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MATHEMATICS

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Galois Theory in a Category of Modules over an Elementary Topos

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ABSTRACT. **The present paper reports that in the case of module categories over an elementary topos the Galois theory of Chase and Sweedler and the Galois theory of Ligon are equivalent .**

Key words: closed symmetric monoidal category, Galois object, Morita context, locally progenerator.

Presented closed symmetric monoidal category $(\underline{C}, -\otimes-, I)$, $M(\underline{C})$ (resp.CM(C), $CoM(\underline{C})$, $HM(\underline{C})$, $CHM(\underline{C})$) denotes the category of monoids (resp. commutative monoids, comonoids, Hopfmonoids, commutative Hopfmonoids) in $\underline{C}$, where a monoid in $\underline{C}$ is to be understood as in [1], and a Hopfmonoid in $\underline{C}$ is defined in the same way as a Hopfalgebra in K-mod is defined in [2]. A left A-object in $\underline{C}$ with to some $A \in Ob(M(\underline{C}))$ is a pair $(\underline{C}, \nabla_{\underline{C}})$ where $\nabla_{\underline{C}}$ is a left action in the sense of [1]. ${}_A\underline{C}$ (resp. $\underline{C}_A$, ${}_A\underline{C}_B$, ${}^A\underline{C}$, $\underline{C}^A$) denotes the category of left A-objects (resp. right A-objects, A-B-biobjects, left A-coobjects, right A-coobjects). Similarly one has the notion of groups, cogroups etc.

The categories $\underline{C}$, ${}_A\underline{C}$, ${}_A\underline{C}_B$ etc are related by the following two functors, provided $\underline{C}$ has equalizers and coequalizers (cp. [3]):

Let A, $B \in Ob(M(\underline{C}))$ then any $P \in Ob(\underline{C}_A)$ defines a functor

$$P\otimes_A - :\ {}_A\underline{C} \to \underline{C}$$

and any $P \in Ob({}_B\underline{C})$ defines a functor

$${}_B[P,-]:\ {}_B\underline{C} \to \underline{C}$$

where ${}_B[P, P]$ again becomes a monoid .

If now $P \in {}_A\underline{C}_B$, these functors may be interpreted as functors between categories $\underline{C}_A$, ${}_B\underline{C}$, ${}_A\underline{C}_B$ etc. in various ways, such that become adjoint [3].

Using these facts, one gets for $A \in Ob(M(\underline{C}))$, $P \in Ob({}_A\underline{C})$ a morphism $g_A : {}_A[P,A] \otimes_{{}_A[P,P]} P \to A$ corresponding to $1_{{}_A[P,A]}$, and a morphism $f_A : P \otimes_A [P,A]_A \to {}_A[P,P]$ corresponding to $1_P \otimes g_A$.

Definition 1 [4]. *P is called*

a) finite over A, if f_A is an isomorphism;

b) faithfully projective over A, if P is finite and g_A is an isomorphism.

Let

$$A \in Ob(M(\underline{C})), H \in Ob(HM(\underline{C})), \alpha_A \in M(\underline{C})(A, A\otimes H), \quad (A,\alpha_A) \in Ob(\underline{C}^H),$$

$$\gamma_A = (\nabla_A \otimes 1_A)(1_A \otimes \alpha_A) \in M(\underline{C})(A \otimes A, A \otimes H).$$

Definition 2 [4]. *A is called H-Galois over I if A is faithfully projective over I, and* γ_A *is an isomorphism in* $\underline{C}$.

Let $H \in Ob(HM(\underline{C}))$ be finite over I and $(A,\alpha_A) \in Ob(\underline{C}^H)$. The fix -object A^{H^*} is defined as the equalizer of the following pair of morphisms $(\alpha_A, 1_A \otimes \eta_H)$.

In the case $A^{H^*} \approx I$ in [4] the Morita -context (*D, I, A, Q, f, g*) is defined, where

$$D = A\# H^*, Q = D^{H^*}, \varphi = \nabla_A(1_A \otimes \beta_A) \in \underline{C}(D \otimes A, A),$$

$$f = \nabla_D(j_A \otimes J_Q) \in {}_D\underline{C}_D(A \otimes Q, D),\ g = \varphi(j_Q \otimes j_A) \in \underline{C}(Q \otimes_D A, I)$$

(here the morphism $\beta_A \in C(H^* \otimes A, A)$ is obtained from $\alpha_A \in \underline{C}(A, A \otimes H^*)$ by the bijection $\underline{C}(H^* \otimes A, A) \approx \underline{C}(A, A \otimes H^*)$, and $j_A : A \to D$ and $j_Q : Q \to D$ are canonical inclusions) .

