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Cohomology of Algebraic Theories

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1. INTRODUCTION

Cohomology theory for associative algebras over a field is due to Hochschild [9]. Generalization of this theory for associative algebras over a commutative ring K posed considerable complications. Several definitions have been proposed. For example, in Cartan and Eilenberg's monograph [5], the groups $\mathrm{Ext}^*_{R^e}(R, M)$ are named as candidates for cohomology of the K-algebra R with coefficients in the R–R-bimodule M; here $R^e = R \otimes_K R^{op}$ is the enveloping algebra of R. In MacLane's book [14] Hochschild cohomology is defined in the framework of relative homological algebra,

$$\mathrm{Hoch}^*(R; M) = \mathrm{Ext}^*_{R^e, K}(R, M),$$

where the subscript K signifies that only those extensions which split over K are considered. Still another definition was proposed by Shukla [22], whose cohomology is denoted $\mathrm{Shukla}^*(R; M)$. All these cohomologies are connected by natural homomorphisms:

$$\mathrm{Hoch}^*(R; M) \to \mathrm{Ext}^*_{R^e}(R; M) \to \mathrm{Shukla}^*(R; M).$$

These homomorphisms are iso in dimensions 0 and 1, while if R is a projective K-module, isomorphism holds in all dimensions. In dimension 2, the group $\mathrm{Shukla}^2(R; M)$ classifies arbitrary singular extensions of the ring R by M, while $\mathrm{Hoch}^2(R; M)$ classifies those singular extensions of K-algebras, which split as K-module extensions [14]. The groups $\mathrm{Ext}^*_{R^e}(R, M)$ have no good relation to algebra extensions, but they constitute a universal connected exact sequence of functors instead, unlike the others. And yet the Shukla cohomology is considered as the most "correct"—not only does it describe extensions well, but it also behaves well with respect to the first argument—namely, it can be described as Barr and Beck's [2] cotriple cohomology (see [1]); Quillen's approach to the construction of the cohomology in "good" categories also yields Shukla cohomology [19].

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There is still another theory for rings, i.e., when $K = \mathbb{Z}$ – Mac Lane cohomology $H^*(R; M)$ from [13]. There are homomorphisms

$$\text{Shukla}^*(R; M) \to H^*(R; M)$$

which are iso in dimensions $\leqslant 2$. Mac Lane cohomologies are closely related to stable cohomologies of Eilenberg–Mac Lane spaces [7].

We make the domain of applicability of the Mac Lane cohomology wider in order to ensure, by suitable choice of that widened domain, that the Mac Lane cohomology provides a universal connected exact sequence of functors. This can be done thanks to the existence of the isomorphism

$$H^*(R; M) \cong \operatorname{Ext}^*_{\mathscr{F}(R)}(I, M \otimes_R -), \tag{1.1}$$

where $\mathscr{F}(R)$ denotes the category of all functors from the category $\underline{M}_R$ of free finitely generated left R-modules to the category R-$\underline{\text{mod}}$ of all left R-modules, and where

$$I: \underline{M}_R \to R\text{-}\underline{\text{mod}}$$

is the obvious embedding.

The isomorphism (1.1) is a corollary of Theorem A, proved in Section 2.

There is a full embedding

$$R^e\text{-}\underline{\text{mod}} \to \mathscr{F}(R)$$

that assigns to an R–R-bimodule M the functor

$$M \otimes_R -: \underline{M}_R \to R\text{-}\underline{\text{mod}},$$

establishing an equivalence of the category of R–R-bimodules with that full subcategory in $\mathscr{R}(R)$ which consists of additive functors. Consequently objects of $\mathscr{F}(R)$ can be viewed as certain generalized "non-additive" bimodules, and conversely, real bimodules can be identified with additive functors from $\mathscr{F}(R)$. In the course of this identification the inclusion I corresponds to the R–R-bimodule R, and the functor $M \otimes_R$ to M. By (1.1), Mac Lane cohomology appears to be a somewhat modified Cartan–Eilenberg-type cohomology: the modification consists in taking Ext not in the category of bimodules, i.e., of additive functors, but in the larger category of all functors from $\underline{M}_R$ to R-$\underline{\text{mod}}$. So the isomorphism (1.1) motivates the following

DEFINITION 1.2. The cohomology of an associative ring R with coefficients in an arbitrary functor

$$T: \underline{M}_R \to R\text{-}\underline{\text{mod}}$$

is defined by the equality

$$H^*(R; T) = \operatorname{Ext}^*_{\mathscr{F}(R)}(I, T).$$

Hence, by the very definition our cohomologies

$$H^*(R; -): \mathscr{F}(R) \to \underline{Ab}$$

constitute a universal exact connected sequence of functors. For additive T's they recover the Mac Lane cohomology, through the aforementioned identification.

In particular, $H^2(R; T)$, for additive functors T, classifies arbitrary singular extensions of the ring R by the R–R-bimodule $T(R)$. What can be said about non-additive T's? To answer this question, let us note that Theorem B of Section 3 implies the existence of isomorphisms

$$H^*(R; T) \cong H^*(\underline{M}_R; \mathscr{H}om_R(I, T)),$$

where the groups on the right denote the Hochschild–Mitchell cohomology [3, 16] of the category of free finitely generated left R-modules with coefficients in the bifunctor $\mathscr{H}om_R(I, T)$, given for $X, Y \in |M_R|$ by

$$(\mathscr{H}om_R(I, T))(X, Y) = \operatorname{Hom}_R(X, TY).$$

But, according to [3] the second Hochschild–Mitchell cohomology group of a small category $\underline{M}_R$ with coefficients in a bifunctor $\mathscr{H}om_R(I, T)$ classifies linear extensions of the category $\underline{M}_R$ by the bifunctor:

$$\mathscr{H}om_R(I, T) + \to \underline{E} \to \underline{M}_R.$$

We prove that here $\underline{E}$ will always be equivalent to the category of all finitely generated free models of some uniquely determined algebraic theory in Lawvere's sense [12, 21, 25]. The functor

$$\mathscr{R}ings \to \mathscr{T}heories$$

from the category of associative rings with unit to the category of algebraic theories, which assigns the theory of left R-modules to the ring R, is known to be a full embedding. This enables us to identify rings with corresponding theories. Hence it turns out that $H^2(R; T)$ classifies extensions of R in the category of algebraic theories.

The above considerations make it clear that the natural domain of objects for our cohomology must be the category of algebraic theories, rather than rings. Section 4 is devoted to their construction. In that section a number of alternative approaches are presented and it is proved that they lead to the same result. In Section 5 examples of calculations of the cohomology for free theories, theories of groups, theories of monoids, theories of G-sets for a monoid G, etc. are given, and in Section 6 some open problems are listed.

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2. On the Mac Lane Cohomology

In this section we prove Theorem A, which is concerned with the relationship between Mac Lane cohomology of rings and Ext groups in functor categories; related questions are discussed.

Let us recall the definition of the Mac Lane cohomology from [13]. Consider the sets C_n with 2^n elements—n-tuples $(\varepsilon_1, ..., \varepsilon_n)$, where $\varepsilon_i = 0$ or 1, for $n \geqslant 0$ and $i \leqslant n$, and the 0-tuple ( ) for $n = 0$. For convenience C_n can be visualized as the set of vertices of an n-cube, the product of n copies of the 1-cube with vertices 0 and 1.

