M) such that the category $\mathrm{Ab}(\mathbb{T}\text{-}\mathbf{mod}(\mathbf{S})/M)$ is equivalent to the category of internal left $\mathscr{U}(M)$-modules in $\mathbf{S}$.*

Proof. The object of objects of $\mathscr{U}(M)$ will be M. Here as usual we identify a model M with its underlying object in $\mathbf{S}$, so that M is viewed as a $\mathrm{T}_{\mathbf{S}}$-algebra $\mu : \mathrm{T}_{\mathbf{S}}(M) \to M$.

In presence of the natural numbers object, we can present morphisms of an internal ringoid such as $\mathscr{U}(M)$ by generators and relations. The generating object for morphisms will be

$$G = \sum_n \mathbb{T}(n) \times M^n \times n.$$

Here we have used notation from [20], so that n stands for the generic finite cardinal. In more detail, n is the object of $\mathbf{S}/\mathrm{N}$ given by the composite

$$\mathrm{N} \times \mathrm{N} \xrightarrow{+} \mathrm{N} \xrightarrow{s} \mathrm{N},$$

where s is the successor map of the object of natural numbers; and G is given by

$$G = \coprod_{\mathrm{N}} \mathrm{T}_{\mathbf{S}/\mathrm{N}}(n) \times (\mathrm{N}^*M)^n \times n,$$

where $\mathrm{T}_{\mathbf{S}/\mathrm{N}}$ is the monad induced by $\mathbb{T}$ on $\mathbf{S}/\mathrm{N}$. In the internal language, elements of G are of the form $\langle u, m_1, ..., m_n, i\rangle$ with an n-ary operation u of $\mathbb{T}$, an n-tuple of elements of M, and $1 \leqslant i \leqslant n$. This element represents a morphism from m_i to $u(m_1, ..., m_n)$, thus the domain and codomain maps $G \rightrightarrows M$ are given, respectively, by

$$\coprod_{\mathrm{N}} \mathrm{T}_{\mathbf{S}/\mathrm{N}}(n) \times (\mathrm{N}^*M)^n \times n \xrightarrow{\coprod_{\mathrm{N}}(\text{projection})}$$
$$\coprod_{\mathrm{N}} (\mathrm{N}^*M)^n \times n \xrightarrow{\coprod_{\mathrm{N}}(\text{evaluation})} \coprod_{\mathrm{N}} \mathrm{N}^*M \xrightarrow{\text{counit}} M$$

and

$$\coprod_{\mathrm{N}} \mathrm{T}_{\mathbf{S}/\mathrm{N}}(n) \times (\mathrm{N}^*M)^n \times n \xrightarrow{\coprod_{\mathrm{N}}(\text{projection})}$$
$$\coprod_{\mathrm{N}} \mathrm{T}_{\mathbf{S}/\mathrm{N}}(n) \times (\mathrm{N}^*M)^n \xrightarrow{\coprod_{\mathrm{N}}(\mathrm{T}_{\mathbf{S}/\mathrm{N}}(n)\times\text{strength})}$$
$$\coprod_{\mathrm{N}} \mathrm{T}_{\mathbf{S}/\mathrm{N}}(n) \times \mathrm{T}_{\mathbf{S}/\mathrm{N}}(\mathrm{N}^*M)^{\mathrm{T}_{\mathbf{S}/\mathrm{N}}(n)} \xrightarrow{\text{evaluation}} \coprod_{\mathrm{N}} \mathrm{T}_{\mathbf{S}/\mathrm{N}}(\mathrm{N}^*M) \xrightarrow{\cong}$$
$$\coprod_{\mathrm{N}} \mathrm{N}^*\mathrm{T}_{\mathbf{S}}(M) \xrightarrow{\coprod_{\mathrm{N}} \mathrm{N}^*\mu} \coprod_{\mathrm{N}} \mathrm{N}^*M \xrightarrow{\text{counit}} M.$$

The object of relations is

$$R = \sum_{n,k} \mathbb{T}(n) \times \mathbb{T}(k)^n \times M^k \times k,$$

where n, k now stands for the generic pair of finite cardinals. We will present relations in the internal language only; it is straightforward to convert them into the diagrammatic form. The relation corresponding to $\langle u, v_1, ..., v_n, m_1, ..., m_k, j\rangle$ for u in $\mathbb{T}(n)$, $v_1, ..., v_n$ in $\mathbb{T}(k)$, $m_1, ..., m_k$ in M and $1 \leqslant j \leqslant k$ has the form

$$\langle u(v_1, ..., v_n), m_1, ..., m_k, j\rangle = \sum_{i=1}^{n} \langle u, v_1(m_1, ..., m_k), ..., v_n(m_1, ..., m_k), i\rangle \circ \langle v_i, m_1, ..., m_k, j\rangle\,,$$

with $\circ$ denoting the composition in the ringoid.

To construct a functor Φ from $\mathrm{Ab}(\mathbb{T}\text{-}\mathbf{mod}(\mathbf{S})/M)$ to $\mathscr{U}(M)$-modules, observe that any object of $\mathrm{Ab}(\mathbb{T}\text{-}\mathbf{mod}(\mathbf{S})/M)$ can be viewed in particular as an internal abelian group $A \to M$ in $\mathbf{S}/M$, i. e. as a family $(A_m)_{m\in M}$ of internal abelian groups indexed by M. The $\mathbb{T}$-model structure on this object results further in the family of maps of the form

$$u_A : A_{m_1} \times ... \times A_{m_n} \to A_{u(m_1,...,m_n)},$$

where n varies over N and u over n-ary operations of $\mathbb{T}$. Compatibility between abelian group structure and $\mathbb{T}$-model structure means that all the u_A's are abelian group homomorphisms. Hence they are uniquely determined by their composites with canonical inclusions

$$A_{m_i} \to A_{m_1} \times ... \times A_{m_n},\ 1 \leqslant i \leqslant n.$$

We then declare these composites to be actions on $\Phi(A \to M)$ of morphisms $\langle u, m_1, ..., m_n, i\rangle$ of $\mathscr{U}(M)$, values of $\Phi(A \to M)$ on objects $m \in M$ of $\mathscr{U}(M)$ being A_m. Conditions on A ensuring that it is a $\mathbb{T}$-model then amount precisely to satisfaction of relations from R by $\Phi(A \to M)$.

It is also clear how to construct the functor Ψ in the opposite direction: given an $\mathscr{U}(M)$-module

$$\big((A_m)_{m\in M}, (A_{m_i} \to A_{u(m_1,...,m_n)})_{\langle u,m_1,...,m_n,i\rangle\in G}\big)\,,$$

one obtains an internal abelian group $\coprod_m A_m \to M$ in $\mathbf{S}/M$; one then defines maps u_A as above by

$$u_A(a_1, ..., a_n) = \sum_{i=1}^{n} \langle u, m_1, ..., m_n, i\rangle\, a_i.$$

These maps obviously combine to produce maps $u_A : A^n \to A$ for all operations of $\mathbb{T}$, and conditions of $\mathscr{U}(M)$-module structure mean precisely that this indeed defines an object of $\mathrm{Ab}(\mathbb{T}\text{-}\mathbf{mod}(\mathbf{S})/M)$.

It is then straightforward to check that Φ and Ψ are mutually inverse equivalences of categories.

Remark. For $\mathbf{S}$ = Set this fact is well known; for the most general case treated and relevant references see [30].

4.3. **Corollary.** *For an algebraic theory* $\mathbb{T}$ *on a topos with natural numbers there are internal ringoids* $\mathscr{U}(\mathbb{T})$ *in* $[\mathbb{T}]$ *and* $\hat{\mathscr{U}}(\mathbb{T})$ *in* $\widehat{\mathbb{T}}$ *such that* $\mathrm{Ab}(\mathscr{F}(\mathbb{T}))$ *(resp.,* $\mathrm{Ab}(\hat{\mathscr{F}}(\mathbb{T}))$ *– see 1.16) is equivalent to the category of* $\mathscr{U}(\mathbb{T})$*-modules (resp.,* $\hat{\mathscr{U}}(\mathbb{T})$*-modules).*

Proof. In view of 1.4 (resp. 1.17), this is just a particular case of 4.2, with $\mathbf{S} = [\mathbb{T}]$ (resp. $\widehat{\mathbb{T}}$) and $M = U_{\mathbb{T}}$ (resp. $U^{\mathbb{T}}$).

4.4. **Corollary.** *For any algebraic theory* $\mathbb{T}$ *on a Grothendieck topos* $\mathbf{S}$*, and any* $\mathbb{T}$*-model* M *in* $\mathbf{S}$*, the category* $\mathrm{Ab}(\mathbb{T}\text{-}\mathbf{mod}(\mathbf{S})/M)$ *has enough injectives. If* $\mathbf{S}$ *is a presheaf topos, then it also has enough projectives. In particular, the categories* $\mathrm{Ab}(\mathscr{F}_{\mathbf{S}}(\mathbb{T}))$ *and* $\mathrm{Ab}(\hat{\mathscr{F}}_{\mathbf{S}}(\mathbb{T}))$ *have enough injectives, and if moreover* $\mathbf{S}$ *is a presheaf topos, then they also have enough projectives.*

Proof. Composing the forgetful functor from $\mathrm{Ab}(\mathbb{T}\text{-}\mathbf{mod}(\mathbf{S})/M)$ to $\mathrm{Ab}(\mathbf{S}/M)$ with the above equivalence

$$\mathscr{U}(M)\text{-}\mathbf{mod} \simeq \mathrm{Ab}(\mathbb{T}\text{-}\mathbf{mod}(\mathbf{S})/M)$$

obviously gives the forgetful functor from $\mathscr{U}(M)$-**mod** to $\mathrm{Ab}(\mathbf{S}/M)$. The latter functor has both adjoints, hence $\mathscr{U}(M)$-**mod** will have enough injectives (resp., projectives) whenever $\mathrm{Ab}(\mathbf{S}/M)$ does. But it is well known that $\mathrm{Ab}(\mathbf{E})$ has enough injectives for any Grothendieck topos $\mathbf{E}$, and enough projectives when $\mathbf{E}$ is a presheaf topos. And obviously $\mathbf{S}/M$ is a presheaf topos if $\mathbf{S}$ is.

