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Some strange monoidal categories

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Abstract. Monoidal categories of adjoint pairs of endofunctors are calculated for categories of models of some interesting algebraic theories. Notably, for theories generated by constants and for theories of various class two nilpotent algebras and groups.

Currently various mathematicians become interested in examples of monoidal categories which are not necessarily symmetric. Without trying to outline possible applications we will construct some examples of such categories.

One immediate way to produce a monoidal category from a category $\mathcal{C}$ is to consider the category $\mathbf{End}(\mathcal{C})$ of endofunctors of $\mathcal{C}$ and natural transformations between them, the monoidal structure being given by the composition of endofunctors. But monoidal categories arising this way are usually too large and complicated. Taking the full subcategory of $\mathbf{End}(\mathcal{C})$ consisting of functors having a left adjoint often gives much more interesting results. (A good reference to systematic study of the situation is [Freyd].)

Let us restrict ourselves to the case when $\mathcal{C}$ is the category of models of some algebraic theory $\mathbb{T}$ (in the sense of [Kock & Reyes]). This means in particular that $\mathcal{C}$ is equipped with the forgetful functor U to the category of sets and it's left adjoint F assigning to a set X the free model of $\mathbb{T}$ generated by X. Moreover U is representable by the free model $F(1)$ where 1 stands for any set with one element. Hence any adjoint pair $L \dashv R$ in $\mathbf{End}(\mathcal{C})$ will satisfy

$$UR(X) \approx \hom(F(1), R(X)) \approx \hom(LF(1), X),$$

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naturally in X. This means that any such adjoint pair is, up to isomorphism, determined by the object $LF(1)$ having the property that the set $\hom(LF(1), X)$ has a natural $\mathbb{T}$-model structure for any $\mathbb{T}$-model X. Clearly this means that $LF(1)$ is a "co-$\mathbb{T}$-model" in $\mathcal{C}$, i.e. a $\mathbb{T}$-model in the category $\mathcal{C}^\circ$ (opposite to $\mathcal{C}$). Hence one has

Proposition 0. *For any algebraic theory $\mathbb{T}$, let $\mathbb{T}(\mathcal{S})$ denote the category of models of $\mathbb{T}$ in a category $\mathcal{S}$. Then there is an equivalence of categories*

$$\mathbf{End\ Adj}(\mathbb{T}(\mathcal{S}ets)) \simeq \mathbb{T}(\mathbb{T}(\mathcal{S}ets)^\circ)^\circ$$

where $\mathbf{End\ Adj}(\mathcal{C})$ *denotes the category of endoadjunctions of a category* $\mathcal{C}$.

Many known examples of monoidal categories arise from this. First of all,

Example 0. $\mathbb{T}$ = the "initial" theory, i.e. such that $\mathbb{T}(\mathcal{S}) = \mathcal{S}$ for any category $\mathcal{S}$; this gives the (obvious) equivalence

$$\mathbf{End\ Adj}(\mathcal{S}ets) \simeq \mathcal{S}ets$$

and the corresponding monoidal structure is given by the usual cartesian product. In other words, for any adjoint pair $L \dashv R$ in $\mathbf{End}(\mathcal{S}ets)$ there are isomorphisms

$$L(X) \approx I \times X, \quad R(X) \approx X^I,$$

where $I = L(1)$; and if $L_1 \dashv R_1$, $L_2 \dashv R_2$ are two such with $I_1 = L_1(1)$, $I_2 = L_2(1)$, then

$$(L_1 \circ L_2)(X) = L_1(L_2(X)) \approx I_1 \times (I_2 \times X) \approx (I_1 \times I_2) \times X.$$

Example 1. As a slight generalization of the previous example, let $\mathbb{T}$ be a unary theory, i.e. the one generated by it's unary operations. In other words $\mathbb{T}$ is the theory of objects with a left M-action, for some monoid M. Since $\mathbb{T}$-models in the category $\mathcal{C}^\circ$ opposite to a category $\mathcal{C}$ are the same as *right* M-objects in $\mathcal{C}$, i.e. M°-objects, where M° is the monoid with the opposite multiplication, one has

$$\mathbf{End\ Adj}(\mathcal{S}ets^M) \simeq \mathcal{S}ets^{M \times M^\circ}.$$

Moreover the monoidal structure is in this case the known "tensor product over M": given two $M \times M^\circ$-sets X and Y, their product w.r.t. this structure is the quotient of $X \times Y$ by the equivalence relation generated by

$$(xm, y) \sim (x, my),$$

for all $x \in X$, $m \in M$, $y \in Y$. M itself with actions given by multiplication is the neutral object for this structure.

