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Scattered toposes ☆

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Abstract

A class of toposes is introduced and studied, suitable for semantical analysis of an extension of the Heyting predicate calculus admitting Gödel's provability interpretation. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction and preliminaries

In this note some features of the elementary topos semantics for the Amended Intuitionistic Predicate Logic and the kindred intuitionistic modal system will be presented. A particular class of toposes, called scattered, will be described, which provides natural environment for modelling this kind of calculi.

Before entering into the matter of the subject let us indicate some motivation for bringing up this topic. An amendment to the standard quantifier extension QHC of the Heyting propositional calculus HC was inspired by the provability interpretation of the Intuitionistic Logic (via Gödel's modal translation and Solovay's arithmetical completeness theorem).

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Our Amended Calculus Q^+HC [2] is obtained from the usual QHC by postulating the following modified version of the rule of Universal Generalization:

$$\frac{\vdash (p(a) \Rightarrow \forall_x p(x)) \Rightarrow \forall_x p(x)}{\vdash \forall_x p(x)}. \tag{+}$$

An alternative definition is expressed by:

Statement 1. The calculus Q^+HC is deductively equivalent to the calculus obtained from the usual QHC by accepting as an additional axiom the following "relativised Kuroda principle":

$$\forall_x[(p(x) \Rightarrow \forall_y p(y)) \Rightarrow \forall_y p(y)] \Rightarrow \forall_x p(x). \tag{rKP}$$

The latter is so called because of its relation to the Kuroda principle

$$\forall_x \neg\neg p(x) \Rightarrow \neg\neg\forall_x p(x), \tag{KP}$$

which can be seen as (rKP) with the added assumption $\neg\forall_x p(x)$. Indeed, one can see that the following formula is intuitionistically valid:

$$\{\forall_x \neg\neg p(x) \Rightarrow \neg\neg\forall_x p(x)\}$$
$$\Leftrightarrow \{(\neg\forall_x p(x)) \Rightarrow (\forall_x[(p(x) \Rightarrow \forall_y p(y)) \Rightarrow \forall_y p(y)] \Rightarrow \forall_x p(x))\}.$$

Recall the remark of Heyting [4, p. 104] in connection to $\neg\neg\forall_x p(x) \Rightarrow \forall_x \neg\neg p(x)$, implication reverse to (KP):

> It is one of the most striking features of intuitionistic logic that the inverse implication does not hold, especially because the formula of the propositional calculus which results if we restrict x to a finite set, is true.

And slightly later:

> It has been conjectured [7, p. 46] that the formula
>
> $$\forall_x \neg\neg p(x) \Rightarrow \neg\neg\forall_x p(x)$$
>
> is always true if x ranges over a denumerable infinite species, but no way of proving the conjecture presents itself at present.

Statement 2. The Kuroda principle

$$\forall_x \neg\neg p(x) \Rightarrow \neg\neg\forall_x p(x)$$

and consequently also the biconditional $\forall_x \neg\neg p(x) \Leftrightarrow \neg\neg\forall_x p(x)$ is provable in Q^+HC.

Recall that the Proof-Intuitionistic propositional logic $HC^\Box$ (= Kuznetsov–Muravitsky Calculus [8]) is the calculus that results when the additional axioms

$$p \Rightarrow \Box p, \quad \Box p \Rightarrow (q \vee (q \Rightarrow p)), \quad (\Box p \Rightarrow p) \Rightarrow p$$

are added to the Heyting propositional calculus HC. It is known that the modal operator $\Box$ of $HC^{\Box}$ can be interpreted under suitable conditions, as the provability predicate of the classical Peano arithmetic PA.

Let us denote by $QHC^{\Box}$ the standard quantifier extension of the proof-intuitionistic logic $HC^{\Box}$.

Statement 3. The relativized Kuroda Principle rKP is provable in the calculus $QHC^{\Box}$ (and, hence, admits a provability interpretation).

One possible approach to semantical analysis of the above calculi is via topos theory. It is well known that elementary toposes correspond to higher order intuitionistic type theories, and in particular provide interpretation of various logical calculi in categories with specific properties. In particular, categories of sheaves on a topological space, or presheaves on a small category, can be used for this purpose. Hence a natural question arises, to characterize the spaces and small categories for which the corresponding logical principles are valid. A sort of standard examples for such kind of investigation can be found in the work of Johnstone, e.g. in [5] de Morgan's law is related in this way to extremal disconnectedness of the space and to the Ore condition for the category; however, also for a general elementary topos it is related there to a very natural and important property – that the two-element lattice $\{0 \leqslant 1\}$ is complete.