For the second part, just note that both $[\mathbb{T}]$ and $\widehat{\mathbb{T}}$ are Grothendieck toposes for a Grothendieck topos $\mathbf{S}$, and presheaf toposes for a presheaf topos $\mathbf{S}$.

Thus for the topos $\mathbf{X} = [\mathbb{T}]$, where $\mathbb{T}$ is an algebraic theory over a Grothendieck topos $\mathbf{S}$, the cohomology groups H^n from 4.1 indeed exist and coincide with right derived functors of $H^0 = \Gamma$. We will write simply $H^n(\mathbb{T}; A)$ instead of $H^n([\mathbb{T}]; A)$. In this case we also define, for an internal abelian group D in $\hat{\mathscr{F}}(\mathbb{T})$,

$$H^n(\mathbb{T}; D) = \mathrm{R}^n\Gamma(D),$$

with $\Gamma : \mathrm{Ab}(\hat{\mathscr{F}}(\mathbb{T})) \to \mathrm{Ab}$ the global section functor $\hom(1_{F_{\mathbb{T}}}, \_)$.

4.5. **Examples.**

4.5.1. Let M be a monoid, and consider the theory $\mathbb{T}_M$ whose models are objects with an action of M. Thus the classifying topos $[\mathbb{T}_M]$ contains the generic M-object U_M, whereas $\mathscr{F}(\mathbb{T}_M)$ is the category $[\mathbb{T}_M]^M/U_M$. Now for any category $\mathbf{S}$ with finite limits and any M-object $M \times X \to X$ in $\mathbf{S}$, there is an internal category $M \rtimes X$ in $\mathbf{S}$ such that $\mathbf{S}^M/X$ is equivalent to the category $\mathbf{S}^{M \rtimes X}$ of internal functors on $M \rtimes X$. Explicitly, the object of objects of $M \rtimes X$ is X, the object of morphisms is $M \times X$, the source map is the projection, and the target – the action.

On the other hand, $[\mathbb{T}_M]$ itself is the category of internal functors from finitely presented M-objects to Set. Hence the internal category $M \rtimes U_M$ of $[\mathbb{T}_M]$ can be identified with the functor $M_{\_}$ from finitely presented M-objects to small categories, which assigns to the M-object X the category $M \rtimes X$. We then obtain that $[\mathbb{T}_M]^{M \rtimes U_M}$ is equivalent to the category of internal functors on the Grothendieck construction $\mathbb{U}_M$ of the functor $M_{\_}$. Explicitly, we have $\mathscr{F}(\mathbb{T}_M) \simeq \mathrm{Set}^{\mathbb{U}_M}$, where objects of the category $\mathbb{U}_M$ are elements $x \in X$ of various finitely presented M-sets; and morphisms from $x \in X$ to $y \in Y$ are pairs (f, m), where $f : X \to Y$ is an M-equivariant homomorphism, and $m \in M$ is such that $y = mf(x)$. Accordingly,

$$\mathrm{Ab}(\mathscr{F}(\mathbb{T}_M)) \simeq \mathrm{Ab}^{\mathbb{U}_M}.$$

A similar but simpler description is available for $\hat{\mathscr{F}}(\mathbb{T}_M)$. In this case one has to replace finitely presented M-sets by finitely generated free ones. Now any such M-set is clearly isomorphic to the one of the form $\{1, ..., n\} \times M$, with M acting on itself via multiplication and trivially on the first factor. Thus $\mathbb{U}_M$ gets replaced by its full subcategory on elements of such objects. It is easy to see that this subcategory is in fact equivalent to the product $\mathbb{S}^{\cdot} \times (M \rtimes M)$, where $\mathbb{S}^{\cdot}$ denotes the category of finite pointed sets. Hence we have

$$\hat{\mathscr{F}}(\mathbb{T}_M) \simeq (\mathscr{R}(\mathrm{Set})/U_{\mathrm{Set}})^{M \rtimes M}.$$

4.5.2. Let $\mathbb{G}$ be the theory of groups, considered over Set. Thus the classifying topos $[\mathbb{G}]$ is the category of functors from finitely presented groups $\mathbb{G}^{\mathrm{fp}}$ to Set. So $\mathfrak{Top}_{\mathrm{Set}}([\mathbb{G}], [\mathbb{G}])$ is the category $\mathrm{Gr}([\mathbb{G}])$ of internal groups in $[\mathbb{G}]$, and the generic group $U_{\mathbb{G}}$ corresponds to the inclusion of finitely presented groups into all groups. Thus

$$\mathrm{Ab}(\mathscr{F}([\mathbb{G}])) = \mathrm{Ab}(\mathrm{Gr}([\mathbb{G}])/U_{\mathbb{G}}).$$

Now it is well known that for any group G in any category with finite products $\mathbf{S}$, the category $\mathrm{Ab}(\mathrm{Gr}(\mathbf{S})/G)$ is equivalent to the category of

G-modules in $\mathbf{S}$. Thus also

$$\mathrm{Ab}(\mathscr{F}([\mathbb{G}])) \simeq U_{\mathbb{G}}\text{-}\mathbf{mod}.$$

Identifying $U_{\mathbb{G}}$ with the inclusion $\mathbb{G}^{\mathrm{fp}} \to \mathrm{Gr}$, we see that objects A of $\mathrm{Ab}(\mathscr{F}([\mathbb{G}]))$ can be identified with assignments, to each finitely presented group G, of a G-module A_G, and to each homomorphism $f : G \to H$ between such groups, of an f-equivariant homomorphism $A_f : A_G \to A_H$, in a functorial way.

Similar description is valid for rings, Lie algebras, ... in place of groups. In particular, for commutative rings one can view this from the point of view of algebraic geometry, as in the next example.

4.5.3. Let k be a commutative ring, and let $\mathbb{A}_k$ be the theory of commutative k-algebras, i. e. commutative rings with the additional set of nullary operations, one for each element of k, and equations ensuring that these determine a homomorphism from k to the ring. Thus $[\mathbb{A}_k]$ is the category of covariant functors from finitely presented k-algebras to sets. Observe that there is a full embedding

$$\mathrm{Sch}/\mathrm{Spec}(k) \hookrightarrow [\mathbb{A}_k]$$

of k-schemes into $[\mathbb{A}_k]$, which sends a scheme X to the corresponding functor of points, $A \mapsto \hom_{\mathrm{Sch}/\mathrm{Spec}(k)}(\mathrm{Spec}(A), X)$.

Moreover the (underlying set-valued functor of the) generic k-algebra $U_{\mathbb{A}_k}$ is clearly in the image of this embedding – it is in fact representable by the polynomial algebra $k[t]$, so the corresponding scheme is $\mathrm{Spec}(k[t]) \to \mathrm{Spec}(k)$, i. e. the projection $\mathrm{Spec}(k) \times \mathbb{A}^1 \to \mathrm{Spec}(k)$. Then the fact that $U_{\mathbb{A}_k}$ carries an internal k-algebra structure in $[\mathbb{A}_k]$ corresponds to the fact that the affine line $\mathbb{A}^1$ is a ring scheme, and moreover $\mathrm{Spec}(k) \times \mathbb{A}^1 \to \mathrm{Spec}(k)$ is a k-algebra scheme. Indeed k-algebra scheme structure on a ring scheme R amounts to a point in the k-fold cartesian power R^k of R which is a ring homomorphism in the obvious sense. And in our case this is $\mathrm{Spec}(k) \to (\mathbb{A}^1)^k$ sending $x \in k$ to x considered as a regular function $\mathrm{Spec}(k) \to \mathbb{A}^1$.

Thus we see that any module scheme over the ring scheme $\mathrm{Spec}(k) \times \mathbb{A}^1 \to \mathrm{Spec}(k)$ in $\mathrm{Sch}/\mathrm{Spec}(k)$ determines an object of $\mathrm{Ab}(\mathscr{F}(\mathbb{A}_k))$.

For classifying toposes of algebraic theories, it is possible to make coefficients for cohomology more explicit.

4.6. **Definition.** For an algebraic theory $\mathbb{T}$, a *coefficient system* $A_{\_}$ for $\mathbb{T}$ consists of a family of abelian groups $(A_u)_{u:X^n \to X}$ indexed by all operations from $\mathbb{T}$, and homomorphisms $\_(v_1, ..., v_n) : A_u \to A_{u(v_1,...,v_n)}$

and $u(v_1, ..., v_{i-1}, \_, v_{i+1}, ..., v_n) : A_{v_i} \to A_{u(v_1,...,v_n)}$ for each n-ary operation u, each n-tuple of operations $v_1, ..., v_n$ of equal arity, and each $1 \leqslant i \leqslant n$, which satisfy the following equalities

$$a(v_1, ..., v_n)(w_1, ..., w_m) = a(v_1(w_1, ..., w_m), ..., v_n(w_1, ..., w_m));$$
$$u(v_1, ..., a_i, ..., v_n)(w_1, ..., w_m) =$$
$$u(v_1(w_1, ..., w_m), ..., a_i(w_1, ..., w_m), ..., v_n(w_1, ..., w_m));$$
$$u(v_1, ..., v_n)(w_1, ..., a_j, ..., w_m) =$$
$$\sum_{i=1}^{n} u(v_1(w_1, ..., w_m), ..., v_i(w_1, ..., a_j, ..., w_m), v_n(w_1, ..., w_m))$$

for all u, v_i, w_j and all $a \in A_u$, $a_i \in A_{v_i}$, $a_j \in A_{w_j}$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$.

With the obvious notion of morphism, coefficient systems form a category, and one has

4.7. **Proposition.** *For an algebraic theory* $\mathbb{T}$ *the category of coefficient systems for* $\mathbb{T}$ *is equivalent to* $\mathrm{Ab}(\hat{\mathscr{F}}(\mathbb{T}))$.