Example 2. If $\mathbb{T}$ is the theory of abelian groups, one obtains

$$\mathbf{End\ Adj}(\mathcal{A}b) \simeq \mathcal{A}b,$$

since for any adjoint pair $L \dashv R$ over $\mathcal{A}b$ one has $L(X) \cong A \otimes X$, $R(X) \cong \mathrm{Hom}(A, X)$. The corresponding monoidal structure is given by tensor product (and the neutral object is $\mathbb{Z}$, the group of integers).

Example 3. Again slightly generalizing one may let $\mathbb{T}$ be the theory of left R-modules, for an associative ring R. Then one obtains

$$\mathbf{End\ Adj}(R\text{-}\mathbf{mod}) \simeq R\text{-}\mathbf{mod}\text{-}R,$$

and the monoidal structure is given by $- \otimes_R -$, tensor product over R.

Before proceeding towards our main theme, let us mention one more unusual example, in which $\mathbb{T}$ is a "nullary" theory, i.e. a theory generated by it's constants; a $\mathbb{T}$-model structure on a set X is the same as a map $I \to X$ where I is the set of constants of $\mathbb{T}$. So the category $\mathbb{T}(\mathcal{S}ets)$ is the comma category $I \downarrow \mathcal{S}ets$. In this case the proposition 0 above gives

Proposition 1. *For any set I, the category*

$$\mathbf{End\ Adj}(I \downarrow \mathcal{S}ets)$$

is equivalent to the category of factorizations of the map

$$\mathrm{const} : I \to I^I$$

assigning to $i \in I$ the constant self-map with value i. In detail, the objects of the latter category are pairs

$$(e : X \to I^I, f : I \to X)$$

satisfying $e \circ f = \mathrm{const}$, a morphism from (e, f) to (e', f') being a map x satisfying $x \circ f = f'$, $e' \circ x = e$. The corresponding monoidal structure is given by

$$(I \xrightarrow{f_X} X \xrightarrow{e_X} I^I) \circ (I \xrightarrow{f_Y} Y \xrightarrow{e_Y} I^I) = (I \xrightarrow{f_\wedge} X \wedge_I Y \xrightarrow{e_\wedge} I^I),$$

where

$$X \wedge_I Y = X \times Y / \left((x, f_Y(i)) \sim (f_X(e_X(x)i), y)\right)$$

for $x \in X$, $i \in I$, $y \in Y$; $f_\wedge$ carries i to $f_X(i) \wedge f_Y(i)$, i.e. to the equivalence class of $(f_X(i), f_Y(i))$, and $e_\wedge(x \wedge y) = e_X(x) \circ e_Y(y)$. Finally, the neutral object for this structure is

$$\tilde{I} = (I \xrightarrow{\mathrm{const}} \mathrm{const}(I) \cup \{\mathrm{id}_I\} \xrightarrow{\subseteq} I^I),$$

where const *is the map assigning to $i \in I$ the constant map with value i while* id_I *denotes the identity map of I.*

Sketch of proof. An object (e, f) as above determines the functor

$$R_{(e,f)} : I \downarrow \mathcal{S}ets \to I \downarrow \mathcal{S}ets$$

carrying an object $s : I \to S$ to $t : I \to \{u | u \circ f = s\}$ with $t(i)(x) = s(e(x)i)$. This functor possesses a left adjoint $L_{(e,f)}$ given by

$$L_{(e,f)}(f_Y : I \to Y) = (f_\wedge : I \to X \wedge_I Y),$$

as above; and one checks that composing these functors corresponds to applying the monoidal structure declared.

NOTE. More familiar is the case when I has one element; then the category under consideration is the category of pointed sets and the monoidal structure is the "smash product".