In the present work we are going to present another example of such activity, this time with formulae involving quantifiers, such as rKP, and with modalities as in proof-intuitionistic logic.

For Heyting algebras of all open sets of a (sufficiently separated) topological space, the natural choice of a modal operator $\Box$ as above is dictated by its intuitive precursor – the operator dual to the Cantor–Bendixson derivative δ, i.e. $\Box U = -\delta - U$, where "–" denotes complement, and δS is the set of limit points of a subset S. Thus, $\Box U$ adds to U those points which are "entirely surrounded by U" – i.e. they possess a neighborhood containing no other points outside U.

For complete Heyting algebras, Simmons in [11] has constructed an analog of the Cantor–Bendixson derivative, coinciding with the usual one for lattices of open sets of sufficiently good spaces. This allows for an analog of the above $\Box$ in this context:

$$\Box a = \bigwedge \{d \geqslant a \mid d \text{ is dense in } [a, 1]\},$$

where an element d in a lattice is called dense if $d \wedge x = \bot$ implies $x = \bot$ for any x. In a Heyting algebra, an element is dense iff it is of the form $b \vee \neg b$ for some b. Thus if the Heyting algebra is complete one can define

$$\Box a = \bigwedge_{b} (b \vee (b \rightarrow a)).$$

But more generally still, a Heyting algebra represented as a lattice of subobjects of an object in an elementary topos also admits such an operator, since this lattice is realized as the set of global elements of an internal complete Heyting algebra, and the constructions of Simmons can be performed with any internally complete Heyting algebra in an elementary topos.

Thus everywhere in the sequel we work inside an elementary topos; we will use interchangeably the external categorical descriptions and internal language of the topos. For example, the operator $\Box$ mentioned above can be defined internally on the subobject classifier Ω using the internal language as

$$\Box\varphi = \forall_\psi(\psi \vee (\psi \Rightarrow \varphi)).$$

The subobject of Ω classified by $\Box$ has many remarkable properties. It has been first investigated by Denis Higgs; see [3, 1]. It is the object of those $\varphi \in \Omega$ satisfying $[\varphi, \top] = \{\varphi, \top\}$, i.e. such that anything between φ and the top of Ω is either φ or the top.

We will use the following notation for nuclei on a frame $\mathscr{T}$: for $U, V \in \mathscr{T}$:

$$o_U(V) = U \to V,$$

$$c_U(V) = U \vee V,$$

$$d_U(V) = (V \to U) \to U.$$

Risking confusion, nuclei will be identified with sublocales of the corresponding locale, i.e. elements of the lattice of quotient frames of $\mathscr{T}$. We will identify $U \in \mathscr{T}$ with the open sublocale corresponding to o_U. Recall that for a nucleus j, its interior $\mathrm{int}(j)$ is defined as interior of the corresponding sublocale, i.e. the largest U with $j \leqslant o_U$.

One has $\mathrm{int}(j) = \bigwedge\{j(U) \to U \mid U \in \mathscr{T}\}$. For $\mathscr{T} = \Omega$ this is clearly the same as $\bigwedge\{\varphi \mid j(\varphi)\}$. In particular,

$$\mathrm{int}(d_U) = \Box U. \qquad (*)$$

Note also that for $U \leqslant V$, one has $V \leqslant \Box U$ iff the frame $[U, V]$ is Boolean. In particular, $\Box\bot$ corresponds to the largest Boolean open sublocale. It also coincides with the meet of all dense elements of the frame.

Denote

$$(Boo) = \forall_\varphi(\varphi \vee \neg\varphi)$$

(it is also equivalent to $\forall_\varphi(\neg\neg\varphi \Rightarrow \varphi)$).