Theorem 1 [4]. *If* $H \in Ob(CHM(\underline{C}))$ *is finite over* I, $A \in Ob(CM(\underline{C}))$ *and* $(A, \alpha_A) \in Ob(\underline{C}^H)$, *then the following statements are equivalent:*

a) A is H-Galois over I;

b) A is faithfully projective over I and the morphism $A\# H^* \to [A, A]$ *induced by the left* $A\# H^*$ *- monoid structure on A is an isomorphism;*

c) $A^{H^*} \approx I$ *and the Morita context (D, I, A, Q, f, g) is strict.*

Now, let $\underline{E}$ be an arbitrary category with finite products and equalizers. An object $A \in Ob(\underline{E})$ is called faithfull if the functor $A \times - : \underline{E} \to \underline{E}$ creates isomorphisms. If this functor preserves coequalizers then A will be called coflat in $\underline{E}$.

Definition 3 [5]. *Let G be a group in* $\underline{E}$. *A faithfull object A is called Galois G-object if there exists a morphism* $\beta_A : G \times A \to A$ *in* $\underline{E}$ *such that* $(A, \beta_A) \in {}_G\underline{E}$ *and the morphism* $\gamma_A : G \times A \to A \times A$ *defined by the product diagram is an isomorphism.*

Let $\underline{E}$ be an elementary topos with a natural numbers object. If R is a commutative ring object with identity in $\underline{E}$, then $(R - \text{mod}, - \otimes_R -, R)$ is a symmetric monoidal closed category [6,7].

Definition 4. $P \in Ob(R - \text{mod})$ *will be called R - progenerator if there are natural numbers n, m such that* $P^{[n]}$ *is a retract of R and* $R^{[m]}$ *is a retract of P in R -* mod.

Theorem 2. *An R - module P is faithfully projective over R in the monoidal category* $(R - \text{mod}, - \otimes_R -, R)$ *if and only if P locally is R - progenerator [8].*

Corollary 1. *For arbitrary* $P \in Ob(R - \text{mod})$ *the Morita context ([P, P], R, P, [P, R], f, g),* *where f, g are morphisms from definition 1, is strict if and only if P locally is R - progenerator.*

By $CR(\underline{E})$ we denote the category of commutative R-algebras in $\underline{E}$. Then $CR(\underline{E})^{op}$ is a category with finite products and coequalizers [7].

Proposition 1. *A group in $CR(\underline{E})^{op}$ is a commutative Hopf algebra with antipode in $CR(\underline{E})$, and if it is finite over R, then it locally is R - progenerator.*

Proposition 2. *a) An object $A \in Ob\left(CR(\underline{E})^{op}\right)$ is faithfull in $CR(\underline{E})^{op}$ if and only if the following condition holds : whenever $f: M \to N$ is a homomorphism of R-modules such that $1_A \otimes_R f$ is an isomorphism, then f likewise is an isomorphism.*

b) A is a coflat object in $CR(\underline{E})^{op}$ if and only if A is a flat R-module.

Let $H \in Gr\left(CR(\underline{E})^{op}\right)$ be finite over R.

Theorem 3 [8]. *Let A be a Galois H-object in $CR(\underline{E})^{op}$. The following statements are equivalent:*

a) A is a coflat Galois H-object in $CR(\underline{E})^{op}$;

b) A locally is finitely generated projective R-module, a faithfull object in $CR(\underline{E})^{op}$ and the homomorphism $A\#H^* \to [A, A]$ is an isomorphism ;

c) $A^{H^*} \approx R$ and the Morita context (D, R, A, Q, f, g) is strict.

Proposition 3 [8]. *Let A be a Galois H-object in $CR(\underline{E})^{op}$. Then A is a coflat object in $CR(\underline{E})^{op}$.*

Combining the results , we obtain the following

Theorem 4. *In the closed symmetric monoidal category $\left(R-\text{mod},-\otimes_R -,R\right)$ the Galois theory of Ligon [4] and Galois theory of Chase and Sweedler [5] are equivalent.*

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