Proof. To a coefficient system $A\_$ in the above sense there corresponds a natural system $D = D(A\_)$ on $\mathbb{T}$ given by

$$D_{f:X^n \to X^k} = A_{x_1 f} \oplus ... \oplus A_{x_k f},$$

where $x_1, ..., x_k : X^k \to X$ are the projections, with the left actions given by

$$(a_1, ..., a_k)(v_1, ..., v_n) = (a_1(v_1, ..., v_n), ..., a_k(v_1, ..., v_n))$$

and right actions given by

$$\begin{aligned}(u_1, ..., u_k)(a_1, a_2, ..., a_n) = \\ (u_1(a_1, v_2, ..., v_n) + u_1(v_1, a_2, ..., v_n) + ... + u_1(v_1, v_2, ..., a_n), ..., \\ u_k(a_1, v_2, ..., v_n) + u_k(v_1, a_2, ..., v_n) + ... + u_k(v_1, v_2, ..., a_n)).\end{aligned}$$

On the other hand $\hat{\mathscr{F}}(\mathbb{T}) = \mathfrak{Top}_{\mathbf{S}}(\widehat{\mathbb{T}}, [\mathbb{T}])/I(\mathbb{T}) \simeq \mathbb{T}\text{-}\mathbf{mod}^{\mathbb{T}^{\mathrm{op}}}/I_{\mathbb{T}}$, where $I_{\mathbb{T}}$ sends X^n to the free model on n generators. Thus we can identify $\hat{\mathscr{F}}(\mathbb{T})$ with the full subcategory of $\mathrm{Set}^{\mathbb{T}^{\mathrm{op}} \times \mathbb{T}} / \hom_{\mathbb{T}}$ consisting of those $p : B \to \hom_{\mathbb{T}}$ for which the bifunctor B preserves finite products in the second variable. One can produce from this a natural system D with $D_f = p^{-1}(f)$. Then clearly internal abelian groups in $\hat{\mathscr{F}}(\mathbb{T})$ will correspond to those D which are natural systems of abelian groups.

Moreover the finite product preservation condition corresponds exactly to the requirement that the canonical map

$$D_{f:X^n\to X^k} \to D_{x_1 f} \oplus \ldots \oplus D_{x_k f}$$

is an isomorphism, just as in 1.9. Thus we can define a coefficient system $A_{\_}$ with $A_u = D_u$ for $u : X^n \to X$ in $\mathbb{T}$. It is straightforward to check that this gives an inverse equivalence for $A_{\_} \mapsto D(A_{\_})$ above.

In terms of coefficient systems, we have an explicit complex for calculating cohomology.

4.8. **Definition.** For an algebraic theory $\mathbb{T}$ and a coefficient system $A_{\_}$, let $C^*(\mathbb{T}; A_{\_})$ be the cochain complex with $C^0(\mathbb{T}; A_{\_}) = A_{1_X}$ and

$$C^n(\mathbb{T}; A_{\_}) = \prod_{X \xleftarrow{u} X^{k_1} \xleftarrow{u^{(1)}} X^{k_2} \xleftarrow{u^{(2)}} \ldots \xleftarrow{u^{(n-1)}} X^{k_n}} A_{uu^{(1)}u^{(2)}\ldots u^{(n-1)}}$$

for $n > 0$, with the differentials $\partial : C^n(\mathbb{T}; A_{\_}) \to C^{n+1}(\mathbb{T}; A_{\_})$ given for $f \in C^n(\mathbb{T}; A_{\_})$ by:

$$\begin{aligned}
&(\partial f)(u, u^{(1)}, \ldots, u^{(n)}) = \\
&\sum_{i=1}^{k_1} u(x_1 u^{(1)} \cdots u^{(n)}, \ldots, f(x_i u^{(1)}, u^{(2)}, \ldots, u^{(n)}), \ldots, x_{k_1} u^{(1)} \cdots u^{(n)}) \\
&\qquad - f(uu^{(1)}, u^{(2)}, \ldots, u^{(n)}) + f(u, u^{(1)}u^{(2)}, \ldots, u^{(n)}) - \ldots \\
&\qquad\qquad \pm f(u, u^{(1)}, \ldots, u^{(n-1)}u^{(n)}) \mp f(u, u^{(1)}, \ldots, u^{(n-1)})u^{(n)}
\end{aligned}$$

for $n > 0$ and

$$(\partial a)u = \sum_{i=1}^{n} u(x_1, \ldots, ax_i, \ldots, x_n) - au$$

for $a \in A_{1_X} = C^0(\mathbb{T}; A_{\_})$, $u \in \hom_{\mathbb{T}}(X^n, X)$.

One then has

4.9. **Proposition.** *For a coefficient system $A_{\_}$ on an algebraic theory $\mathbb{T}$ there are isomorphisms*

$$H^n(\mathbb{T}; D(A_{\_})) \cong H^n(C^*(\mathbb{T}; A_{\_})).$$

Proof. Recall from [4] the cochain complex for calculating cohomology of any small category $\mathbb{C}$ with coefficients in any natural system D on

$\mathbb{C}$. It has

$$C^n(\mathbb{C}; D) = \prod_{c_0 \xleftarrow{\gamma_1} c_1 \xleftarrow{\gamma_2} \dots \xleftarrow{\gamma_n} c_n} D_{\gamma_1 \cdots \gamma_n}$$

and the "Hochschild-type" differential $\partial : C^n(\mathbb{C}; D) \to C^{n+1}(\mathbb{C}; D)$ with

$$(\partial f)(\gamma_1, ..., \gamma_{n+1}) = \gamma_1 f(\gamma_2, ..., \gamma_{n+1}) - f(\gamma_1\gamma_2, ..., \gamma_{n+1}) + ... \\ \pm f(\gamma_1, ..., \gamma_n\gamma_{n+1}) \mp f(\gamma_1, ..., \gamma_n)\gamma_{n+1}.$$

Comparing this with $\mathbb{C} = \mathbb{T}$, $D = D(A_{\text{-}})$ to our complex shows that there is a natural surjective map of complexes

$$\pi : C^*(\mathbb{T}; D(A_{\text{-}})) \to C^*(\mathbb{T}; A_{\text{-}})$$

given by projecting onto those components $X^{k_0} \leftarrow X^{k_1} \leftarrow ... \leftarrow X^{k_n}$ with $k_0 = 1$. This situation thus falls into the scope of [3], namely, $C^*(\mathbb{T}; A_{\text{-}})$ is the sum normalized complex for $D(A_{\text{-}})$ in terms of that paper.

Moreover one has

4.10. **Proposition.** *Let D be an internal abelian group in $\hat{\mathscr{F}}(\mathbb{T})$, for an algebraic theory $\mathbb{T}$, and let $A(D)_{\text{-}}$ be the coefficient system corresponding to it by 4.7. Then there are isomorphisms*

$$H^*(\mathbb{T}; D) \cong H^*(C^*(\mathbb{T}; A(D)_{\text{-}})).$$

Proof. First note that the forgetful functor from $\mathrm{Ab}(\mathbb{T}\text{-}\mathbf{mod}(\mathbf{S})/M)$ to $\mathbb{T}\text{-}\mathbf{mod}(\mathbf{S})/M$ has a left adjoint L_M for any algebraic theory $\mathbb{T}$ and any $\mathbb{T}$-model M. This follows, e. g., from the adjoint lifting theorem of [16]: one has the commutative diagram

$$\begin{array}{ccccc}
 & & \mathrm{Ab}(\mathbb{T}\text{-}\mathbf{mod}(\mathbf{S})/M) & \longrightarrow & \mathbb{T}\text{-}\mathbf{mod}(\mathbf{S})/M \\
 & \overset{\simeq}{\diagup} & & & \\
\mathscr{U}(M)\text{-}\mathbf{mod} & & & & \big\downarrow \\
\big\downarrow & & & & \\
\mathrm{Ab}(\mathbf{S}/M) & & \longrightarrow & & \mathbf{S}/M
\end{array}$$

with vertical functors monadic, and the lower horizontal one having a left adjoint; since all categories in sight are cocomplete, the aforementioned lifting theorem from [16] applies to produce the required left adjoint L_M to the upper horizontal functor.

Recall further that the Barr-Beck cotriple resolution $\mathrm{T}^*(M)$ of M is an augmented simplicial object in $\mathbb{T}$-$\mathbf{mod}(\mathbf{S})$, with n-th component $\mathrm{T}_{\mathbf{S}}^{n+1}(M)$ and augmentation $\mathrm{T}_{\mathbf{S}}(M) \to M$, where $\mathrm{T}_{\mathbf{S}}$ is the finitary monad on $\mathbf{S}$ induced by $\mathbb{T}$. Thus it can be considered as a simplicial object in $\mathbb{T}$-$\mathbf{mod}(\mathbf{S})/M$ which yields a simplicial object $L_M\mathrm{T}^*(M)$ in $\mathrm{Ab}(\mathbb{T}\text{-}\mathbf{mod}(\mathbf{S})/M)$. Now it is a standard fact that $\mathrm{T}^*(M)$, when viewed as a simplicial object of $\mathbf{S}/M$, is contractible (there are extra degeneracy maps $e(\mathrm{T}_{\mathbf{S}}^n(M)) : \mathrm{T}_{\mathbf{S}}^n(M) \to \mathrm{T}_{\mathbf{S}}^{n+1}(M)$, although they are not $\mathrm{T}_{\mathbf{S}}$-algebra homomorphisms; here e is the unit of the monad $\mathrm{T}_{\mathbf{S}}$). It thus follows that both $\mathrm{T}^*(M)$ and $L_M\mathrm{T}^*(M)$ are homotopically trivial simplicial objects; hence the chain complex $CL_M\mathrm{T}^*(M)$, obtained from $L_M\mathrm{T}^*(M)$ by taking alternating sums of face operators in the standard way, provides an acyclic resolution for $L_M(1_M)$ in $\mathrm{Ab}(\mathbb{T}\text{-}\mathbf{mod}(\mathbf{S})/M)$.