Lemma. *An operator $\Phi : \Omega \to \Omega$ preserves meets iff it is of the form o_φ for some φ. In fact, $\varphi = \bigwedge\{\psi \mid \Phi(\psi)\}$.*

Proof. Since o_φ has a left adjoint $\varphi\&-$, it preserves meets. Conversely, suppose Φ preserves meets. Firstly, it is then monotone. Now, consider, the meet $\varphi = \bigwedge\{\psi \mid \Phi(\psi)\}$. So, for any ψ one has

$$\Phi(\psi) \Rightarrow (\varphi \Rightarrow \psi).$$

But Φ must preserve this meet, so $\Phi(\varphi)$ holds. Since Φ is monotone, this implies

$$(\varphi \Rightarrow \psi) \Rightarrow \Phi(\psi)$$

for any ψ.

2. Scatteredness

Theorem 1. *The following conditions on an elementary topos* **X** *are equivalent*:

(i) *the Kuroda principle*

$$(\forall_x \neg\neg p(x)) \Rightarrow \neg\neg\forall_x p(x) \tag{KP}$$

holds in **X**;

(ii) *the smallest dense subtopos* $sh_{\neg\neg}(\mathbf{X})$ *is open in* **X**;

(iii) **X** *has a Boolean open dense subtopos*;

(iv) *interior of a dense subtopos of* **X** *is dense*;

(v) $\neg\neg(Boo)$ *holds in* **X**.

Proof. In Ω, meets are equivalent to universal quantification; that is, $\forall_x p(x)$ is the meet of $p(x)$ considered as a family of propositions indexed by x. Hence by the lemma, (i) is equivalent to requiring that $\neg\neg$ equals o_φ for some φ (in fact, for $\varphi = \bigwedge\{\psi \mid \neg\neg\psi\} = \bigwedge\{\chi \vee \neg\chi \mid \chi \in \Omega\} = \Box\bot$). In terms of nuclei, this means that $\neg\neg$ is an open nucleus, i.e. coincides with its interior (which is $\Box\bot$). So (KP) is equivalent to (ii). Clearly (ii) implies (iii); and converse holds as restriction of the nucleus $\neg\neg$ to any open subtopos is $\neg\neg$ there. On the other hand, o_φ is dense iff $\neg\neg\varphi$ and it is Boolean iff $\varphi \Rightarrow (\psi \vee (\psi \Rightarrow \neg\varphi))$ for all ψ. Hence a dense o_φ is Boolean iff $\varphi \Rightarrow \Box\bot$. So (iii) is equivalent to $\neg\neg\Box\bot$, which is (v). Finally (iv) follows since a subtopos is dense iff it contains $\neg\neg$.

We call a topos $\bot$-scattered, if it satisfies one of the equivalent conditions of the theorem.

Examples. (1) To characterize those small categories **C** for which the topos $\mathrm{Set}^{\mathbf{C}}$ of set-valued functors on **C** is $\bot$-scattered, we need some terminology: call an object m of **C** *maximal* if any morphism with domain m is a split monomorphism. Then, one sees easily that $\mathrm{Set}^{\mathbf{C}}$ is a $\bot$-scattered topos iff every object of **C** admits a morphism to a maximal object.

(2) In the frame of open sets of a topological space X, $\Box\bot$ is the largest open subspace whose frame of opens is Boolean; as a space it is thus determined by an equivalence relation on the set of its points, its open (at the same time, closed) sets being any unions of equivalence classes. In particular, if X is T_0 then $\Box\bot$ is precisely the subspace of open points of X. Thus, the topos of sheaves on a T_0 space X is $\bot$-scattered iff open points are dense in X.

We may now introduce scattered topos as a substitute for the classical Cantor notion of scattered topological space (i.e. one with no nonempty subspace that is dense in itself): namely, let us call an elementary topos **X** *scattered* if one of the equivalent conditions listed below is satisfied.

Theorem 2. *The following conditions on an elementary topos* **X** *are equivalent:*
(i) *the relativized Kuroda principle*

$$\forall_x[(p(x)\Rightarrow\forall_y p(y))\Rightarrow\forall_y p(y)]\Rightarrow\forall_x p(x) \tag{rKP}$$

holds in **X**;
(i′) *for any* φ *and* $p(x)$ *with* $\varphi\Rightarrow\forall_x p(x)$ *one has*

$$[\forall_x((p(x)\Rightarrow\varphi)\Rightarrow\varphi)]\Rightarrow[((\forall_x p(x))\Rightarrow\varphi)\Rightarrow\varphi];$$

(ii) *every closed subtopos of* **X** *is* $\perp$*-scattered*;
(iii) *the Löb principle*

$$(\Box\varphi\Rightarrow\varphi)\Rightarrow\varphi \tag{LP}$$

holds in **X**;
(iv) *the principle*

$$[(\psi\Rightarrow\varphi)\Rightarrow\varphi]\Leftrightarrow[\Box\varphi\Rightarrow(\varphi\vee\psi)]$$

holds in **X**.