Let us now return to the example 2 showing that tensor product may be determined by purely categorical considerations, as corresponding to composition in a certain category of endofunctors. Immediately a question arises: what monoidal categories can be obtained using non-abelian groups? Situation with the category of all groups is quite amazing:

EXAMPLE 4.
From [Eckmann&Hilton] immediately follows

$$\mathbf{End\ Adj}(\mathcal{G}roups) \simeq \mathcal{S}ets.$$

Indeed, it is proved in [Eckmann&Hilton] that any comultiplication

$$G \to G * G$$

on a group G which has a two-sided counit, is isomorphic to

$$F(S) \xrightarrow{F(\Delta)} F(S \times S) \subseteq F(S) * F(S),$$

where $\Delta : S \to S \times S$ is the diagonal and the rightmost inclusion picks the "cartesian subgroup", i.e. the kernel of the canonical homomorphism $F(S) * F(S) \to F(S) \times F(S)$. In other words, for any adjunction $L \dashv R$ in $\mathbf{End}(\mathcal{G}roups)$, $R(G)$ is isomorphic to G^S for some fixed set S. Hence the monoidal category obtained here is the same as in example 0.

The moral is that one has to stay closer to the category of abelian groups. Let us start with the theory of nilcube (that is, class 2 nilpotent) rings (without unit), i.e. those satisfying

$$(xy)z = x(yz) = 0$$

for all elements x, y, z. The category of these will be denoted by $T^2\mathcal{R}ings$.

THEOREM 1. *There is an equivalence of categories*

$$\mathbf{Adj}(T^2\mathcal{R}ings, \mathcal{A}b) \simeq \mathcal{A}b;$$

the category

$$\mathbf{End\ Adj}(T^2\mathcal{R}ings)$$

is equivalent to the category whose objects are triples (A, m_0, m_1) *where* A *is an abelian group and* $m_0, m_1 : A \to A \otimes A$ *are homomorphisms. A morphism from* (A, m_0, m_1) *to* (A', m'_0, m'_1) *is a homomorphism* $f : A \to A'$ *satisfying* $m'_i f = f(m_i \otimes m_i)$, $i = 0, 1$. *The monoidal structure corresponding to the composition of endoadjunctions is given by*

$$(A, m_0, m_1) \circ (B, n_0, n_1) =$$
$$= (A \otimes B, \tau_{23}(m_0 \otimes n_0 + (\tau m_1) \otimes n_1), \tau_{23}((\tau m_0) \otimes n_1 + m_1 \otimes n_0))$$

where $\tau : A \otimes A \to A \otimes A$ *and* $\tau_{23} : A \otimes A \otimes B \otimes B \to A \otimes B \otimes A \otimes B$ *are the canonical isomorphisms given by* $\tau(a \otimes a') = a' \otimes a$ *and* $\tau_{23}(a \otimes a' \otimes b \otimes b') = a \otimes b \otimes a' \otimes b'$. *The neutral object is* $(\mathbb{Z}, 1, 0)$ *where* $i : \mathbb{Z} \to \mathbb{Z} \otimes \mathbb{Z}$ *acts as* $(n \mapsto in(1 \otimes 1))$.

Proof. For any object X of $T^2\mathcal{R}ings$, let $X^2 \subseteq X$ denote the subgroup generated by elements x_1x_2, $x_1, x_2 \in X$, and by X_{ab} the quotient X/X^2. Up to isomorphism the additive structure of X is determined by a symmetric 2-cocycle of X_{ab} with values in X^2: given such a cocycle χ, X is isomorphic as an abelian group to $X^2 \times X_{\mathrm{ab}}$ with addition

$$(\xi, x) + (\eta, y) = (\xi + \eta + \chi(x, y), x + y).$$

As to the multiplicative structure, it is determined by any surjective homomorphism $X_{\mathrm{ab}} \otimes X_{\mathrm{ab}} \twoheadrightarrow X^2$.