Next observe that for any $A \to M$ in $\mathrm{Ab}(\mathbb{T}\text{-}\mathbf{mod}(\mathbf{S})/M)$, one has

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Ab}(\mathbb{T}\text{-}\mathbf{mod}(\mathbf{S})/M)}(L_M\mathrm{T}^{n+1}(M), A) &\cong \hom_{\mathbb{T}\text{-}\mathbf{mod}(\mathbf{S})/M}(\mathrm{T}^{n+1}(M), A)\\ &\cong \hom_{\mathbf{S}/M}(\mathrm{T}^n(M), A).\end{aligned}$$

Recall also the explicit formula for $\mathrm{T}_{\mathbf{S}}$ from [20]; it is given by coequalizer

$$\coprod_{k_1 \to k_0} \mathbb{T}(k_1) \times X^{k_0} \rightrightarrows \coprod_{k} \mathbb{T}(k) \times X^k \twoheadrightarrow \mathrm{T}_{\mathbf{S}}(X),$$

where $\mathbb{T}(k) = \hom_{\mathbb{T}}(k, 1)$. Since obviously also $\mathrm{T}_{\mathbf{S}}(k) = \mathbb{T}(k)$, one easily obtains all the iterates $\mathrm{T}_{\mathbf{S}}^n(X)$, $n \geqslant 2$, in a similar way, as quotients of

$$\coprod_{k_1,k_2,\dots,k_n} \mathbb{T}(k_1) \times \mathbb{T}(k_2)^{k_1} \times \cdots \times \mathbb{T}(k_n)^{k_{n-1}} \times X^{k_n}$$

by equivalence relations which, in the internal language, can be stated as

$$\begin{aligned}&(u, u^{(1)}, \dots, u^{(i-1)}u^i, u^{(i+1)}, \dots, u^{(n)}, (x_j))\\ &\qquad\qquad\sim (u, u^{(1)}, \dots, u^{(i-1)}, u^i u^{(i+1)}, \dots, u^{(n)}, (x_j))\end{aligned}$$

for all $1 \leqslant i < n$ and

$$(u, u^{(1)}, \dots, u^{(n)}\mathbb{T}(\varphi), (x_j)) \sim (u, u^{(1)}, \dots, u^{(n)}, (x_{\varphi(j)}))$$

for all φ in $\mathbb{S}^{\mathrm{fp}}$, where $\mathbb{T}(\varphi)$ stands for the image of φ under the canonical functor $\mathbb{S}^{\mathrm{fp}} \to \mathbb{T}$.

Let us now turn to the case $\mathbf{S} = \widehat{\mathbb{T}}$, with $M = U^{\mathbb{T}}$. Then by 4.7, abelian groups in $\mathbb{T}$-$\mathbf{mod}(\mathbf{S})/M$ can be identified with coefficient systems for $\mathbb{T}$. It is then straightforward to check, using the above formula for $\mathrm{T}^n_{\mathbf{S}}$, that there are isomorphisms

$$C^n(\mathbb{T}; A) \cong \hom_{\widehat{\mathbb{T}}/U^{\mathbb{T}}}(\mathrm{T}^n_{\widehat{\mathbb{T}}}(U^{\mathbb{T}}), A),$$

for all n and any coefficient system A, where $C^n(\mathbb{T}; A)$ is the n-th component of the complex from 4.8. Moreover the differentials in $C^*(\mathbb{T}; A)$ correspond under these isomorphisms precisely with the differentials induced on

$$\mathrm{Hom}_{\mathrm{Ab}(\hat{\mathscr{F}}(\mathbb{T}))}(CL_{U^{\mathbb{T}}}\mathrm{T}^*_{\widehat{\mathbb{T}}}(U^{\mathbb{T}}), A).$$

Finally observe that all functors

$$C^n(\mathbb{T}; \_) : \mathrm{Ab}(\hat{\mathscr{F}}(\mathbb{T})) \to \mathrm{Ab},\ n \geqslant 0,$$

are obviously exact; since we have shown that these functors are represented by $L_{U^{\mathbb{T}}}\mathrm{T}^{n+1}_{\widehat{\mathbb{T}}}(U^{\mathbb{T}})$, the latter object of $\mathrm{Ab}(\hat{\mathscr{F}}(\mathbb{T}))$ is projective. It follows that, on one hand, $CL_{U^{\mathbb{T}}}\mathrm{T}^*_{\widehat{\mathbb{T}}}(U^{\mathbb{T}})$ is a projective resolution of $L_{U^{\mathbb{T}}}(1_{U^{\mathbb{T}}})$ and, on the other hand, applying to this resolution the functor $\mathrm{Hom}_{\mathrm{Ab}(\hat{\mathscr{F}}(\mathbb{T}))}(\_, A)$ yields $C^*(\mathbb{T}; A)$ for any coefficient system A. This implies our proposition.

The complex 4.8 enables one to describe the cohomology groups in low dimensions. For H^0 one immediately obtains

$$H^0(\mathbb{T}; D(A_{\_})) =$$
$$\{\ a \in A_x \mid \forall n \forall u \in \hom_{\mathbb{T}}(X^n, X)\ \ au = \sum_{i=1}^{n} u(x_1, ..., ax_i, ..., x_n)\ \}$$

To deal with H^1 we introduce

4.11. **Definition.** For a coefficient system $A_{\_}$ on a theory $\mathbb{T}$, a *derivation* d of $\mathbb{T}$ with values in $A_{\_}$ is an assignment, to each operation $u : X^n \to X$ of $\mathbb{T}$, of an element $d(u) \in A_u$, in such a way that the following equality holds

$$d(u(v_1, ..., v_n)) = (du)(v_1, ..., v_n) + \sum_{i=1}^{n} u(v_1, ..., dv_i, ..., v_n)$$

for all n-ary u and all $v_1, ..., v_n$ of same arity. Derivations form an abelian subgroup $\operatorname{Der}(\mathbb{T}; A_-)$ of $\prod_u A_u$. Its subgroup $\operatorname{Ider}(\mathbb{T}; A_-)$ consists of derivations of the form

$$d_a(u) = au - \sum_{i=1}^{n} u(x_1, ..., ax_i, ..., x_n)$$

for $a \in A_x$, called *inner* derivations. Here as before $x_i : X^n \to X$ are the projections in $\mathbb{T}$, and $x : X \to X$ the identity.

One then has

4.12. **Proposition.** *For a coefficient system A_- on a theory $\mathbb{T}$, with the corresponding object $D(A_-)$ of $\operatorname{Ab}(\mathscr{F}(\mathbb{T}))$, one has*

$$H^1(\mathbb{T}; D(A_-)) \cong \operatorname{Der}(\mathbb{T}; A_-)/\operatorname{Ider}(\mathbb{T}; A_-).$$

Proof. This is clear from 4.9.

The second cohomology group turns out to be related to extensions of theories. To introduce them, consider the category $\mathscr{Th}(\mathbf{S})$ over the base topos $\mathbf{S}$, i. e., according to 1.14, the category of finitary monads on the object $\mathscr{R}(\mathbf{S})$ of the 2-category $\mathfrak{Top}/\mathbf{S}$. Recall that $\mathscr{R}(\mathbf{S})$ is an internal topos, in the suitable sense, in that category; in particular, the (strict) monoidal category of its endomorphisms has all finite limits. This implies that $\mathscr{Th}(\mathbf{S})$ has finite limits too. In fact one has (cf. [2, Lemma 1.3]):

4.13. **Proposition.** *For any monoidal category $(\mathscr{V}, \otimes, I)$, the forgetful functor $\operatorname{Mon}(\mathscr{V}) \to \mathscr{V}$ from monoids of $\mathscr{V}$ reflects limits.*

Proof. Consider any diagram

$$((M_i)_{i\in I}, (f_\iota : M_i \to M_{i'})_{\iota: i\to i'})$$

in $\operatorname{Mon}(\mathscr{V})$, where M_i are monoids with units η_i and multiplication μ_i. Suppose we are given a limiting cone $(f_i : M \to M_i)_{i\in I}$ over this diagram, considered as a diagram in $\mathscr{V}$. Since

$$\left(M \otimes M \xrightarrow{f_i \circ f_i} M_i \otimes M_i \xrightarrow{\mu_i} M_i\right)_{i\in I}$$

and

$$\left(I \xrightarrow{\eta_i} M_i\right)_{i\in I}$$

are cones in $\mathscr{V}$, they determine maps $\mu : M \otimes M \to M$ and $\eta : I \to M$, respectively. It is straightforward to show that this is a limiting cone in $\mathrm{Mon}(\mathscr{V})$.

We are going to introduce extensions of a theory $\mathbb{T}$ as torsors (principal homogeneous objects) under internal abelian groups in $\mathscr{T\!h}(\mathbf{S})/\mathbb{T}$. Now concerning these abelian groups, one in fact has

4.14. **Proposition.** *For a monoid (M, μ, η) in a monoidal category with finite limits $(\mathscr{V}, \otimes, I)$, there is an equivalence of categories*

$$\mathrm{Ab}(\mathrm{Mon}(\mathscr{V})/M) \simeq \mathrm{Ab}({}^M\mathscr{V}^M/M),$$

where ${}^M\mathscr{V}^M$ is the category of M-M-biobjects in $\mathscr{V}$.

For a proof, see [2, Prop. 1.5].