Define maps $0_i, 1_i: C_n \to C_{n+1}$, $1 \leqslant i \leqslant n+1$, by the equalities

$$0_i(\varepsilon_1, ..., \varepsilon_n) = (\varepsilon_1, ..., \varepsilon_{i-1}, 0, \varepsilon_{i+1}, ..., \varepsilon_n),$$
$$1_i(\varepsilon_1, ..., \varepsilon_n) = (\varepsilon_1, ..., \varepsilon_{i-1}, 1, \varepsilon_{i+1}, ..., \varepsilon_n).$$

For an abelian group A and a set S, let $A[S]$ denote the sum of S copies of the group A. Since the sets C_n are finite, the group $A[C_n]$ can be identified with the group of all maps

$$t: C_n \to A.$$

Let $Q'_n(A)$ be the free abelian group generated by the set $A[C_n]$, i.e.,

$$Q'_n(A) = \mathbb{Z}[A[C_n]].$$

Following Mac Lane [13], define for $i = 1, 2, ..., n$ the homomorphisms

$$R_i, S_i, P_i: Q'_n(A) \to Q'_{n-1}(A)$$

by

$$R_i = \mathbb{Z}[\bar{R}_i], \qquad S_i = \mathbb{Z}[\bar{S}_i], \qquad P_i = \mathbb{Z}[\bar{P}_i],$$

where

$$\bar{R}_i, \bar{S}_i, \bar{P}_i: A[C_n] \to A[C_{n-1}]$$

are homomorphisms defined for $e \in C_{n-1}$ and $t \in A[C_n]$ by

$$(\bar{R}_i t)(e) = t(0_i e); \qquad (\bar{S}_i t)(e) = t(1_i e);$$
$$(\bar{P}_i t)(e) = t(0_i e) + t(1_i e).$$

In [13], Mac Lane defines the boundary homomorphism

$$\partial: Q'_n(A) \to Q'_{n-1}(A)$$

by the equality

$$\partial = \sum_{i=1}^{n} (-1)^i (P_i - R_i - S_i).$$

A generator $t: C_n \to A$ of the group $Q'_n(A)$ is called a slab when $t(\ ) = 0$, for $n = 0$, and an i-slab, $i = 1, ..., n$, for $n > 0$, if either $t(0_i e) = 0$ for all $e \in C_{n-1}$ or $t(1_i e) = 0$ for all $e \in C_{n-1}$; t is called an i-diagonal if for all $(\varepsilon_1, ..., \varepsilon_n) \in C_n$ with $\varepsilon_i \neq \varepsilon_{i+1}$, we have

$$t(\varepsilon_1, ..., \varepsilon_n) = 0, \qquad n > 1,\ 1 \leqslant i \leqslant n-1.$$

Let $N_n(A)$ denote the subgroup of $Q'_n(A)$ generated by all the slabs and diagonals. It is easily seen that $\partial\partial = 0$ and $\partial(N_n(A)) \subset N_{n-1}(A)$; i.e., $Q'_*(A)$ is a complex, with the subcomplex $N_*(A)$. So we obtain the complex

$$Q_*(A) = Q'_*(A)/N_*(A).$$

Define an augmentation $\eta: Q_*(A) \to A$ by $\eta t = 0$ if t is a positive degree generator, and $\eta t = t(\ )$ for generators t of degree zero.

According to [6] and [13], the homology of the chain complex $Q_*(A)$ is isomorphic to the stable homology of Eilenberg–Mac Lane spaces corresponding to A [7]; i.e., $H_q Q_*(A) \cong H_{n+q}(K(A, n))$, $n > q \geqslant 0$, where $K(A, n)$ is the Eilenberg–Mac Lane space.

In the case where A is a left module over the ring R, Dixmier (private communication to Mac Lane) has defined a product

$$Q_*(R) \otimes Q_*(A) \to Q_*(A)$$

in the following way [13]. For $t \in R[C_m]$, $u \in A[C_n]$ define the map $tu: C_{n+m} \to A$ by

$$(tu)(\varepsilon_1, ..., \varepsilon_{n+m}) = t(\varepsilon_1, ..., \varepsilon_m)\, u(\varepsilon_{m+1}, ..., \varepsilon_{m+n}),$$

where $\varepsilon_i = 0$ or 1, $1 \leqslant i \leqslant m+n$.

This product equips $Q_*(R)$ with the structure of a differential graded (DG) ring, and $Q_*(A)$ with the structure of a left DG module over $Q_*(R)$.

The augmentation $\eta: Q_*(R) \to R$ is a morphism of DG rings, if R is given a grading concentrated in degree zero, and the trivial ∂. In particular, R becomes a $Q_*(R)$–$Q_*(R)$-bimodule.

Recall that when X and Y are left and right modules over a DG ring Λ, their two-sided bar construction $B(X, \Lambda, Y)$ is defined (see, e.g., [4]), with

$$B(X, \Lambda, Y) = \sum_{n \geqslant 0} X \otimes \Lambda^{\otimes n} \otimes Y.$$

Also $B(R, Q_*(R), R)$ evidently has the structure of an R–R-bimodule.

DEFINITION 2.1 [13]. For a ring R and an R–R-bimodule M, the Mac Lane cohomology of R with coefficients in M is defined by the equality

$$H^n(R; M) = H^n(\mathrm{Hom}_{R\text{-}R}(B(R, Q_*(R), R), M)).$$

It remains to state some auxiliary theorems for the proof of the main theorem of this section.

Recall the definition of the cross-effects [7] of a functor $T: \underline{A} \to \mathscr{B}$ from an additive category $\underline{A}$ to the abelian category $\mathscr{B}$. For objects $A, A_1, ..., A_n$ of $\underline{A}$ the cross-effects can be determined by the functorial decompositions

$$T_0 = T(0), \qquad T(A) \cong T_0 \oplus T_1(A),$$
$$T_1(A_1 \oplus A_2) \cong T_1(A_1) \oplus T_1(A_2) \oplus T_2(A_1, A_2),$$
$$T_2(A_1 \oplus A_2, A_3) \cong T_2(A_1, A_3) \oplus T_2(A_2, A_3) \oplus T_3(A_1, A_2, A_3), \cdots.$$

For $A_1 = \cdots = A_n = A$ the object $T_n(A_1, ..., A_n)$ is denoted by $T_n^d(A)$ for brevity.

For an arbitrary finite set S, the number of its elements is denoted by $|S|$ and the set of its subsets by $P(S)$.

The following proposition is contained in [7].