Now suppose X carries structure of a model of the theory of abelian groups in the category $T^2\mathcal{R}ings^{\circ}$, i.e. X has a co-abelian group structure in $T^2\mathcal{R}ings$, with coaddition

$$\Delta : X \to X * X$$

and cozero

$$0 : X \to 0$$

where * denotes coproduct in $T^2\mathcal{R}ings$. For any objects X, Y this coproduct fits in a short exact sequence

$$0 \to X_{\mathrm{ab}} \otimes Y_{\mathrm{ab}} \oplus Y_{\mathrm{ab}} \otimes X_{\mathrm{ab}} \xrightarrow{\iota} X * Y \to X \times Y \to 0$$

where $\iota([x] \otimes [y]) = \iota_X(x)\iota_Y(y)$ and $\iota([y] \otimes [x]) = \iota_Y(y)\iota_X(x)$, ι_X and ι_Y being the coproduct structure embeddings. Taking into account that the coaddition has a two-sided cozero, one sees that $\Delta(x)$ for $x \in X$ may be uniquely written in the form

$$\Delta(x) = \iota_l(x) + \lambda(x) + \rho(x) + \iota_r(x)$$

where $\iota_l, \iota_r : X \to X * X$ are the corresponding structure morphisms and $\lambda(x) \in X_{\mathrm{ab}} \otimes X_{\mathrm{ab}} \cong \iota_l(X)\iota_r(X)$, $\rho(x) \in X_{\mathrm{ab}} \otimes X_{\mathrm{ab}} \cong \iota_r(X)\iota_l(X)$. Now the fact that Δ is a multiplicative homomorphism implies

$$\lambda(xy) + \rho(xy) = \iota_l(x)\iota_r(y) + \iota_r(x)\iota_l(y) \in X_{\mathrm{ab}} \otimes X_{\mathrm{ab}} \oplus X_{\mathrm{ab}} \otimes X_{\mathrm{ab}},$$

which means that the diagram

(D)
$$\begin{array}{ccc} X_{\mathrm{ab}} \otimes X_{\mathrm{ab}} & \xrightarrow{\text{diag}} & X_{\mathrm{ab}} \otimes X_{\mathrm{ab}} \oplus X_{\mathrm{ab}} \otimes X_{\mathrm{ab}} \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\iota} \\ X & \xrightarrow{\lambda+\rho} & X * X \end{array}$$

commutes, where μ is the multiplication map. Since the diagonal and ι are monos, this yields a short exact sequence

(S)
$$0 \to X_{\mathrm{ab}} \otimes X_{\mathrm{ab}} \xrightarrow{\mu} X \to X_{\mathrm{ab}} \to 0,$$

i.e. $X^2 \cong X_{\mathrm{ab}} \otimes X_{\mathrm{ab}}$; let's identify these groups from now on. Now additivity of Δ implies that both λ and ρ are additive, and composing the diagonal in (D) with either of the projections

$$\pi_l, \pi_r : X_{\mathrm{ab}} \otimes X_{\mathrm{ab}} \oplus X_{\mathrm{ab}} \otimes X_{\mathrm{ab}} \to X_{\mathrm{ab}} \otimes X_{\mathrm{ab}}$$

shows that both λ and ρ provide retractions for μ in (S). This means that X is isomorphic to the "truncated tensor algebra" of X_{ab},

$$T(X_{\mathrm{ab}}) = X_{\mathrm{ab}} \oplus X_{\mathrm{ab}} \otimes X_{\mathrm{ab}}.$$

So up to isomorphism an adjoint pair between $T^2\mathcal{R}ings$ and $\mathcal{A}b$ is determined by a single abelian group; moreover natural transformations between such pairs are easily seen to be determined by homomorphisms of the corresponding abelian groups.

Now let us add into consideration the comultiplication; suppose an X as above, determined by $X_{\mathrm{ab}} = A$, is equipped with a morphism

$$T(A) \to T(A) * T(A),$$

or equivalently

$$A \to U(T(A) * T(A)),$$

where U is the forgetful functor right adjoint to T. If this has to define a coring structure on X, comultiplying from either side by cozero must give cozero, i.e. the above map lands in

$$\mathrm{Ker}(T(A) * T(A) \to T(A) \times T(A)),$$

i.e. is determined by $(m_0, m_1) : A \to A \otimes A \oplus A \otimes A$. For a nilcube ring Z, multiplication on $\mathrm{Hom}(A, U(Z))$ determined by these data is given by

$$(f \cdot g)a = (fg)(m_0 a) + (gf)(m_1 a),$$

in other terms, for $f, g \in \mathrm{Hom}(A, U(Z))$, their product is the composition

$$A \xrightarrow{(m_0,m_1)} A \otimes A \oplus A \otimes A \xrightarrow{f\otimes g+g\otimes f} Z \otimes Z \oplus Z \otimes Z \xrightarrow{\mu\oplus\mu} Z \oplus Z \xrightarrow{+} Z.$$

It remains to calculate the effect of composing the adjunctions on the corresponding corings, which is straightforward.