To apply this to the monoidal category $\mathrm{End}_{\mathfrak{Top}/\mathbf{S}}(\mathscr{R}(\mathbf{S}))$, note that the latter category is equivalent to $\mathscr{R}(\mathbf{S})$ itself. To describe the induced monoidal structure on $\mathscr{R}(\mathbf{S}) = \mathbf{S}^{\mathbb{S}^{\mathrm{fp}}}$, recall from [17] that composition of geometric morphisms between presheaf toposes can be described using *tensor product* of corresponding bifunctors. The theory of sets $\mathbb{S}$ is equivalent to the opposite of the category of finite sets, so that $\mathbb{S}^{\mathrm{op}} = \mathbb{S}^{\mathrm{fp}}$, and any $F \in \mathbf{S}^{\mathbb{S}^{\mathrm{fp}}}$ determines uniquely a bifunctor $\widetilde{F} : \mathbb{S} \times \mathbb{S}^{\mathrm{op}} \to \mathbf{S}$ which preserves finite limits in the first argument, namely

$$\widetilde{F}(X^m, X^n) = F(n)^m.$$

Then the monoidal operation $\otimes$ on $\mathscr{R}(\mathbf{S})$ corresponding to composition under the identification of $\mathscr{R}(\mathbf{S})$ with $\mathrm{End}_{\mathfrak{Top}/\mathbf{S}}(\mathscr{R}(\mathbf{S}))$ is determined by

$$\widetilde{F \otimes G} = \widetilde{F} \underset{\mathbb{S}}{\otimes} \widetilde{G}.$$

Now for a theory $\mathbb{T}$ corresponding to a finitary monad $\mathrm{T} = (T, m, e)$ the bifunctor $\widetilde{T}$ is easily seen to be given by

$$\widetilde{T}(m, n) = \hom_{\mathbb{T}}(X^n, X^m),$$

with the $\otimes$-monoid structure on T corresponding to composition in $\mathbb{T}$. Thus T-T-biobjects with respect to $\otimes$ correspond to bifunctors $\mathbb{T}^{\mathrm{op}} \times \mathbb{T} \to \mathbf{S}$, and biobjects with a morphism to T – to natural systems on $\mathbb{T}$. Moreover since bifunctors of the form $\widetilde{F}$ must preserve finite limits in the first variable, taking into account 1.10 and 1.15 gives

4.15. **Theorem.** *For a theory* $\mathbb{T}$, *there is an equivalence*

$$\mathrm{Ab}(\mathscr{Th}(\mathbf{S})/\mathbb{T}) \simeq \mathrm{Ab}(\hat{\mathscr{F}}(\mathbb{T})).$$

We next recall the notion of *linear extension* from [4].

4.16. **Definition.** For a natural system D on a category $\mathbb{C}$ with values in abelian groups, a linear extension of $\mathbb{C}$ by D is a functor $P : \mathbb{E} \to \mathbb{C}$ that is identity on objects, together with transitive and effective actions $D_\gamma \times P^{-1}(\gamma) \to P^{-1}(\gamma)$, $(x, e) \mapsto x + e$, for all $\gamma : c \to c'$ in $\mathbb{C}$, such that for any composable morphisms e_1, e_2 in $\mathbb{E}$ and any $x_i \in D_{P(e_i)}$, $i = 1, 2$, one has

$$(x_1 + e_1)(x_2 + e_2) = (x_1 P(e_2) + P(e_1)x_2) + e_1 e_2.$$

Two linear extensions are equivalent if there is a functor between them over $\mathbb{C}$ which is equivariant with respect to actions.

An example of a linear extension by a natural system D is given by the *trivial* linear extension $\mathbb{C} \rtimes D$ with

$$\hom_{\mathbb{C}\rtimes D}(c, c') = \coprod_{\gamma : c \to c'} D_\gamma,$$

composition $x_1 x_2 = x_1\gamma_2 + \gamma_1 x_2$ for $x_1 \in D_{\gamma_1}$, $x_2 \in D_{\gamma_2}$, identities $0 \in D_{1_c}$, $c \in \mathbb{C}$, and the actions $D_\gamma \times D_\gamma \to D_\gamma$ given by the group law in D_γ. As noticed in [14], the trivial extension $\mathbb{C} \rtimes D \to \mathbb{C}$ has the structure of an internal abelian group in $\mathscr{Cat}/\mathbb{C}$, and moreover linear extensions of $\mathbb{C}$ by D are in one-to-one correspondence with torsors (principal homogeneous objects) under this abelian group. More precisely, one has

4.17. **Proposition.** (cf. [14]) *For a linear extension* $\mathbb{E} \to \mathbb{C}$ *of a small category* $\mathbb{C}$ *by a natural system* D *there is an action*

$$\mathbb{E} \times_{\mathbb{C}} (\mathbb{C} \rtimes D) \to \mathbb{E}$$

such that the induced functor

$$\mathbb{E} \times_{\mathbb{C}} (\mathbb{C} \rtimes D) \xrightarrow{\text{(projection, action)}} \mathbb{E} \times_{\mathbb{C}} \mathbb{E}$$

is an equivalence; conversely, any such action on a functor $\mathbb{E} \to \mathbb{C}$ *which is bijective on objects and surjective on morphisms, turns it into a linear extension of* $\mathbb{C}$ *by* D.

It is proved in [4] that equivalence classes of linear extensions of $\mathbb{C}$ by D are in one-to-one correspondence with elements of the group $H^2(\mathbb{C}; D)$, i. e. the second cohomology group of the complex $C^*(\mathbb{C}; D)$ from 4.9.

On the other hand one has (cf. [15, (6.1)], [14, 2.5])

4.18. **Proposition.** *A natural system of abelian groups D on a category with finite products $\mathbb{T}$ is cartesian iff for any linear extension $P : \mathbb{T}' \to \mathbb{T}$ of $\mathbb{T}$ by D, the category $\mathbb{T}'$ also has finite products, and P preserves them.*

In particular, for a cartesian natural system D on a theory $\mathbb{T}$, the trivial extension $\mathbb{T} \rtimes D \to \mathbb{T}$ is an internal abelian group in $\mathscr{T}\!\!h/\mathbb{T}$. Moreover by 4.15 and 4.17 one has

4.19. **Corollary.** *For a cartesian natural system D on a theory $\mathbb{T}$, the group of equivalence classes of $(\mathbb{T} \rtimes D \to \mathbb{T})$-torsors in $\mathscr{T}\!\!h/\mathbb{T}$ is isomorphic to $H^2(\mathbb{T}; D)$.*

There are lots of examples of linear extensions of theories in [15]. Let us mention just some of them.

4.20. **Examples.**

4.20.1. Consider the functor from theories to monoids given by $\mathbb{T} \mapsto \hom_{\mathbb{T}}(X, X)$. This functor has a full and faithful right adjoint assigning to a monoid M, the theory $\mathbb{T}_M$ of M-sets. Thus the category of monoids can be identified with a full subcategory of $\mathscr{T}\!\!h$ closed under limits there. In particular, groups, torsors, herds, natural systems, linear extensions, etc. of monoids (considered as categories with one object) can be identified with those of the corresponding theories. In other words, a morphism of theories $P : \mathbb{T}_N \to \mathbb{T}_M$ induced by a homomorphism of monoids $p : N \to M$ is a linear extension iff p, considered as a functor between categories with one object, is a linear extension – i. e. p is an abelian extension in the category of monoids. The corresponding natural system on M consists of abelian groups D_x, for x in M, and actions $x(-) : D_y \to D_{xy}$, $(-)y : D_x \to D_{xy}$. It can be also considered as an "M-graded M-M-bimodule". The corresponding extensions of theories are simple iff all the D_x are equal.

4.20.2. Any homomorphism of rings $p : S \to R$ gives rise to a morphism $P : \mathbb{T}_S \to \mathbb{T}_R$ from the theory of (left) S-modules to that of R-modules. This morphism is a linear extension iff p is a *singular extension*, i. e. $\mathrm{Ker}(p) = B$ is a square zero ideal in S. In [15], an isomorphism is obtained

$$H^2(\mathbb{T}_R; D_B) \cong H^2(R; B)$$

from the group of (simple) linear extensions of $\mathbb{T}_R$ by the bifunctor given by

$$D_B(X^n, X^k) = \mathrm{Hom}_{R\text{-}\mathbf{mod}}(\mathbb{T}_R(k), B \underset{R}{\otimes} \mathbb{T}_R(n)) \cong (B^{\oplus n})^k,$$

to the second MacLane cohomology group of R with coefficients in B.

4.20.3. It is proved in [15] that for each n there is a linear extension from the theory of $(n+1)$-nilpotent groups to that of n-nilpotent ones; similarly for groups replaced by Lie rings, associative rings without unit, or associative commutative rings without unit.

4.20.4. For a left module M over a ring R, let $M/(R\text{-}\mathbf{mod})$ be the coslice category of modules under M, with objects of the form $M \to N$ and obvious commutative triangles as morphisms. Consider the functor $P : M/(R\text{-}\mathbf{mod}) \to R\text{-}\mathbf{mod}$ sending $f : M \to N$ to $\mathrm{Coker}(f)$. It has a right adjoint U_P given by $U_P(N) = 0 : M \to N$. It is then easy to see that this adjoint pair is induced by a morphism of theories $P : \mathbb{T}_{R;M} \to \mathbb{T}_R$, where $\mathbb{T}_{R;M}$ is the opposite of the full subcategory of $M/(R\text{-}\mathbf{mod})$ on objects of the form $(1,0) : M \to M \oplus R^n$ for $n \geqslant 0$. In particular, $M/(R\text{-}\mathbf{mod})$ is equivalent to $\mathbb{T}_{R;M}\text{-}\mathbf{mod}$.

Now this P in fact presents $\mathbb{T}_{R;M}$ as a trivial linear extension of $\mathbb{T}_R$, by the bifunctor H_M given by composition

$$\mathbb{T}_R^{\mathrm{op}} \times \mathbb{T}_R \xrightarrow{\text{projection}} \mathbb{T}_R \xrightarrow{I_{\mathbb{T}_R}^{\mathrm{op}}} (R\text{-}\mathbf{mod})^{\mathrm{op}} \xrightarrow{\mathrm{Hom}_R(-,M)} \mathrm{Ab},$$

that is,

$$H_M(X^n, X^k) = \mathrm{Hom}_R(R^k, M) \cong M^k.$$

Indeed, the trivial extension $P : \mathbb{T}_R \rtimes H_M \to \mathbb{T}_R$ can be easily calculated; one has

$$\mathrm{hom}_{\mathbb{T}_R \rtimes H_M}(X^n, X^k) = \mathrm{Hom}_R(R^k, M \oplus R^n).$$

One can represent the latter group also as

$$\mathrm{hom}_{M/(R\text{-}\mathbf{mod})}(M \xrightarrow{(1,0)} M \oplus R^k, M \xrightarrow{(1,0)} M \oplus R^n),$$

which is precisely $\mathrm{hom}_{\mathbb{T}_{R;M}}(X^n, X^k)$.