PROPOSITION 2.2. *Let T be an arbitrary functor from an additive category $\underline{A}$ to an abelian category $\mathscr{B}$. For a finite set S and an object A of $\underline{A}$, let $A[S]$ be the sum of S copies of A. Then there exists a natural isomorphism*

$$T(A[S]) \cong \bigoplus_{L \in P(S)} T^d_{|L|}(A).$$

COROLLARY 2.3. *Let $T: \underline{A} \to \mathscr{B}$ be as above. Given a finite set S and some of its subsets $S_1, ..., S_n$ define functors*

$$T_S: \underline{A} \to \mathscr{B}, \qquad T_{S/\{S_i\}}: \underline{A} \to \mathscr{B}$$

by the equalities

$$T_S(X) = T(X[S]); \qquad T_{S/\{S_i\}} = \mathrm{Coker}\left(\bigoplus_{i=1}^{n} T_{S_i} \to T_S\right),$$

for $X \in |\underline{A}|$ an object of A, while $T_{S_i} \to T_S$ is induced by the inclusion $S_i \subsetneq S$, $i = 1, ..., n$. Then the natural projection $T_S \to T_{S/\{S_i\}}$ has a section.

Proof. By Proposition 2.2 we have natural isomorphisms

$$T_S(X) \cong \bigoplus_{L \in P(S)} T^d_{|L|}(X), \qquad T_{S_i}(X) \cong \bigoplus_{L \in P(S_i)} T^d_{|L|}(X),$$

for $X \in |\underline{A}|$, $i = 1, ..., n$. Since the isomorphisms of Proposition 2.2 are natural in S, we obtain natural isomorphisms

$$T_{S/\{S_i\}}(X) = \bigoplus_{L \in \Omega} T^d_{|L|}(X), \tag{2.4}$$

where Ω is the set of those subsets L of S which are not contained in any of the S_i. Consequently $T_{S/\{S_i\}}$ is a direct summand of T_S.

In the rest of the paper we repeatedly use the following direct consequence of the Yoneda Lemma [15].

PROPOSITION 2.5. *Consider an arbitrary category $\underline{C}$, an object $c \in |\underline{C}|$, and an associative ring with unit R. Denote by $\mathscr{F}$ the category of all functors from $\underline{C}$ to R-$\underline{\mathrm{mod}}$. Then the functor $R[\underline{C}(c, -)]: \underline{C} \to R$-$\underline{\mathrm{mod}}$ is a projective object of $\mathscr{F}$, and, for any functor $T: \underline{C} \to R$-$\underline{\mathrm{mod}}$, there is a natural bijection*

$$\mathrm{Hom}_{\mathscr{F}}(R[\underline{C}(c, -)], T) \approx T(c).$$

Moreover any projective object of $\mathscr{F}$ is a retract of a sum of functors of type $R[\underline{C}(c, -)]$.

PROPOSITION 2.6. *For any ring R and a natural number n, denote by $Q_n, Q'_n: R$-$\underline{\mathrm{mod}} \to \mathscr{Ab}$ the functors assigning to a left R-module X the abelian groups $Q_n(X)$ and $Q'_n(X)$, respectively. Then for any small full subcategory $\underline{A}$ of R-$\underline{\mathrm{mod}}$, containing the free module R^{2^n}, restrictions of Q_n and Q'_n to $\underline{A}$ are projective objects of the category of all functors from $\underline{A}$ to $\mathscr{Ab}$.*

Proof. Recall that by definition, for $X \in |R\text{-}\underline{\mathrm{mod}}|$,

$$Q_n'(X) = \mathbb{Z}[X[C_n]];$$

so since $X[C_n] \cong \mathrm{Hom}_R(R^{2^n}, X)$, projectivity of $Q_n'|_{\underline{A}}$ follows from Proposition 2.5. Hence it is clear that the proposition will follow if one shows that the projection $Q_n' \to Q_n$ has a section. To this end, put

$$\begin{aligned} S_i &= \{(\varepsilon_1, ..., \varepsilon_n) \in C_n, \varepsilon_i = 0\}, & 1 \leqslant i \leqslant n, \\ L_j &= \{(\varepsilon_1, ..., \varepsilon_n) \in C_n, \varepsilon_j = 1\}, & 1 \leqslant j \leqslant n, \\ D_k &= \{(\varepsilon_1, ..., \varepsilon_n) \in C_n, \varepsilon_k = \varepsilon_{k+1}\}, & 1 \leqslant k < n. \end{aligned}$$

Denote by T: R-$\underline{\mathrm{mod}} \to \mathscr{A}\mathscr{b}$ the functor determined by

$$TX = \mathbb{Z}[X].$$

Clearly $Q_n' = T_{C_n}$; by definition of Q_n,

$$\begin{aligned} Q_n &= T_{C_n/\{S_i, L_j, D_k\}} & \text{for } n > 1 \\ Q_n &= T_{C_n/\{S_i, L_j\}} & \text{for } n = 1 \\ Q_n &= T_{C_n/\{0\}} & \text{for } n = 0, \end{aligned} \tag{2.7}$$

and the section exists by Proposition 2.5.

In the following, restrictions of functors from R-$\underline{\mathrm{mod}}$ to full subcategories have identical notations, if no confusion is caused.

PROPOSITION 2.8. *Let $\underline{A}$ be a small full subcategory of the category of left modules over the ring R containing the free modules R^i for $0 \leqslant i \leqslant 2^n$, and let F: R-$\underline{\mathrm{mod}} \to \mathscr{A}\mathscr{b}$ be an additive functor. Then*

$$\mathrm{Hom}_{\mathscr{A}\mathscr{b}^{\underline{A}}}(Q_i, F) = \begin{cases} 0, & 0 < i \leqslant n, \\ F(R), & i = 0. \end{cases}$$

Proof. For a positive integer k, let $[k]$ denote the set $\{1, ..., k\}$. By virtue of (2.4), for any functor F: R-$\underline{\mathrm{mod}} \to \mathscr{A}\mathscr{b}$ there is an isomorphism

$$F_k^d \cong F_{[k]/\{[k]-\{1\}, ..., [k]-\{k\}\}}, \qquad k \geqslant 0.$$

If one uses T to denote the same functor here as in the proof of Proposition 2.6, one has

$$\mathrm{Hom}_{\mathscr{A}\mathscr{b}^{\underline{A}}}(T_k^d, F) \cong F_k^d(R), \qquad 0 \leqslant k \leqslant 2^n. \tag{2.9}$$

Indeed, as we said in (2.3), there is an exact sequence

$$0 \leftarrow T_k^d \leftarrow T_{[k]} \leftarrow \bigoplus_{i=1}^{k} T_{[k]-\{i\}}$$

which gives

$$\mathrm{Hom}_{\mathscr{A}b^{\underline{A}}}(T_k^d, F) \cong \mathrm{Ker}(\mathrm{Hom}_{\mathscr{A}b^{\underline{A}}}(T_{[k]}, F) \to \bigoplus_{i=1}^{k} \mathrm{Hom}_{\mathscr{A}b^{\underline{A}}}(T_{[k]-\{i\}}, F)).$$

But for any finite set S one has

$$T_S = \mathbb{Z}[\mathrm{Hom}_R(R[S], -)],$$

and for $0 \leqslant k \leqslant 2^n$, R^k, $R^{k-1} \in |\underline{A}|$; hence by Proposition 2.5 one has

$$\mathrm{Hom}_{\mathscr{A}b^{\underline{A}}}(T_k^d, F) \cong \mathrm{Ker}(F(R^k) \to \bigoplus_{i=1}^{k} F(R^{k-1})) \cong F_k^d(R).$$

Using (2.4) and (2.7), we obtain

$$Q_0 = T_1, \qquad Q_1 = T_2^d, \qquad Q_n = \bigoplus_{L \in V_n} T_{|L|}^d, n > 1,$$

where

$$V_n = P(C_n) - \left(\bigcup_{i=1}^{n} (P(S_i) \cup P(L_i)) \cup \bigcup_{k=1}^{n-1} P(D_k) \right).$$

For any functor F: R-$\underline{\mathrm{mod}} \to \mathscr{A}b$, (2.9) implies that

$$\mathrm{Hom}_{\mathscr{A}b^{\underline{A}}}(Q_0, F) = F_1(R),$$
$$\mathrm{Hom}_{\mathscr{A}b^{\underline{A}}}(Q_1, F) = F_2^d(R),$$
$$\mathrm{Hom}_{\mathscr{A}b^{\underline{A}}}(Q_n, F) = \bigoplus_{L \in V_n} F_{|L|}^d(R), \qquad n > 1.$$

For additive functors, these equalities imply our proposition, since for $n \geqslant 2$, any $L \subset V_n$ satisfies $|L| \geqslant 2$.