Proof. Indeed, (iii) is precisely condition (v) of the previous theorem for all closed subtoposes, hence (iii)$\Leftrightarrow$(ii). For (iii)$\Leftrightarrow$(iv), note that

$$[(\psi\Rightarrow\varphi)\Rightarrow\varphi]\Rightarrow[\Box\varphi\Rightarrow(\varphi\vee\psi)]$$

is equivalent to

$$\Box\varphi\Rightarrow[(((\varphi\vee\psi)\Rightarrow\varphi)\Rightarrow\varphi)\Rightarrow(\varphi\vee\psi)]$$

which is true by definition of $\Box$. Whereas

$$[\Box\varphi\Rightarrow(\varphi\vee\psi)]\Rightarrow[(\psi\Rightarrow\varphi)\Rightarrow\varphi]$$

is equivalent to

$$[(\Box\varphi\Rightarrow(\varphi\vee\psi))\&((\varphi\vee\psi)\Rightarrow\varphi)]\Rightarrow\varphi$$

which is clearly equivalent to Löb principle (LP).

Since $(\psi\Rightarrow\varphi)\Rightarrow\varphi$ is equal to $((\varphi\vee\psi)\Rightarrow\varphi)\Rightarrow\varphi$, (iv) is equivalent to the statement that in the lattice $[\varphi,\text{true}]$ the double negation nucleus is open. By the lemma this is equivalent to requiring that in this lattice the double negation nucleus commutes with universal quantifiers. The latter statement is precisely (i′).

Clearly (i′) implies (i) if one takes $\varphi = \forall_x p(x)$. Finally, considering (i) as a statement in the closed subtopos determined by $\forall_x p(x)$ one can see that (rKP) says that in any closed subtopos of **X** one has

$$\neg\forall_x p(x) \Rightarrow [\forall_x(p(x) \Rightarrow \bot) \Rightarrow \bot] \Rightarrow \bot,$$

which by $[\alpha \Rightarrow (\beta \Rightarrow \gamma)] \Leftrightarrow [\beta \Rightarrow (\alpha \Rightarrow \gamma)]$ is equivalent to $[\forall_x \neg\neg p(x)] \Rightarrow \neg\neg\forall_x p(x)$, i.e. to (KP). So (i) is equivalent to (ii).

Another interesting characterization of scattered toposes, in terms of the lattice of nuclei, can be readily produced from [11]: that paper is about frames, but the arguments we need are all constructive and hence valid internally in a topos – in particular, they can be applied to the subobject classifier. For this, let $\mathcal{N}$ denote the subobject of Ω^Ω consisting of nuclei, i.e. Lawvere–Tierney topologies in the topos. That is, $\mathcal{N}$ is the subobject of those $j : \Omega \to \Omega$ which satisfy

$$j(\top) = \top, \quad jj\varphi \Rightarrow j\varphi$$

(as shown in [6], these two conditions imply that j preserves binary meets). It is well known that under pointwise order $\mathcal{N}$ is an internal complete Heyting algebra. The key property of $\mathcal{N}$ needed for our characterization is Lemma 2.2 from [11]: in $\mathcal{N}$, one has

$$\neg d_\varphi = c_{\Box\varphi} \wedge o_\varphi. \tag{†}$$

Using this, it is not difficult to prove:

Proposition 1. *A topos is scattered if and only if the internal lattice $\mathcal{N}$ of its nuclei is a Boolean algebra.*

Proof. Clearly $\mathcal{N}$ is Boolean iff the only $j \in \mathcal{N}$ with $\neg j = \bot_{\mathcal{N}}$ is the constant map with value $\top$. But it is well known that for any nucleus j one has

$$j = \bigwedge\{d_\varphi \mid \varphi \in \mathrm{Fix}\, j\}.$$

Hence $\mathcal{N}$ is Boolean iff the only φ with $\neg d_\varphi = \bot_{\mathcal{N}}$ is $\varphi = \top_\Omega$. But by (†), $\neg d_\varphi = \bot_{\mathcal{N}}$ is equivalent to $c_{\Box\varphi} \wedge o_\varphi = \bot_{\mathcal{N}}$, i.e. $c_{\Box\varphi} \leqslant \neg o_\varphi = c_\varphi$, which is easily seen to be equivalent to $\Box\varphi \Rightarrow \varphi$. It follows that Booleanness of $\mathcal{N}$ is equivalent to (LP) above.