5. Abelian theories

For theories $\mathbb{T}_R$ of modules over a ring R, the category $\mathscr{F}(\mathbb{T}_R)$ simplifies. One has

5.1. **Proposition.** *For a ring R, the category $\mathrm{Ab}(\mathscr{F}(\mathbb{T}_R))$ is equivalent to the category of functors from finitely presented R-modules to all R-modules.*

Similarly, $\mathrm{Ab}(\hat{\mathscr{F}}(\mathbb{T}_R))$ is equivalent to the category of functors from finitely generated free R-modules to R-modules.

Proof. By 1.5 one has

$$\mathrm{Ab}(\mathscr{F}(\mathbb{T}_R)) \simeq \mathrm{Ab}((R\text{-}\mathbf{mod}(\mathrm{Set}))^{\mathbb{T}^{\mathrm{fp}}}/I_{\mathbb{T}}).$$

But for any object I of any additive category $\mathscr{A}$, the category $\mathrm{Ab}(\mathscr{A}/I)$ is equivalent to $\mathscr{A}$. Similarly for $\hat{\mathscr{F}}$.

Similar simplifications are possible for some more general "abelian" theories.

5.2. **Definition.** An *abelian Maltsev theory* is a finitary algebraic theory such that among its operations there is a ternary operation m which satisfies identities

$$m(x, x, y) = y$$
$$m(x, y, z) = m(z, x, y)$$
$$m(x, y, m(z, t, u)) = m(m(x, y, z), t, u)$$

For such theories, we will give more detailed description of low dimensional cohomology and its interpretation by derivations and extensions. We will first deal with abelian theories without constants, i. e. nullary operations. Abelian theories with constants are much simpler and will be treated in the end.

It follows from [14, Prop. 3.2] that for a model A of an abelian Maltsev theory the operation $m : A^3 \to A$ is actually a homomorphism, and that any two such operations coincide. It will be convenient to fix such an operation throughout and denote it by

$$m(x, y, z) = x +_y z.$$

Thus all models of an abelian theory are *abelian herds*, and all of their operations are homomorphisms of abelian herds. To describe such theories, we need the following notions.

5.3. **Definition.** A *left linear form* consists of an associative ring R with unit, a left R-module M, and a homomorphism $\partial : M \to R$ of left R-modules.

In fact usually we will omit the word "left", as it is customary with modules.

5.4. **Definition.** An *affinity* over a linear form $\partial : M \to R$ is an abelian herd A together with maps $R \times A \times A \to A$ and $M \times A \to A$, denoted, respectively, $(r, a, b) \mapsto r_a b$ and $(x, a) \mapsto \varphi_a(x)$, such that the following identities hold:

- For each $a \in A$, the operations $(-) +_a (-)$ and $(-)_a(-)$ turn A into a left R-module (with zero a), and φ_a into a module homomorphism. In other words, for any $a, b, c, d \in A$, $r, s \in R$, $x, y \in M$ one has

$$\begin{aligned}
&b +_a (c +_a d) = (b +_a c) +_a d,\\
&a +_a b = b,\\
&b +_a c = c +_a b,\\
&b -_a b = a,\\
&r_a(b +_a c) = r_a b +_a r_a c,\\
&(r + s)_a b = r_a b +_a s_a b,\\
&1_a b = b,\\
&r_a(s_a b) = (rs)_a b,\\
&\varphi_a(x + y) = \varphi_a(x) +_a \varphi_a(y),\\
&\varphi_a(rx) = r_a \varphi_a(x),
\end{aligned}$$

  where we have denoted $b -_a c = b +_a ((-1)_a c)$.
- ("Coordinate change".) These structures are related by the identities

$$\begin{aligned}
&b +_{a'} c = ((b -_a a') +_a (c -_a a')) +_a a',\\
&r_{a'} b = r_a(b -_a a') +_a a',\\
(*)\qquad &\varphi_{a'}(x) = \varphi_a(x) +_a (1 - \partial x)_a a'.
\end{aligned}$$

A homomorphism between affinities A, A' is a map $f : A \to A'$ preserving all this, i. e. satisfying

$$f(a +_b c) = f(a) +_{f(b)} f(c),$$
$$f(r_a b) = r_{f(a)} f(b),$$
$$f(\varphi_a(x)) = \varphi_{f(a)}(x).$$

Obviously the category ∂-**aff** of affinities over a linear form ∂ is the category of models of a suitable abelian theory $\mathbb{T}_\partial$. Here is an explicit description of this theory.

5.5. **Proposition.** *The theory $\mathbb{T}_\partial$ of affinities over $\partial : M \to R$ can be described as follows:*

$$\hom_{\mathbb{T}_\partial}(X^n, X) = \begin{cases} \varnothing, & n = 0; \\ M \times R^{n-1}, & n > 0. \end{cases}$$

The projections $a_0, a_1, a_2, ... : X^n \to X$ are given, respectively, by the elements $\langle 0,0,0,...\rangle$, $\langle 0,1,0,...\rangle$, $\langle 0,0,1,...\rangle$, ...; and, composition is given by

$$\langle x, r_1, r_2, ...\rangle \left(\langle x_0, s_0, t_0, ...\rangle, \langle x_1, s_1, t_1, ...\rangle, \langle x_2, s_2, t_2, ...\rangle, ...\right) = \langle x', s', t', ...\rangle,$$

where

$$x' = x + (1 - \partial(x))x_0 + r_1(x_1 - x_0) + r_2(x_2 - x_0) + ...,$$
$$s' = (1 - \partial(x))s_0 + r_1(s_1 - s_0) + r_2(s_2 - s_0) + ...,$$
$$t' = (1 - \partial(x))t_0 + r_1(t_1 - t_0) + r_2(t_2 - t_0) + ...,$$
$$...$$

Proof. Take as basic operations the ternary $(-) +_{(-)} (-)$, the family of binaries $r_{(-)}(-)$ indexed by $r \in R$, and unaries $\varphi_{(-)}(x)$ indexed by $x \in M$. Using the affinity identities 5.4, one can write any composite of these operations in the form

$$\langle x, r, s, ...\rangle (a, b, c, ...) = \varphi_a(x) +_a r_a b +_a s_a c +_a \cdots$$

in a unique way. The rest is straightforward verification.

Define now a morphism of left linear forms from $\partial : M \to R$ to $\partial' : M' \to R'$ to be an equivariant homomorphism, i. e. a pair $(f : R \to R', g : M \to M')$ of additive maps such that the obvious square commutes, that f is a unital ring homomorphism, and that $g(rx) =$

$f(r)g(x)$ holds for any $r \in R$, $x \in M$. This clearly defines the category **Lf** of left linear forms. We then have

5.6. **Theorem.** *The category of abelian Maltsev theories without constants is equivalent to the category* **Lf** *of left linear forms.*

Proof. Define the functor $\mathbb{T}_{(-)}$ from **Lf** to algebraic theories by sending an object ∂ of **Lf** to the corresponding theory $\mathbb{T}_\partial$ described above in 5.5. It is clear from that description that any morphism in **Lf** determines a morphism of the corresponding theories.

Conversely, given an abelian Maltsev theory $\mathbb{T}$, define the left linear form $\partial_\mathbb{T} : M_\mathbb{T} \to R_\mathbb{T}$ as follows: let $M_\mathbb{T}$ be the set of all unary operations of $\mathbb{T}$, with the abelian group structure given by $(x + y)(a) = m(x(a), a, y(a))$, where m is the Maltsev operation of $\mathbb{T}$. Let $R_\mathbb{T}$ be the set of *convex* binary operations of $\mathbb{T}$, i. e. those binary operations r satisfying the identity $r(a, a) = a$. Define the ring structure on it by taking zero 0 to be $0(a, b) = a$, unit 1 to be $1(a, b) = b$, addition to be $(r + s)(a, b) = m(r(a, b), a, s(a, b))$, additive inverse $(-r)(a, b) = m(a, r(a, b), a)$, and multiplication $(rs)(a, b) = r(a, s(a, b))$. Let $R_\mathbb{T}$ act on $M_\mathbb{T}$ via $(rx)(a) = r(a, x(a))$, and let the crossing $M_\mathbb{T} \to R_\mathbb{T}$ be $(\partial_\mathbb{T} x)(a, b) = m(x(a), x(b), b)$. It is then straightforward to check that this defines a left linear form, that any morphism of theories gives rise to a morphism in **Lf** in a functorial way, and that if one starts from a theory of the form $\mathbb{T}_\partial$, then one recovers the original ∂ back. Finally for the second way round, observe that for any operation $u : X^n \to X$ in an abelian theory, with $n > 0$, one has

$$
\begin{aligned}
u(a, b, c, ...) &= u(a{+_a}a{+_a}a{+_a}\cdots, a{+_a}b{+_a}a{+_a}\cdots, a{+_a}a{+_a}c{+_a}\cdots, ...)\\
&= u(a, a, a, ...) +_{u(a,a,a,...)} u(a, b, a, ...) +_{u(a,a,a,...)} u(a, a, c, ...) +_{u(a,a,a,...)} \cdots\\
&= u(a, a, a, ...) +_a (a +_{u(a,a,a,...)} u(a, b, a, ...))\\
&\qquad +_a (a +_{u(a,a,a,...)} u(a, a, c, ...)) +_a \cdots\\
&= x(a) +_a r(a, b) +_a s(a, c) +_a \cdots,
\end{aligned}
$$

with x in $M_\mathbb{T}$ and $r, s, ...$ in $R_\mathbb{T}$. This implies easily that including $M_\mathbb{T}$ and $R_\mathbb{T}$ in $\mathbb{T}$ extends to an isomorphism of theories from $\mathbb{T}_{\partial_\mathbb{T}}$ to $\mathbb{T}$.