We also need

PROPOSITION 2.10 [13, Theorem 6]. *The morphism*

$$B(R, Q_*(R), Q_*(A)) \to B(R, R, A) \to A$$

induced by the augmentations $Q_*(R) \to R$ and $Q_*(A) \to A$ is a quasi-

isomorphism; i.e., it induces isomorphisms in homology, for any ring R and R-module A.

Remark. In [13], the DG-module $B(R, Q_*(R); Q_*(A))$ is denoted by $M_R(A)$.

THEOREM A. *Let $\underline{A}$ be a small full additive subcategory of the category of left modules over a ring R, containing the module R. Let I, T: $\underline{A} \to R\text{-}\underline{\text{mod}}$ be the inclusion I and an arbitrary additive functor T. Then there is an isomorphism*

$$H^*(R; T(R)) \cong \operatorname{Ext}^*(I, T),$$

where the Mac Lane cohomology groups of R with coefficients in the obvious R–R-bimodule T(R) are on the left, while the Ext *groups on the right are taken in the category of all functors from $\underline{A}$ to R-modules.*

Proof. According to Proposition 2.10 we have a resolution of I in the category of all functors from $\underline{A}$ to R-modules of the form

$$B(R, Q_*(R), Q_*(-)) \to I. \tag{2.11}$$

We claim that this is a projective resolution. Indeed, all the abelian groups $Q_i(R)$, $i \geqslant 0$, are free, while $Q_n(-)$ are projective objects of the category of all functors from $\underline{A}$ to abelian groups (by Proposition 2.6). Hence for all $i_1, i_2, ..., i_k$, $n \in \mathbb{N}$, the functor

$$Q_{i_1}(R) \otimes \cdots \otimes Q_{i_k}(R) \otimes Q_n(-)$$

is a projective object of the category of all functors from $\underline{A}$ to abelian groups, and hence

$$R \otimes Q_{i_1}(R) \otimes \cdots \otimes Q_{i_k}(R) \otimes Q_n(-) \tag{2.12}$$

is also a projective object in all functors from $\underline{A}$ to R-modules. But every component of the complex $B(R, Q_*(R), Q_*(-))$ is precisely a sum of functors of type (2.12), so (2.11) really is a projective resolution.

By virtue of Proposition 2.8, the group of natural transformations from

$$Q_{i_1}(R) \otimes \cdots \otimes Q_{i_k}(R) \otimes Q_n(-)$$

to the additive functor $T: A \to R\text{-}\underline{\text{mod}}$, if we view both of them as functors to abelian groups, is trivial for $n > 0$ and equal to

$$\operatorname{Hom}_R(Q_{i_1}(R) \otimes \cdots \otimes Q_{i_k}(R) \otimes R, T(R))$$

for $n = 0$. Hence in the category of functors from $\underline{A}$ to R-modules, the group of transformations from $R \otimes Q_{i_1}(R) \otimes \cdots \otimes Q_{i_k}(R) \otimes Q_n(-)$ to T is trivial for $n > 0$ and coincides with

$$\mathrm{Hom}_{R\text{-}R}(R \otimes Q_{i_1}(R) \otimes \cdots \otimes Q_{i_k}(R) \otimes R, T(R))$$

for $n = 0$.

Summing up, we obtain

$$\begin{aligned}\mathrm{Ext}^*(I, T) &= H^*(\mathrm{Hom}(B(R, Q_*(R), Q_*(-)), T)) \\ &= H^*(\mathrm{Hom}_{R\text{-}R}(B(R, Q_*(R), R), T(R)) = H^*(R; M),\end{aligned}$$

where Ext and Hom are taken in the category of all functors from $\underline{A}$ to R-modules.

Recall that in Section 1 we defined the cohomology $H^*(R; T)$ of the ring R with coefficients in an arbitrary functor T from the category $\underline{M}_R$ of finitely generated free left R-modules to the category of all R-modules R-$\underline{\text{mod}}$ by the equality

$$H^*(R; T) = \mathrm{Ext}^*_{\mathscr{F}(R)}(I, T),$$

where $\mathscr{F}(R)$ is the category of all functors from $\underline{M}_R$ to R-$\underline{\text{mod}}$, and $I: \underline{M}_R \to R$-$\underline{\text{mod}}$ is the inclusion. In that section we also identified the category of R–R-bimodules with that full subcategory of $\mathscr{F}(R)$ consisting of additive functors; clearly, the bimodule $T(R)$ corresponds to the additive functor T in this way. From this point of view, Theorem A states that by restricting our cohomology

$$H^*(R; -): \mathscr{F}(R) \to \mathscr{A}b$$

to the category of R–R-bimodules, we obtain the Mac Lane cohomology.

Cohomology with coefficients in functors has good stability properties. Namely, one has

PROPOSITION 2.13. *For natural numbers r, n, let $\underline{M}_R(r)$ denote the full subcategory of $\underline{M}_R$ with objects the R-modules $0, R, R^2, ..., R^r$, and let $\mathscr{F}(R, r)$ be the category of all functors from $\underline{M}_R(r)$ to R-$\underline{\text{mod}}$. Let $I, T: \underline{M}_R \to R$-$\underline{\text{mod}}$ be the inclusion and an arbitrary functor, respectively. Then, for $r \geqslant 2^n$, the homomorphism*

$$\mathrm{Ext}^n_{\mathscr{F}(R)}(I, T) \to \mathrm{Ext}^n_{\mathscr{F}(R,r)}(I_r, T_r),$$

induced by the exact functor $(\ )_r: \mathscr{F}(R) \to \mathscr{F}(R, r)$ that assigns to $T: \underline{M}_R \to R$-$\underline{\text{mod}}$ its composition with the embedding $\underline{M}_R(r) \subsetneq \underline{M}_R$, is an isomorphism.