Our next example concerns the theory of nilcube exterior rings, i.e. rings in which the identities

$$x^2 = xyz = 0$$

are satisfied. Denoting the category of these by $\Lambda^2\mathcal{R}ings$ one has

THEOREM 2. *The category*

$$\mathbf{Adj}(\Lambda^2\mathcal{R}ings, \mathcal{A}b)$$

is equivalent to the category of modules over

$$\mathbb{Z}[\delta] = \mathbb{Z}[t]/(2t);$$

the category

$$\mathbf{End\ Adj}(\Lambda^2\mathcal{R}ings)$$

is equivalent to the category with objects looking like

$$(\delta : C \to C, \mu : C \to \Sigma^2 C)$$

where C is an abelian group, δ emposes a $\mathbb{Z}[\delta]$-structure, i.e. it is an endomorphism with $2\delta = 0$, and μ is a homomorphism to symmetric 2-tensors, i.e., to the group

$$\Sigma^2 C = \{x \in C \otimes C | \tau_{C,C}(x) = x\}$$

where $\tau_{X,Y} : X \otimes Y \cong Y \otimes X$ denotes the canonical isomorphisms. Morphisms from (δ_C, μ_C) to $(\delta_{C'}, \mu_{C'})$ are $\mathbb{Z}[\delta]$-module homomorphisms $f : C \to C'$ satisfying $\mu_{C'} f = (\Sigma^2 f)\mu_C$. The monoidal structure corresponding to the composition of endofunctors is

$$(\delta_C, \mu_C) \circ (\delta_D, \mu_D) = (\delta_{C\otimes D}, \mu_{C\otimes D})$$

with

$$\delta_{C\otimes D}(c \otimes d) = c \otimes \delta_D(d) + \delta_C(c) \otimes [\mu_D(d)],$$

where $[\mu_D(d)]$ denotes $\mu_D(d)$ modulo the elements of the form $d_1 \otimes d_2 + d_2 \otimes d_1$ in $\Sigma^2 D$, and

$$\mu_{C\otimes D} = (C \otimes \tau_{D,C} \otimes D)(\mu_C \otimes \mu_D);$$

the neutral object is $(0 : \mathbb{Z} \to \mathbb{Z}, 1 \otimes 1 : \mathbb{Z} \cong \Sigma^2\mathbb{Z})$.

PROOF. Let X be any object in $\Lambda^2\mathcal{R}ings$. Up to isomorphism it is determined by the exact sequence of abelian groups

$$\Lambda^2 C \xrightarrow{m} X \to C \to 0$$

where $C = X/X^2$ and $m(c \wedge c') = xx'$ for $x \in c$, $x' \in c'$. Now suppose X carries a co-abelian group structure in $\Lambda^2\mathcal{R}ings$, with the coaddition

$$\Delta : X \to X \coprod X.$$

Since $X \to 0$ must be a two-sided cozero, one has

$$\Delta(x) = \iota_l(X) + \bar{\Delta}(x) + \iota_r(x),$$

with $\iota_l, \iota_r : X \to X \coprod X$ the coproduct embeddings, and $\operatorname{im}(\bar{\Delta}) \subseteq \iota_l(X)\iota_r(X) \cong C \otimes C$. But Δ must be a multiplicative homomorphism, which implies

$$\bar{\Delta}m(c \wedge c') = c \otimes c' - c' \otimes c.$$

Indeed one has

$$\begin{aligned}\Delta(x)\Delta(x') &= (\iota_l(x) + \bar{\Delta}(x) + \iota_r(x))(\iota_l(x') + \bar{\Delta}(x') + \iota_r(x')) \\ &= \iota_l(xx') + \iota_l(x)\iota_r(x') + \iota_r(x)\iota_l(x') + \iota_r(xx') \\ &= \iota_l(xx') + \iota_l(x)\iota_r(x') + \iota_l(x')\iota_r(x) + \iota_r(xx').\end{aligned}$$