For toposes of sheaves on a space, an interesting characterization of scatteredness can be derived from an exercise in [9]:

Proposition 2. *A topos of sheaves on a topological space X is scattered if and only if each of its nondegenerate subtoposes has a point.*

Proof. According to Exercise IX.8 (p. 524) of [9], the topos of sheaves on X has a smallest pointless subtopos, and the latter can be described as the topos of sheaves on the frame of those open sets U of X whose complement is perfect. But it is easy to see that U has perfect complement iff $\Box U = U$ in the frame of open sets of X. Thus every nondegenerate subtopos has a point iff X is the only open fixed by $\Box$. Since the topos of sheaves on X is generated by subobjects of 1, i.e. by opens, there is a one-to-one correspondence between subtoposes and sublocales of X, i.e. quotient frames of the frame of opens of X, so that the sublocale of opens fixed by $\Box$ is trivial if and only if the internal sublocale of the subobject classifier determined by elements fixed by $\Box$ is trivial. In other words, one must have

$$\{\varphi \mid \varphi = \Box\varphi\} = \{\mathbf{true}\}.$$

And this is exactly (LP).

3. Fixed points

As mentioned in the introduction, the Proof-Intuitionistic calculus $\mathrm{HC}^{\Box}$ is of special interest in connection with the study of the notion of provability in Peano Arithmetic. One of the most remarkable properties of $\mathrm{HC}^{\Box}$ is the existence and uniqueness (up to provable equivalence) of a fixed point for a wide range of formulae. More precisely, if a formula Φ is such that the propositional variable p appears in Φ only under the scope of the modal operator $\Box$, then there exists a formula φ, such that $\Phi(\varphi) \Leftrightarrow \varphi$ is provable in $\mathrm{HC}^{\Box}$. This fact can be viewed as an analog of the Gödel's diagonalization lemma in a modal context. A detailed analysis of the logical phenomenon of diagonalization in terms of a modal system based on the intuitionistic propositional logic can be found in [10].

We here present a topos-theoretic counterpart of fixed point theorems of the above kind. Thus a formula such as $\Phi(p)$ above can be replaced by an endomorphism f of the object of truth values Ω, and the fixpoint φ by an element $b: 1 \to \Omega$ with $fb = b$. However, in fact, more generally endomorphisms of any object X may be considered. The restriction on Φ above corresponds to a certain property of f which we now describe.

Let us call a map $f: X \to X'$ *unchanging* if the equivalence relation

$$R = \{(x, y) \in X \times X \mid f(x) = f(y)\}$$

is a dense element of the lattice [diagonal(X), $X \times X$]. (One might call equivalence relations R with this property *undistinguishing*.) In other words, f is unchanging if one has

$$\forall_{x,y \in X}(fx = fy \Rightarrow x = y) \Rightarrow x = y.$$

Clearly any constant map, i.e. one which factors through a subterminal object, is unchanging. In a Boolean topos the converse also holds, however this is far from being true in general. In fact, Theorem 2 shows that the map $\Box: \Omega \to \Omega$ is unchanging in any scattered topos; and this map is constant iff the topos is Boolean.