Proof. As we have seen in the proof of Theorem A,

$$B(R, Q_*(R), Q_*(-)) \to I$$

is a projective resolution of I in the category $\mathscr{F}(R)$. Consider the restriction of this resolution to the category $\underline{M}_R(r)$:

$$B(R, Q_*(R), Q_*(-))_r \to I_r.$$

Proposition 2.6 implies that all components of $B(R, Q_*(R), Q_*(-))_r$ up to n are projective objects of $\mathscr{F}(R, r)$. By the isomorphism (2.9) there are isomorphisms (2.9) there are isomorphisms

$$\begin{aligned} &H^i(\mathrm{Hom}_{\mathscr{F}(R)}(B(R, Q_*(R), Q_*(-)), T) \\ &\quad \cong H^i(\mathrm{Hom}_{\mathscr{F}(R,r)}(B(R, Q_*(R), Q_*(-))_r, T_r)), \qquad i \leqslant n. \end{aligned}$$

Hence

$$\mathrm{Ext}^i_{\mathscr{F}(R)}(I, T) \cong \mathrm{Ext}^i_{\mathscr{F}(R,r)}(I_r, T_r), \qquad \text{for} \quad i \leqslant n.$$

It follows from the definition that our cohomology

$$H^n(R; -) \colon \mathscr{F}(R) \to \mathscr{A}b, \qquad n \geqslant 0$$

vanishes on injective objects for $n > 0$. We now describe another sufficiently large class of objects of $\mathscr{F}(R)$, where these cohomologies vanish. For that purpose consider

DEFINITION 2.14. A functor F from an additive category $\underline{A}$ to another additive category $\underline{B}$ is called diagonalizable if it can be represented in the form $F = T \circ \Delta$, where $\Delta \colon \underline{A} \to \underline{A} \times \underline{A}$ is the diagonal and $T \colon \underline{A} \times \underline{A} \to \underline{B}$ is a bifunctor satisfying $T(0, X) = 0 = T(X, 0)$ for every object X from $\underline{A}$.

The following proposition was proved in [17].

PROPOSITION 2.15. *Let $\underline{A}$ be a small additive category, and let $\mathscr{F}$ be the category of all functors from $\underline{A}$ to the category of modules over a ring R. Consider the functors $U, F \colon \underline{A} \to R$-$\underline{\mathrm{mod}}$ with U additive and F diagonalizable. Then*

$$\mathrm{Ext}^*_{\mathscr{F}}(U, F) = 0 = \mathrm{Ext}^*_{\mathscr{F}}(F, U).$$

COROLLARY 2.16. *If F is a diagonalizable functor from the category of free left R-modules of finite type to the category of left R-modules, then*

$$H^*(R; F) = 0.$$

With the aid of Corollary 2.16 we apply the methods of [23] to calculate cohomology groups in low dimensions of commutative rings, with coefficients in some quadratic functors:

Let R be a commutative ring. For an R-module M, $S^2_R M$ denotes the symmetric square of M and $\Lambda^2_R M$ the exterior square of M. By definition,

$$S^2_R M = M \otimes_R M/U(M), \qquad \Lambda^2_R M = M \otimes_R M/V(M),$$

where $U(M)$ and $V(M)$ are submodules of $M \otimes_R M$ generated by elements of type $m \otimes n - n \otimes m$ and $m \otimes m$, respectively, for $m, n \in M$. The image of the element $m \otimes n$ in the quotient module $S^2_R M$ is denoted by $m \vee n$ and that in the module $\Lambda^2_R M$ by $m \wedge n$.

Define the homomorphisms

$$\alpha\colon \Lambda^2_R M \to M \otimes_R M, \qquad \beta\colon S^2_R M \to V(M)$$

by

$$\alpha(m \wedge n) = m \otimes n - n \otimes m, \qquad \beta(m \vee n) = m \otimes n + n \otimes m.$$

Denote by ${}_2\bar{R}$ and $\overline{R/2R}$ the following R–R-bimodules: As left R-modules they coincide with

$${}_2R = \{r \in R,\ 2r = 0\}$$

and $R/2R$, respectively, while the right actions of R on them are defined by

$$x \cdot r = r^2 x, \qquad r \in R,\ x \in {}_2R, \text{ or } x \in R/2R.$$

It is proved in [23] that for a flat R-module M the sequences

$$0 \longrightarrow \Lambda^2_R M \xrightarrow{\alpha} M \otimes_R M \longrightarrow S^2_R M \longrightarrow 0,$$
$$0 \longrightarrow V(M) \longrightarrow M \otimes_R M \longrightarrow \Lambda^2_R M \longrightarrow 0,$$
$$0 \longrightarrow {}_2\bar{R} \otimes_R M \longrightarrow S^2_R M \xrightarrow[\beta]{} V(M) \longrightarrow \overline{R/2R} \otimes_R M \longrightarrow 0$$

are exact.

By varying M over the category $\underline{M}_R$ we obtain exact sequences in the category $\mathscr{F}(R)$:

$$\begin{gathered} 0 \longrightarrow \Lambda^2_R \longrightarrow \otimes^2_R \longrightarrow S^2_R \longrightarrow 0 \\ 0 \longrightarrow V \longrightarrow \otimes^2_R \longrightarrow \Lambda^2_R \longrightarrow 0 \\ 0 \longrightarrow ({}_2\bar{R} \otimes_R -) \longrightarrow S^2_R \xrightarrow[\beta]{} V \longrightarrow (\overline{R/2R} \otimes_R -) \longrightarrow 0. \end{gathered} \tag{2.17}$$

Since $\otimes_R^2$ is a diagonalizable functor, we have $H^*(R; \otimes_R^2)=0$. Hence (2.17) yields isomorphisms

$$\begin{aligned} &H^0(R; V)=H^1(R; V)=0; \qquad H^0(R; \Lambda_R^2)=0, \\ &H^{n+2}(R; V)=H^{n+1}(R; \Lambda_R^2)=H^n(R; S_R^2), \qquad n\geq 0, \end{aligned} \tag{2.18}$$

and long exact sequences

$$0\to H^0(R; {}_2\bar{R})\to H^0(R; S_R^2)\to H^0(R; \mathrm{Im}(\beta))\to H^1(R; {}_2\bar{R})\to \cdots, \tag{2.19}$$

$$0\to H^0(R; \mathrm{Im}(\beta))\to H^0(R; V)\to H^0(R; \overline{R/2R})\to H^1(R; \mathrm{Im}(\beta))\to \cdots. \tag{2.20}$$

By combining (2.20) with (2.18) we obtain

$$H^0(R; \mathrm{Im}(\beta))=0, \qquad H^1(R; \mathrm{Im}(\beta))=H^0(R; \overline{R/2R}).$$

Hence (2.19) implies an isomorphism $H^0(R; S_R^2)\cong H^0(R; {}_2\bar{R})$ and an exact sequence

$$0\to H^1(R; {}_2\bar{R})\to H^1(R; S_R^2)\to H^0(R; \overline{R/2R})\to H^2(R; {}_2\bar{R}). \tag{2.20a}$$

If we also assume that ${}_2R=0$, then (2.17) and (2.18) give

$$H^0(R; S_R^2)=0, \qquad H^1(R; S_R^2)\cong H^0(R; \overline{R/2R})$$

and an exact sequence

$$0\to H^1(R; \overline{R/2R})\to H^2(R; S_R^2)\to H^2(R; V).$$

By (2.18), $H^2(R; V)=H^0(R; S_R^2)=0$. Hence $H^2(R; S_R^2)\cong H^1(R; \overline{R/2R})$. The above considerations together prove

PROPOSITION 2.21. *Let R be a commutative ring. Then*

$$\begin{aligned} &H^0(R; \Lambda_R^2)=0, \\ &H^1(R; \Lambda_R^2)\cong H^0(R; S_R^2)\cong H^0(R; {}_2\bar{R}), \\ &H^{n+1}(R; \Lambda_R^2)\cong H^n(R; S_R^2), \qquad n\geq 0. \end{aligned}$$

Moreover, there is an exact sequence (2.20a). *If, in addition, ${}_2R=0$, then*

$$H^2(R; S_R^2)\cong H^1(R; \overline{R/2R}).$$

In particular, $H^2(\mathbb{Z}; \Lambda^2)=\mathbb{Z}/2\mathbb{Z}$, $H^2(\mathbb{Z}; S^2)=0$.