Hence in particular we see that m is a monomorphism, since $\bar{\Delta}m$ is. Furthermore mutativity (= cocommutativity) of Δ implies that $\bar{\Delta}$ takes values in the subgroup $A^2C \subseteq C \otimes C$ of antisymmetric tensors, and hence induces

$$\delta : C = X/\Lambda^2C \to A^2C/\Lambda^2C \cong {}_2C = \{x \in C | 2x = 0\}$$

where the latter isomorphism may be deduced from the short exact sequence

$$0 \to A^2C/\Lambda^2C \to C \otimes C/\Lambda^2C \to C \otimes C/A^2C \to 0,$$

as $C \otimes C/\Lambda^2C \cong S^2C$ is the symmetric square and

$$C \otimes C/A^2C = C \otimes C/\operatorname{Ker}(\mathrm{id}_{C\otimes C} + \tau_{C,C}) \cong \operatorname{Im}(\mathrm{id}_{C\otimes C} + \tau_{C,C}).$$

It follows that there is a commutative square

$$\begin{array}{ccc} X & \xrightarrow{\bar{\Delta}} & A^2C \\ \downarrow & & \downarrow \\ C & \xrightarrow{\delta} & {}_2C \end{array}$$

and since the induced homomorphism on kernels of vertical surjections is an isomorphism, a standard diagram-chasing shows that the square is a pullback. Hence the whole co-abelian group X is up to isomorphism determined by $\delta : C \to {}_2C$. Moreover any morphism $f : X \to Y$ between co-abelian groups which preserves coaddition is determined by the induced map $X/X^2 \to Y/Y^2$ since the upper left corner map in a morphism between pullback squares is determined by the remaining maps.

Now let us turn to the comultiplication, which is another morphism in $\Lambda^2\mathcal{R}ings$

$$M : X \to X \coprod X.$$

Since comultiplying by cozero must be cozero, it follows that

$$\operatorname{Im}(M) \subseteq \operatorname{Ker}(X \coprod X \to X \times X) = \iota_l(X)\iota_r(X),$$

hence $X^2 \subseteq \operatorname{Ker}(M)$. So M factors as

$$X \twoheadrightarrow C \xrightarrow{\mu} C \otimes C \rightarrowtail X \coprod X$$

where μ may be any homomorphism. Moreover the comultiplication must be coexterior, i.e. satisfy the coidentity dual to $x^2 = 0$, which means

$$\mathrm{Im}(\mu) \subseteq \mathrm{Ker}(C \otimes C \rightarrowtail X \coprod X \xrightarrow{\nabla} X)$$

where ∇ is the codiagonal. But the latter composition is easily seen to coincide with

$$C \otimes C \twoheadrightarrow \Lambda^2 C \overset{m}{\rightarrowtail} X$$

so that $\mathrm{Im}(\mu) \subseteq \mathrm{Ker}(1 - \tau) = \Sigma^2 C$.

It remains to check the effect of the composition of endoadjunctions on the corresponding coexterior riings. This is straightforward and yields the theorem.

Closely related to nilcube exterior rings is the theory of class 2 nilpotent groups: it is known [Jibladze & Pirashvili] that these theories are linear extensions in the sense of [Baues & Wirsching] of the theory of abelian grups by the same functor Λ^2. Surprisingly enough their endoadjunction categories look entirely different. To formulate the result we need to recall the definition of the universal degree 2 map from an abelian group A. It is a map

$$p_2 : A \to P^2(A)$$

where the latter is the abelian group generated by symbols $p_2(a)$, $a \in A$, subject to the relations

$$p_2(x + y + z) = p_2(x + y) + p_2(x + z) + p_2(y + z) - p_2(x) - p_2(y) - p_2(z).$$

We'll need also the natural short exact sequence

$$0 \to S^2(A) \xrightarrow{\iota_A} P^2(A) \xrightarrow{\pi_A} A \to 0$$

with $\iota_A(xy) = p_2(x + y) - p_2(x) - p_2(y)$, $xy \in S^2(A)$, and $\pi_A p_2(a) = a$.