Theorem 3. *Let f be any unchanging endomorphism of an object X. Then the subobject* $\mathrm{Fix}(f)$ *of fixed points of f is a maximal subterminal subobject of X; moreover its support is dense in the support of X.*

Proof. Simplest is to show uniqueness of fixed points: given $f(x)=x$ and $f(y)=y$ one has

$$f(x)=f(y) \Rightarrow x=f(x)=f(y)=y,$$

which by hypothesis implies $x=y$. So $\mathrm{Fix}(f)$ is a subterminal subobject of X. Let

$$\mathrm{Prefix}(f)=f^{-1}(\mathrm{Fix}(f))=\{x \mid f(f(x))=f(x)\}.$$

Then $\mathrm{Fix}(f) \subseteq \mathrm{Prefix}(f)$ and $f(\mathrm{Prefix}(f))=\mathrm{Fix}(f)$, so $\mathrm{Fix}(f)$ and $\mathrm{Prefix}(f)$ have equal support. By hypothesis one has

$$(f(x)=f(f(x)) \Rightarrow x=f(x)) \Rightarrow x=f(x),$$

i.e.

$$\mathrm{Prefix}(f) \to \mathrm{Fix}(f) \leqslant \mathrm{Fix}(f) \qquad (**)$$

in the lattice of subobjects of X.

We now can show that $\mathrm{Fix}(f)$ is dense in the support of X: let U denote support of $\mathrm{Fix}(f)$, which coincides with the support of $\mathrm{Prefix}(f)$ by the above. Suppose given a subterminal V inside support of X with $V \cap \mathrm{Supp}(\mathrm{Fix}(f))=\bot$, then

$$V \times X \cap \mathrm{Prefix}(f)=\bot \leqslant \mathrm{Fix}(f)$$

in the lattice of subobjects of X, hence $V \times X \leqslant (\mathrm{Prefix}(f) \to \mathrm{Fix}(f))$, which by $(**)$ above implies $V \times X \leqslant \mathrm{Fix}(f)$, hence $V \leqslant \mathrm{Supp}(\mathrm{Fix}(f))$ and $V=\bot$.

To show that $\mathrm{Fix}(f)$ is a maximal subterminal subobject, suppose given a subterminal subobject W of X with $\mathrm{Fix}(f) \leqslant W$ in the lattice of subobjects of X. Then for any $x \in \mathrm{Prefix}(f) \cap W$ one has $f(x) \in \mathrm{Fix}(f) \subseteq W$. Since W is subterminal, one has $x_1=x_2$ for any $x_1, x_2 \in W$, in particular $x=f(x)$, i.e. $x \in \mathrm{Fix}(f)$. Thus $\mathrm{Prefix}(f) \cap W \leqslant \mathrm{Fix}(f)$, or equivalently $W \leqslant (\mathrm{Prefix}(f) \to \mathrm{Fix}(f))$, which by $(**)$ implies $W=\mathrm{Fix}(f)$.

Scattered toposes are particularly rich in unchanging maps, as the following corollary shows.

Corollary. *Let $f: X \to X$ be an endomorphism in a scattered topos satisfying*

$$\forall_{x,y \in X} \Box(x=y) \Rightarrow f(x)=f(y).$$

Then f satisfies the condition of Theorem 3, *so that the object of fixpoints of f is a maximal subterminal subobject of X whose support is dense in the support of X.*

Proof. By Theorem 2(iv) above, a topos is scattered if and only if for any φ, ψ with $\varphi \Rightarrow \psi$, the formula $(\psi \Rightarrow \varphi) \Rightarrow \varphi$ is equivalent to $(\Box\varphi) \Rightarrow \psi$. Taking here $\varphi =$ '$x = y$' and $\psi =$ '$f(x) = f(y)$' gives that f is unchanging.

Note that, for any object X in a scattered topos, the morphism $\Box : \Omega^X \to \Omega^X$ satisfies the condition in the above corollary. And if in a composition one of the terms satisfies this condition, then the whole composition also does.

Note also that any maximal subterminal subobject of an *injective* object is a global element. Hence in this case Theorem 3 implies existence of an actual unique fixpoint.

However, in general, one cannot say that Fix(f) is isomorphic to the support of X. Indeed, consider the topos of presheaves on the partially ordered set $0 < 1 < 2 < \cdots < \infty$. On the presheaf X which sends n to itself (i.e. to $\{i \mid i < n\}$) and has restriction maps $m \to n$, for $m \geqslant n$ given by

$$i \mapsto i \cap n = \begin{cases} i, & i \leqslant n, \\ n, & i \geqslant n, \end{cases}$$

consider the endomap $f : X \to X$ given on the nth level, $n \leqslant \infty$, by

$$f(i) = \begin{cases} i+1, & i+1 < n, \\ i, & i+1 = n. \end{cases}$$

Then the object of fixed points does not have global support, although the map clearly satisfies the condition of Theorem 3.

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