3. Connections with the Cohomology of Small Categories

In this section we prove Theorem B and some auxiliary propositions which are needed in Section 4.

First let us recall basic facts about the Hochschild–Mitchell cohomology theory of small categories [3, 16].

Suppose we are given a small category $\underset{\sim}{C}$ and a bifunctor $D: \underset{\sim}{C}^{op} \times \underset{\sim}{C} \to \mathscr{Ab}$. For a morphism $\alpha: A \to B$ in $\underset{\sim}{C}$, objects X, Y of $\underset{\sim}{C}$, and elements

$$a \in D(X, A), \qquad b \in D(B, Y),$$

images of these elements under the homomorphisms

$$D(1_X, \alpha): D(X, A) \to D(X, B), \qquad D(\alpha, 1_Y): D(B, Y) \to D(A, Y)$$

are denoted by $\alpha_* a$ and $\alpha_* b$, respectively.

$N_* \underset{\sim}{C}$ denotes the nerve of C_* [20]; it is that simplicial set whose n-simplices are diagrams

$$A_0 \xleftarrow{\lambda_1} A_1 \longleftarrow \cdots \xleftarrow{\lambda_n} A_n, \qquad n \geqslant 0,$$

which in the sequel will be denoted simply by $(\lambda_1, ..., \lambda_n)$. Moreover, in this situation $D(\lambda_1, ..., \lambda_n)$, for a bifunctor D, denotes the group $D(A_n, A_0)$.

Definition 3.1 [3]. The Hochschild–Mitchell cohomology of the category $\underset{\sim}{C}$ with coefficients in the bifunctor $D: \underset{\sim}{C}^{op} \times \underset{\sim}{C} \to \mathscr{Ab}$ is the cohomology of the cochain complex $F^*(\underset{\sim}{C}; D)$, whose n-dimensional cochains are elements of the group

$$F^n(\underset{\sim}{C}; D) = \prod_{(\lambda_1, ..., \lambda_n) \in N_n \underset{\sim}{C}} D(\lambda_1, ..., \lambda_n)$$

for $n > 0$ and of

$$F^0(\underset{\sim}{C}; D) = \prod_{A \in |\underset{\sim}{C}|} D(A, A)$$

for $n = 0$, while the coboundary homomorphism $\delta: F^{n-1} \to F^n$ is defined by

$$\begin{aligned}(\delta f)&(\lambda_1, ..., \lambda_n)\\ &= \lambda_{1*} f(\lambda_2, ..., \lambda_n) + \sum_{i=1}^{n-1} (-1)^i f(\lambda_1, ..., \lambda_i \lambda_{i+1}, ..., \lambda_n)\\ &\quad + (-1)^n \lambda_n^* f(\lambda_1, ..., \lambda_{n-1})\end{aligned}$$

for $n \geqslant 1$ and by

$$(\delta f)(\lambda) = \lambda_* f(A) - \lambda^* f(B)$$

for $n = 0$, $(\lambda: A \to B) \in N_1 \underline{C}$.

By the very definition, the group $H^0(\underline{C}; D)$ coincides with the end of the bifunctor D [15]. In particular, we have

PROPOSITION 3.2. *Let $\mathscr{F}$ be the category of all functors from $\underline{C}$ to the category R-$\underline{\text{mod}}$ of left modules over the ring R, and, for $U, T \in |\mathscr{F}|$, let the bifunctor $\mathscr{H}om_R(U, T)$: $\underline{C}^{op} \times \underline{C} \to \mathscr{A}b$ be defined by*

$$(\mathscr{H}om_R(U, T))(X, Y) = \mathrm{Hom}_R(UX, TY), \qquad X, Y \in |\underline{C}|.$$

Then $H^0(\underline{C}; \mathscr{H}om_R(U, T)) \cong \mathrm{Hom}_{\mathscr{F}}(U, T)$.

Cohomology groups of small categories with coefficients in bifunctors are known to constitute a universal exact connected sequence of functors [3, 16], so they are derived functors of ends.

We also need the connection between second cohomology and linear extensions of categories [3].

DEFINITION 3.3 [3]. We say that

$$D + \longrightarrow \underline{E} \xrightarrow{p} \underline{C} \tag{3.3a}$$

is a linear extension of the category $\underline{C}$ by the bifunctor D: $\underline{C}^{op} \times \underline{C} \to \mathscr{A}b$; if $\underline{E}$ is a category with the same objects as $\underline{C}$, p is a functor which is identity on objects and surjective on morphisms; and moreover, for all objects $A, B \in |\underline{C}|$, an effective action of the group $D(B, A)$ on the set $\underline{E}(B, A)$ (denoted by $\lambda_0 + a$ for $\lambda_0: B \to A$ in $\underline{E}$ and $a \in D(B, A)$) satisfying the following is given:

(1) For $\lambda_0, \lambda_1 \in \underline{E}(B, A)$, $p(\lambda_0) = p(\lambda_1)$ iff there is an $a \in D(B, A)$ with $\lambda_1 = \lambda_0 + a$.

(2) For $a \in D(B, A)$, $b \in D(C, B)$, and morphisms $C \to^{\mu_0} B \to^{\lambda_0} A$ in $\underline{E}$, with $p(\lambda_0) = \lambda$, $p(\mu_0) = \mu$, one has

$$(\lambda_0 + a)(\mu_0 + b) = \lambda_0 \mu_0 + \lambda_* b + \mu^* a.$$

The class of all linear extensions of $\underline{C}$ by D has a naturally defined equivalence relation; the set of equivalence classes is denoted by $M(\underline{C}; D)$, and [3] constructs a natural bijection $M(\underline{C}; D) \approx H^2(\underline{C}; D)$.

PROPOSITION 3.4. *Given a linear extension* (3.3a) *of the category* $\underline{C}$ *by the bifunctor* D, *let*

$$A \xrightarrow{\alpha} C \xleftarrow{\beta} B$$

be a diagram in $\underline{E}$ *whose image under* p *is a coproduct diagram in* $\underline{C}$. *Then the original diagram is a coproduct diagram in* $\underline{E}$ *iff for every* $X \in |\underline{C}|$, *with* $\underline{C}(A, X) \times \underline{C}(B, X) \neq \varnothing$, *the homomorphism induced by* α *and* β,

$$D(C, X) \to D(A, X) \oplus D(B, X),$$

is an isomorphism.