THEOREM 3. *The category*

$$\mathbf{End\ Adj}(\Lambda^2\mathcal{G}roups)$$

of endoadjunctions of the category of class 2 nilpotent groups has, up to equivalence, the following description: objects are homomorphisms

$$\sigma : A \to P^2(A)$$

for abelian groups A, satisfying

$$\pi_A \circ \sigma = identity$$

and morphisms from $\sigma : A \to P^2A$ to $\sigma' : A' \to P^2A'$ are homomorphisms $f : A \to A'$ with $\sigma' f = (P^2 f)\sigma$. The monoidal structure is given by

$$(A \xrightarrow{\sigma} P^2A) \circ (B \xrightarrow{\tau} P^2B) =$$
$$= (A \otimes B \xrightarrow{\sigma\otimes\tau} (P^2A) \otimes (P^2B) \xrightarrow{p_{A,B}} P^2(A \otimes B)$$

where the natural transformation p is given by

$$p_{A,B}(p_2(a) \otimes p_2(b)) = p_2(a \otimes b).$$

The neutral object is $(\iota : \mathbb{Z} \to P^2\mathbb{Z})$ *with* $\iota(n) = np_2(1)$.

Proof. Let $\Delta : G \to G{*_2}G$ be a comultiplication homomorphism on a 2-nilpotent group G, where $*_2$ denotes coproduct in $\Lambda^2\mathcal{G}roups$. If Δ has a two-sided counit, then for all $g \in G$

$$\Delta(g) = \iota_l(g)\bar{\Delta}(g)\iota_r(g)$$

where $\iota_l, \iota_r : G \to G *_2 G$ are the coproduct structure maps and

$$\mathrm{Im}(\bar{\Delta}) \subseteq \mathrm{Ker}(G *_2 G \to G \times G) \cong A \otimes A$$

where $A = G_{\mathrm{ab}} = G/[G,G]$. The latter isomorphism relates $a \otimes a'$ to $[\iota_l g, \iota_r g']$ with $g \in a$, $g' \in a'$: more generally $X *_2 Y$ for any X, Y fits in a central extension of groups

$$X_{\mathrm{ab}} \otimes Y_{\mathrm{ab}} \rightarrowtail X *_2 Y \twoheadrightarrow X \times Y.$$

Now $\Delta(xy) = \Delta(x)\Delta(y)$ and $\Delta([x,y]) = [\Delta(x), \Delta(y)]$ imply

$$\begin{aligned} \bar{\Delta}(xy) &= [\iota_l x, \iota_r y]\bar{\Delta}(x)\bar{\Delta}(y) \\ \bar{\Delta}([x,y]) &= [\iota_l x.\iota_r y][\iota_l y, \iota_r x]^{-1}. \end{aligned}$$

This means that $\bar{\Delta}$ factors via $\delta : A \to S^2A$ to fill in the diagram

$$\begin{array}{ccc} G & \xrightarrow{\bar{\Delta}} & [\iota_l(G), \iota_r(G)] \cong A \otimes A \\ \big\downarrow & & \pi\big\downarrow \\ A & \xrightarrow{\delta} & S^2A \end{array}$$

where π is the canonical map; moreover the composition

$$\Lambda^2 A \xrightarrow{[,]} G \xrightarrow{\bar{\Delta}} [\iota_l(G), \iota_r(G)] \cong A \otimes A$$

is the canonical embedding

$$a \wedge a' \mapsto a \otimes a' - a' \otimes a.$$

It follows that the diagram is a pullback of sets and G is up to isomorphism the set $\{(x,a) \in A \otimes A \times A | \pi x = \delta a\}$ with multiplication

$$(x,a)(x',a') = (x + x' + a \otimes a', a + a'),$$

whilst δ must satisfy

$$\delta(x+y) - \delta(x) - \delta(y) = xy,$$

$x, y \in A$. This means that δ factors through p_2 to give a homomorphism

$$\tilde{\delta} : P^2(A) \to S^2(A)$$

satisfying

$$\tilde{\tilde{\delta}} \circ \iota_A = \text{identity},$$

and such homomorphisms are clearly in one-to-one correspondence with sections of π_A. Moreover given a homomorphism between such objects which preserves comultiplication, it will determine a transformation of the corresponding pullback squares, and the upper left corner map of this transformation will be determined by the remaining maps. Finally one calculates the effect of the endofunctor composition on the representing objects, which is tedious but straightforward.

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