Remark. Construction of the ring $R_\mathbb{T}$ from an abelian Maltsev theory $\mathbb{T}$ is obviously well known to universal algebraists, in a slightly different context – see e. g. [6]. It is in fact closely related to the classical coordinatization construction for geometries. The reader might consult, e. g. [12] or [10] for that.

Using our description, we can now find out what kind of linear extensions exist between abelian theories. Indeed, since the Maltsev operation in abelian theories is unique, they are clearly closed under arbitrary finite limits, hence 5.6 together with [14, Prop. 1.6] implies that abelian linear extensions of an abelian theory $\mathbb{T}$ can be identified with torsors under internal abelian groups in $\mathbf{Lf}/\partial_{\mathbb{T}}$. Thus we just have to describe torsors under a linear form $\partial : M \to R$. Consider one such, given by

$$(E) \qquad \begin{array}{ccccc} K & \overset{j}{\rightarrowtail} & N & \overset{q}{\twoheadrightarrow} & M \\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\partial'} & & \downarrow{\scriptstyle\partial} \\ B & \overset{i}{\rightarrowtail} & S & \overset{p}{\twoheadrightarrow} & R. \end{array}$$

Now by [14, Prop. 1.3] we know that this torsor is equipped with a herd structure in $\mathbf{Lf}/\partial$. Thus we have a Maltsev homomorphism $m : \partial' \times_\partial \partial' \times_\partial \partial' \to \partial'$ over ∂; then, by an argument just as in [14, Prop. 3.2], both in N and S one has

$$\begin{aligned} m(x,y,z) &= m(x-y+y, y-y+y, y-y+z) \\ &= m(x,y,y) - m(y,y,y) + m(y,y,z) = x - y + z. \end{aligned}$$

Thus it follows that (p,q) above is a torsor iff this map is a homomorphism. One sees easily that this happens iff $B^2 = BK = 0$. In such case, B becomes naturally an R-R-bimodule, K a left R-module, and restriction δ of ∂' to it – a module homomorphism, via $rb = sb$, $br = bs$, $rk = sk$, for any $b \in B$, $k \in K$, $r \in R$ and $s \in S$ with $p(s) = r$. Moreover there is an R-module homomorphism $B \otimes_R M \to K$, denoted $(b,m) \mapsto b \cdot m$, given by $b \cdot m = bn$ for any $n \in N$ with $q(n) = m$. It clearly satisfies $\delta(b \cdot m) = b\partial(m)$. On the whole, one gets a structure which can be described by

5.7. **Definition.** For a left linear form $\partial : M \to R$, a *∂-bimodule* consists of an R-R-bimodule B, a left R-module K, and R-linear maps $\delta : K \to B$ and $\cdot : B \otimes_R M \to K$ satisfying $\delta(b \cdot m) = b\partial m$ for any $b \in B$, $m \in M$. It will be denoted $\delta^{\cdot} = (B \otimes_R M \to K \to B)$.

Examples of such ∂-bimodules include $R \otimes_R M \cong M \to R$, i. e. ∂ itself, $(B \otimes_R M \to B \otimes_R R \cong B = B)$, for any R-R-bimodule B, which we denote $\mathscr{C}(B)$, and $0 \to K \to 0$ for any left R-module K, which we denote $K[1]$.

It is easy to show that also conversely, internal groups in $\mathbf{Lf}/\partial$ are determined by ∂-bimodules $\delta^{\cdot}$ as above, and that as soon as the map $(x,y,z)\mapsto x-y+z$ is a homomorphism from $\partial'\times_\partial\partial'\times_\partial\partial'$ to ∂' over ∂, then there is a torsor structure on ∂' under the corresponding group. We summarize this as follows:

5.8. **Proposition.** *For a left linear form $\partial : M\to R$, internal groups in $\mathbf{Lf}/\partial$ are in one-to-one correspondence with ∂-bimodules. Moreover the underlying linear form of the corresponding group is $\delta\oplus\partial : K\oplus M\to B\oplus R$, with multiplicative structure $(b,r)(b',r')=(br'+rb',rr')$, $(b,r)(k,m)=(b\cdot m+rk,rm)$. Torsors under this group are in one-to-one correspondence with diagrams such as* (E) *above, where $B\rightarrowtail S\twoheadrightarrow R$ is a singular extension, i. e. the ideal $i(B)$ has zero multiplication in S and the induced R-R-bimodule structure coincides with the original one, and moreover $i(B)j(K)=0$, the induced R-module structure on K is the original one, i. e. $j(p(s)k)=sj(k)$, and finally, the induced action $B\otimes_R M\to K$ coincides with the original one, i. e. $j(b\cdot q(n))=i(b)n$.* $\square$

Translating now all of the above from **Lf** to abelian theories, in view of 4.7 we conclude

5.9. **Proposition.** *For a left linear form $\partial : M\to R$, each internal group $\mathscr{A}=(\delta\oplus\partial\to\partial)$ in $\mathbf{Lf}/\partial$ corresponding to the ∂-bimodule $\delta^{\cdot}=(B\otimes_R M\to K\to B)$ as above, gives rise to a coefficient system $A^{\mathscr{A}}$ on the corresponding abelian theory $\mathbb{T}_\partial$. Explicitly, one has*

$$A^{\mathscr{A}}_{\langle x,r_1,\dots,r_{n-1}\rangle}=K\oplus B^{n-1},$$

with actions given by restricting those in 5.5 for $\mathbb{T}_{\delta\oplus\partial}$ to $K\oplus B^{n-1}\subseteq (K\oplus M)\times(B\oplus R)^{n-1}$.

In view of this, we will in what follows identify internal groups $\mathscr{A}$ in $\mathbf{Lf}/\partial$ with ∂-bimodules and with the corresponding natural systems $D^{\mathscr{A}}$ on $\mathbb{T}_\partial$. In particular, equivalence classes of extensions of $\mathbb{T}_\partial$ by $D^{\mathscr{A}}$ form, by 4.19, an abelian group isomorphic to $H^2(\mathbb{T}_\partial;D^{\mathscr{A}})$, which we can as well denote $H^2(M\to R;\ B\otimes_R M\to K\to B)$, or just by $H^2(\partial;\delta^{\cdot})$.

Now from [15] we know that any short exact sequence $\delta'\rightarrowtail\delta^{\cdot}\twoheadrightarrow\delta''$ induces the exact sequence

$$\begin{aligned}&0\to H^0(\partial;\delta')\to\dots\\&\qquad\to H^1(\partial;\delta'')\to H^2(\partial;\delta')\to H^2(\partial;\delta^{\cdot})\to H^2(\partial;\delta'')\to\dots\end{aligned}$$

which one can use to reduce investigation of cohomologies, in particular linear extensions by a ∂-bimodule, to those by more "elementary" ones. In particular, observing the diagrams

$$\begin{array}{ccccc}
\operatorname{Ker}(\delta) & \rightarrowtail & K & \twoheadrightarrow & \operatorname{Im}(\delta) \\
\downarrow & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\iota} \\
0 & \longrightarrow & B & = & B,
\end{array}$$

and

$$\begin{array}{ccccc}
\operatorname{Im}(\delta) & \overset{\iota}{\rightarrowtail} & B & \twoheadrightarrow & \operatorname{Coker}(\delta) \\
\downarrow & & \| & & \downarrow \\
B & = & B & \longrightarrow & 0,
\end{array}$$

one sees that there are short exact sequences of ∂-bimodules of the form $K'[1] \rightarrowtail \delta^{\cdot} \twoheadrightarrow \iota^{\cdot}$ and $\iota^{\cdot} \rightarrowtail \mathscr{C}(B) \twoheadrightarrow K''[1]$, so that linear extensions by any δ can be described in terms of those by bimodules of the form $K[1]$ and $\mathscr{C}(B)$.

Before dealing with these, just let us make a note about lower cohomologies – they can be expressed using derivations similarly to Hochschild cohomology.

5.10. **Definition.** The group $\operatorname{Der}(\partial;\delta^{\cdot})$ of *derivations* of a linear form $\partial : M \to R$ with values in a ∂-bimodule $\delta^{\cdot} = (B\otimes_R M \to K \to B)$ consists of pairs of abelian group homomorphisms $(d : R \to B, \nabla : M \to K)$ satisfying

$$\begin{aligned}
&d\partial = \delta\nabla,\\
&d(rs) = d(r)s + rd(s),\\
&\nabla(rm) = d(r)m + r\nabla(m),
\end{aligned}$$

under pointwise addition. Its subgroup $\operatorname{Ider}(\partial;\delta^{\cdot})$ consists of *inner* derivations $\operatorname{ad}(k) = (d_k, \nabla_k)$ for $k \in K$, defined by

$$d_k(r) = r\delta(k) - \delta(k)r, \quad \nabla_k(m) = \partial(m)k - \delta(k)\cdot m.$$

Then by analogy with well known classical facts, 4.11 and 4.12 readily give

5.11. **Proposition.** *For a linear form* $\partial : M \to R$ *and a* ∂*-bimodule* $\delta^{\cdot} = (B \otimes_R M \to K \to B)$*, one has an exact sequence*

$$0 \to H^0(\partial; \delta^{\cdot}) \to K \xrightarrow{\text{ad}} \mathrm{Der}(\partial; \delta^{\cdot}) \to H^1(\partial; \delta^{\cdot}) \to 0.$$

In other words, there are isomorphisms

$$H^0(\mathbb{T}_\partial; \delta^{\cdot}) \cong \{c \in K \,|\, \forall m \in M \ (\partial m)c = (\delta c) \cdot m\}$$

and

$$H^1(\mathbb{T}_\partial; \delta^{\cdot}) \cong \mathrm{Der}(\partial; \delta^{\cdot}) / \mathrm{Ider}(\partial; \delta^{\cdot}).$$

□

Now for $\mathscr{C}(B)$, one has

5.12. **Proposition.** *For a linear form* $\partial : M \to R$ *and an* R*-*R*-bimodule* B*, there is an isomorphism*

$$H^2(\mathbb{T}_\partial; \mathscr{C}(B)) \cong H^2(R; B),$$

the latter being the MacLane cohomology group.