Proof. By the given conditions in $\underline{C}$, α and β induce

$$\underline{C}(C, X) \to \underline{C}(A, X) \times \underline{C}(B, X),$$

a bijection. We need to determine the conditions under which

$$\underline{E}(C, X) \to \underline{E}(A, X) \times \underline{E}(B, X)$$

will be bijective too. Hence the proposition follows from the following easy lemma:

LEMMA 3.5. *Let* $\alpha\colon G_1 \to G_2$ *be a homomorphism of groups and let* X_i *be nonempty sets with effective actions of* G_i, $i = 1, 2$. *Suppose we are given an* α-*equivariant map* $\beta\colon X_1 \to X_2$ *that induces a bijection between sets of orbits* $\beta^*\colon X_1/G_1 \xrightarrow{\approx} X_2/G_2$. *Then* β *is a bijection iff* α *is an isomorphism.*

For a functor $p\colon \underline{C}' \to \underline{C}$ and a bifunctor D on $\underline{C}$, composition with p determines a bifunctor on $\underline{C}'$ that will again be denoted by D. So p induces a morphism of the complexes

$$p^*\colon F^*(\underline{C}; D) \to F^*(\underline{C}'; D) \tag{3.6}$$

of D and, so, homomorphisms in cohomology

$$p^*\colon H^*(\underline{C}; D) \to H^*(\underline{C}'; D).$$

Before we formulate the following proposition, recall that an augmented simplicial object $\varepsilon\colon X_* \to X_{-1}$ in some category $\mathbb{K}$ consists of a simplicial object $X_* = (X_n, S_i^n, d_i^n)$, $n \geqslant 0$, $0 \leqslant i \leqslant n$, in $\mathbb{K}$; an object X_{-1} of $\mathbb{K}$; and a morphism $\varepsilon\colon X_0 \to X_{-1}$ in $\mathbb{K}$ with $\varepsilon d_0^1 = \varepsilon d_1^1$. Such an object $\varepsilon\colon X_* \to X_{-1}$ is called contractible if there are morphisms $h_n\colon X_n \to X_{n+1}$, $n \geqslant -1$, satisfying $\varepsilon h_{-1} = 1_{X_{-1}}$, $d_{n+1}^{n+1} h_n = 1$, $d_0^1 h_0 = h_{-1}\varepsilon$, $d_i^{n+1} h_n = h_{n-1} d_{i-1}^n$, $0 \leqslant i \leqslant n \geqslant 1$.

PROPOSITION 3.7. *Let $\varepsilon\colon \underline{C}_* \to \underline{C}_{-1}$ be an augmented simplicial category, such that all the $\underline{C}_n$'s, $n \geqslant -1$, have the same set of objects and all the structure functors involved are the identity on objects. Suppose also that for any $A, B \in |\underline{C}_{-1}|$, the augmented simplicial set*

$$\varepsilon(A, B)\colon \underline{C}_*(A, B) \to \underline{C}_{-1}(A, B) \tag{3.8}$$

is contractible. Then for any bifunctor $D\colon \underline{C}^{op}_{-1} \times \underline{C} \to \mathscr{Ab}$, there is a spectral sequence with

$$E_1^{pq} = H^q(\underline{C}_p; D) \Rightarrow H^{p+q}(\underline{C}_{-1}; D).$$

Proof. Denote by h the contraction of the augmented simplicial set (3.8). For every $n \geqslant 0$, applying the functor $F^n(-; D)$ of Definition 3.1 componentwise to $\varepsilon\colon \underline{C}_* \to \underline{C}_{-1}$ gives the augmented cosimplicial abelian group

$$\varepsilon^n\colon F^n(\underline{C}_{-1}; D) \to F^n(\underline{C}_*; D), \qquad n \geqslant 0,$$

which also has a contraction $\bar{h}$ given by

$$(\bar{h}f)(\lambda_1, ..., \lambda_n) = f(h\lambda_1, ..., h\lambda_n)$$

for $f \in F^n(\underline{C}_m; D)$, $m \geqslant 0$, $(\lambda_1, ..., \lambda_n) \in N_n\underline{C}_{m-1}$.

By varying n we obtain an augmented cosimplicial object in the category of cochain complexes:

$$\varepsilon^*\colon F^*(\underline{C}_{-1}; D) \to F^*(\underline{C}_*; D).$$

The cosimplicial cochain complex $F^*(\underline{C}_*; D)$ can be converted to a bicomplex, whose total complex is denoted $\operatorname{Tot} F^*(\underline{C}_*; D)$. Spectral sequences associated with a bicomplex [5, 14] have in our case the form

$$'E_1^{pq} = H^q(\underline{C}_p; D) \Rightarrow H^{p+q}(\operatorname{Tot} F^*(\underline{C}_*; D)),$$
$$''E_1^{pq} = H^q(F^p(\underline{C}_*; D)) \Rightarrow H^{p+q}(\operatorname{Tot} F^*(\underline{C}_*; D)).$$

Since the augmented cosimplicial abelian groups ε^n are contractible, we get $''E_1^{pq} = 0$ for $q > 0$ and $''E_1^{p0} = F^p(\underline{C}_{-1}; D)$. Hence the second spectral sequence degenerates to yield isomorphisms

$$H^p(\operatorname{Tot}(F^*(\underline{C}_*; D))) \cong {''E_2^{p0}} \cong H^p(\underline{C}_{-1}; D).$$

Substituting this in the first spectral sequence gives the proposition.

The proof of theorem B uses the following lemmas.

LEMMA 3.9. *Given a functor T from the category $\underline{C}$ to the category R-$\underline{\text{mod}}$ of modules over the ring R, define the bifunctor*

$$D\colon \underline{C}^{op} \times \underline{C} \to \mathscr{A}b$$

from an object A of $\underline{C}$ by the equalities

$$D(X, Y) = (TY)^{\underline{C}(A, X)} \qquad \text{for} \quad X, Y \in |\underline{C}|.$$

Then

$$H^n(\underline{C}; D) = 0 \qquad \text{for} \quad n > 0, \qquad H^0(\underline{C}; D) = T(A).$$

Proof. For $a \in T(A)$ define the function $f_a \in F^0(\underline{C}; D)$ by

$$(f_a(X))(\alpha\colon A \to X) = T(\alpha)(a) \qquad \text{for} \quad X \in |\underline{C}|, \alpha \in \underline{C}(A, X).$$

Then the assignment $a \mapsto f_a$ defines a homomorphism $\varepsilon\colon T(A) \to F^0(\underline{C}; D)$, which gives an augmented cochain complex $\varepsilon\colon T(A) \to F^*(\underline{C}; D)$. To show that it is contractible, define the homomorphism $h\colon F^0(\underline{C}; D) \to T(A)$ by

$$h(f) = (f(A))(1_A) \qquad \text{for} \quad f \in F^0(\underline{C}; D).$$

To define $h^n\colon F^{n+1}(\underline{C}; D) \to F^n(\underline{C}; D)$ for $n \geqslant 0$, note first that for any $f \in F^{n+1}(\underline{C}; D)$ and any $(X_0 \xleftarrow{\lambda_1} X_1 \leftarrow \cdots \xleftarrow{\lambda_n} X_n) \in N_n\underline{C}$, each morphism $\lambda\colon A \to X_n$ determines an element

$$(X_0 \xleftarrow{\lambda_1} \cdots \xleftarrow{\lambda_n} X_n \xleftarrow{\lambda} A) \in N_{n+1}\underline{C}$$

so that a map

$$f(\lambda_1, \ldots, \lambda_n, \lambda) \in D(A, X_0) = TX_0^{\underline{C}(A, A)}$$

is determined. Hence one may define h^n by the equality

$$(h^*f)(\lambda_1, \ldots, \lambda_n)(\lambda) = f(\lambda_1, \ldots, \lambda_n, \lambda)(1_A)$$
$$\text{for } f \in F^{n+1}(\underline{C}; D),\ (\lambda_1, \ldots, \lambda_n) \in N_n\underline{C},\ \lambda \in \underline{C}(A, X_n).$$