Proof. Observe the diagram

$$\begin{array}{ccccc} B & \rightarrowtail & N & \twoheadrightarrow & M \\ \| & & \downarrow & & \downarrow{\scriptstyle \partial} \\ B & \rightarrowtail & S & \twoheadrightarrow & R. \end{array}$$

It shows that the right hand square is pullback, so that the upper row is completely determined by the lower one. Thus forgetting the upper row defines an isomorphism, with the inverse which assigns to a singular extension of R by B the pullback as above.

Thus one arrives at a well studied situation here. As for the $K[1]$ case, we have

5.13. **Proposition.** *For a linear form* $\partial : M \to R$ *and a left* R*-module* K*, there is an isomorphism*

$$H^2(\mathbb{T}_\partial; K[1]) \cong \mathrm{Ext}^1_R(M, K).$$

Proof. This is obvious from the diagram

$$\begin{array}{ccccc} K & \rightarrowtail & * & \xrightarrow{p} \!\!\!\!\rightarrow & M \\ \downarrow & & \downarrow{\scriptstyle \partial p} & & \downarrow{\scriptstyle \partial} \\ 0 & \longrightarrow & R & = & R. \end{array}$$

But moreover the diagram

$$\begin{array}{ccccc} \mathrm{Ker}(\partial) & \rightarrowtail & M & \twoheadrightarrow & \mathrm{Im}(\partial) \\ \downarrow & & \downarrow{\scriptstyle \partial} & & \downarrow \\ 0 & \longrightarrow & R & = & R \end{array}$$

shows that $\mathbb{T}_\partial$ is itself a linear extension of a theory corresponding to the linear form of type $\mathfrak{a} \rightarrowtail R$, where $\mathfrak{a}$ is a left ideal in R, by a natural system corresponding to an $(\mathfrak{a} \rightarrowtail R)$-bimodule of the form $K[1]$. And this is clearly the end: one obviously has

5.14. **Proposition.** *An abelian theory without constants cannot be represented nontrivially as a linear extension of another theory if and only if it is of the type* $\mathbb{T}_{\mathfrak{a} \rightarrowtail R}$*, for the left linear form determined by a left ideal* $\mathfrak{a}$ *in a ring* R *which does not have any nontrivial square zero two-sided ideals.*

Proof. The only nontrivial remark to make here is that for any square zero two-sided ideal $\mathfrak{b} \rightarrowtail R$, one gets an extension

$$\begin{array}{ccccc} \mathfrak{k} & \rightarrowtail & \mathfrak{a} & \twoheadrightarrow & \mathfrak{a}/\mathfrak{k} \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{b} & \rightarrowtail & R & \twoheadrightarrow & R/\mathfrak{b} \end{array}$$

for any left ideal $\mathfrak{k}$ with $\mathfrak{b}\mathfrak{a} \subseteq \mathfrak{k} \subseteq \mathfrak{b} \cap \mathfrak{a}$.

Finally, consider an abelian theory $\mathbb{T}$ with constants. It has a largest subtheory $\mathbb{T}_0$ without constants, obtained from $\mathbb{T}$ by removing all morphisms $1 \to X^n$ for $n > 0$. The constants of $\mathbb{T}$ will then reappear in $\mathbb{T}_0$ as *pseudoconstants*, that is, those unary operations $p : X \to X$ satisfying the identity $p(a) = p(b)$. Conversely, if a theory without constants is obtained nontrivially in such way, it must have some pseudoconstants.

We know by 5.6 that $\mathbb{T}_0 = \mathbb{T}_\partial$ for some linear form $\partial : M \to R$. Now pseudoconstants in $\mathbb{T}_\partial$ correspond to elements p of M satisfying the identity $\varphi_a(p) = \varphi_b(p)$ (with a, b as variables). Using the identity $(*)$ from 5.4 this gives $(1 - \partial p)_a b = a$, i. e. $(\partial p)_a b = b$ for any a, b in any affinity. Then taking $a = \langle 0, 0\rangle$, $b = \langle 0, 1\rangle$ in $M \times R$ gives $\partial p = 1$. Now clearly there is a $p \in M$ with $\partial p = 1$ if and only if ∂ is surjective, in which case it is split by $\sigma(r) = rp$. Thus in this case our linear form is isomorphic to (projection): $\mathrm{Ker}(\partial) \oplus R \to R$. Let us fix one such p. We then may declare $\{p + t_0 \,|\, t_0 \in K\}$ to be the set of pseudoconstants corresponding to nullary operations, where K is either empty or any R-submodule of $\mathrm{Ker}(\partial)$. All choices will give equivalent categories of models, the only difference being that for $K = \varnothing$ the empty set is also allowed as a model. Each other model A shall then have at least one element a, and value of the unary operation p on A at a will then be $\varphi_a(p)$, which does not depend on a as we just saw. Denoting this element by 0_A fixes a canonical R-module structure on each non-empty model. Moreover each element of M becomes uniquely written as $m = k + rp$ with $r \in R$ and $k \in \mathrm{Ker}(\partial)$, so $\varphi_a(m) = \varphi_a(k) + r$, i. e. φ_a is completely determined by its restriction to $\mathrm{Ker}(\partial)$. Moreover by $(*)$ of 5.4 it is determined by φ_{0_A} alone. We see that, ignoring the possible empty model, the category $\mathbb{T}_\partial$-**mod** is equivalent to the coslice $(\mathrm{Ker}\,\partial)/R$-**mod**. We thus have proved:

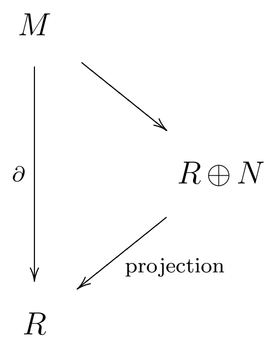
5.15. **Proposition.** *An abelian theory has at least one pseudoconstant if and only if the category of its models is equivalent to the category $K/(R$-**mod**$)$ of left R-modules under K, for some ring R and an R-module K, with the possible difference that the empty set is another model.* □

Now observing 4.20.4 we conclude

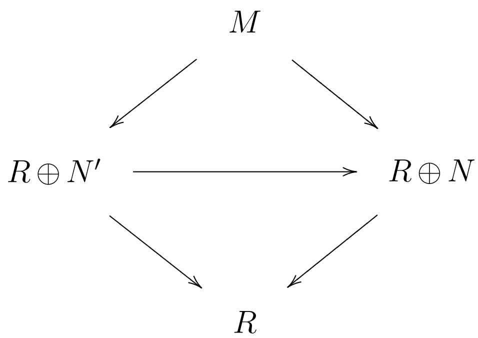
5.16. **Corollary.** *Any abelian theory with constants is isomorphic to $\mathbb{T}_{R;K}$ (defined in 4.20.4) for some ring R and left R-module K.* □

Concerning linear extensions one observes that by 4.20.4, any theory with constants $\mathbb{T}_{R;K}$ is a trivial linear extension of $\mathbb{T}_R$ by the bifunctor constructed there. Also observe that in any linear extension $\mathbb{T}' \to \mathbb{T}$ of abelian theories one has constants if and only if the other does.

On the other hand, a description similar to 5.15 is in fact possible for categories of models of abelian theories without constants too. For any left linear form $\partial : M \to R$, denote (temporarily) by ∂-$\mathbf{aff}'$ the following category: objects are R-module homomorphisms $f : M \to N$; a morphism from $f' : M \to N'$ to $f : M \to N$ is a pair (g, n), where $g : N' \to N$ is an R-module homomorphism and $n \in N$ is an element such that $f(x) - gf'(x) = \partial(x)n$ holds for all $x \in M$. Composition is given by $(g, n)(g', n') = (gg', n + g(n'))$, and identities have form $(\mathrm{Id}, 0)$. Equivalently, one might define objects as commutative triangles



and morphisms as commutative diagrams



in R-**mod**. One then has

5.17. **Proposition.** *The category* $\mathbb{T}_\partial$-**mod** *is equivalent to* ∂-$\mathbf{aff}'$ *with an extra initial object added.*

Proof. Define a functor ∂-$\mathbf{aff}' \to \partial$-$\mathbf{aff}$ as follows: for $f : M \to N$, define an affinity structure on N by $a +_b c = a - b + c$, $r_a b = (1 - r)a + rb$, and $\varphi_a(x) = f(x) + (1 - \partial x)a$. And to a morphism (g, n) assign the homomorphism of affinities $N' \to N$ given by $n' \mapsto n + g(n')$. It is straightforward to check that this defines a full and faithful functor. Moreover any nonempty affinity is isomorphic to one in the image of this functor – just choose an element and use it as zero to define a module structure and a homomorphism from M according to the affinity identities.

This allows to give an example, which looks pleasantly familiar:

Example. Fix a field k, and let the *category of cycles* be defined as follows. Objects are pairs $((V,d),c)$, where (V,d) is a differential k-vector space and $c \in V$ is a cycle, i. e. $dc = 0$. A morphism from $((V,d),c)$ to $((V',d'),c')$ is a pair (φ, x), where $\varphi : V \to V'$ is a k-linear differential map and $x \in V'$ an element with $c' - \varphi(c) = dx$. With the evident identities and composition this forms a category which is clearly of the form $\mathbb{T}_\partial$-**mod**, for the linear form $\partial : \varepsilon k[\varepsilon] \rightarrowtail k[\varepsilon]$, where ε is an indeterminate element with $\varepsilon^2 = 0$.

Now obviously this example admits a linear extension structure over $\mathbb{T}_k$, since $\varepsilon k[\varepsilon]$ is a square zero ideal. But of course 5.14 provides lots of similar (less cute) examples without this property.

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