Direct calculation now shows that the sequence $h, h_1, h_2, \ldots$ determines a contraction for the augmented cochain complex $\varepsilon\colon T(A) \to F^*(\underline{C}; D)$. Hence

$$H^n(\underline{C}; D) = 0 \text{ when } n > 0, \qquad H^0(\underline{C}; D) = T(A).$$

LEMMA 3.10. *For covariant functors P, T from a category $\underline{C}$ to modules over a ring R, let*

$$\mathscr{H}om_R(P, T)\colon \underline{C}^{op} \times \underline{C} \to \mathscr{A}b$$

be the bifunctor assigning the group $\mathcal{H}om_R(PX, TY)$ *to the pair* $(X, Y) \in |\underline{C}^{op} \times \underline{C}|$. *If* P *is a projective object of the category* $\mathcal{F}$ *of all functors from* $\underline{C}$ *to* R-mod, *then*

$$H^0(\underline{C}; \mathcal{H}om_R(P, T)) = \mathrm{Hom}_{\mathcal{F}}(P, T),$$

$$H^n(\underline{C}; \mathcal{H}om_R(P, T)) = 0, \qquad n > 0.$$

Proof. In dimension 0 our assertion is a corollary of Proposition 3.2. For positive dimensions, consider the family of "representable" functors h_A, $A \in |\underline{C}|$; h_A is the functor from $\underline{C}$ to R-mod assigning the free R-module generated by the set $\underline{C}(A, X)$ to the object X of $\underline{C}$. It is know that the h_A's constitute a family of small projective generators for $\mathcal{F}$ (cf. Proposition 2.5). Hence we can restrict ourselves to the case $P = h_A$ for some $A \in |\underline{C}|$. But then the bifunctor $\mathcal{H}om_R(h_A, T)$ coincides with the bifunctor from the previous lemma, so that our assertion follows from Lemma 3.9.

We now prove the main result of this section.

THEOREM B. *For any functors* U, T *from a category* $\underline{C}$ *to modules over a ring* R, *there is a spectral sequence of type*

$$E_2^{pq} = H^p(\underline{C}; \mathcal{E}xt_R^q(U, T)) \Rightarrow \mathrm{Ext}_{\mathcal{F}}^{p+q}(U, T),$$

where $\mathcal{E}xt_R^q(U, T)$ *is the bifunctor* $\underline{C}^{op} \times \underline{C} \to \mathcal{A}b$ *which assigns the group* $\mathrm{Ext}_R^q(UX, TY)$ *to* X, Y *from* $\underline{C}$, *while* Ext *on the right is taken in the category* $\mathcal{F}$ *of all functors from* $\underline{C}$ *to R-modules.*

Proof. Let

$$P_* \to U \to 0$$

be a projective resolution in the category $\mathcal{F}$. It determines a complex of bifunctors $\mathcal{H}om_R(P_*, T)$, whose components $H^*(\underline{C}; -)$ are acyclic by Lemma 3.10. Hence the hypercohomology spectral sequence (cf. [8]) for the functor $H^*(\underline{C}; -)$ and the complex $\mathcal{H}om_R(P_*, T)$ takes the form

$$E_2^{pq} = H^p(\underline{C}; H^q\mathcal{H}om_R(P_*, T)) \Rightarrow H^{p+q}(H^0(\underline{C}; \mathcal{H}om_R(P_*, T))).$$

Now Proposition 3.2 gives the equality

$$H^0(\underline{C}; \mathcal{H}om_R(P_*, T)) = \mathrm{Hom}_{\mathcal{F}}(P_*, T).$$

Hence that spectral sequence has the required abutment $\mathrm{Ext}_{\mathcal{F}}^*(U, T)$. Furthermore the bifunctor

$$H^q(\mathcal{H}om_R(P_*, T)) \colon \underline{C}^{op} \times \underline{C} \to \mathcal{A}b$$

assigns the groups

$$H^q(\mathrm{Hom}_R(P_* X, TY)).$$

to the pairs $(X, Y) \in |\underline{C}^{op} \times \underline{C}|$. Obviously for any object $X \in |\underline{C}|$, $P_* X$ is a projective resolution for UX. So

$$H^q(\mathscr{H}om_R(P_*, T)) = \mathscr{E}xt^q_R(U, T),$$

and the theorem is proved.

COROLLARY 3.11. *Suppose that the functor*

$$U: \underline{C} \to R\text{-}\underline{\mathrm{mod}}$$

takes values in projective modules; then for any $T: \underline{C} \to R\text{-}\underline{\mathrm{mod}}$ *there are isomorphisms*

$$H^*(\underline{C}; \mathscr{H}om_R(U, T)) \cong \mathrm{Ext}^*_{\mathscr{F}}(U, T).$$

By combining Corollary 3.11 with the results of Section 2 (e.g., Proposition 2.13) we obtain

PROPOSITION 3.12. *Let* $\underline{M}_R$ *be the category of free left finitely generated* R*-modules and let* $\underline{M}_R(r)$ *denote, for* $r \in \mathbb{N}$, *the full subcategory of* $\underline{M}_R$ *whose objects are* $R^0, R, R^2, \ldots, R^r$. *Then, for any functor* $T: \underline{M}_R \to R\text{-}\underline{\mathrm{mod}}$ *there are isomorphisms*

$$H^i(R; T) \cong H^i(\underline{M}_R(I, T)) \cong H^i(\underline{M}_R(r); \mathscr{H}om_R(I, T)_r)$$

for $r \geq 2^i$, *where* $\mathscr{H}om_R(I, T)_r$ *is the restriction of* $\mathscr{H}om_R(I, T)$ *to* $\underline{M}_R(r)$.

4. COHOMOLOGY OF ALGEBRAIC THEORIES

Now we generalize the definition of our ring cohomology to algebraic theories; by proving that the cohomology of free theories vanishes in dimensions ≥ 2 we are able, using Proposition 3.7, to prove Theorem C, asserting that the cohomology of algebraic theories is a case of Barr and Beck's cotriple cohomology [2].

For convenience, let us recall briefly the basic notions of the Lawvere approach to algebraic theories. A detailed exposition may be found in [21].

A finitary algebraic theory (simply a theory for us) is a category whose objects are natural numbers $\mathbb{0}, \mathbb{1}, \mathbb{2}, ..., \mathbb{m}, ..., n \in \mathbb{N}$, with distinguished morphisms

$$p_1^n, ..., p_n^n : \mathbb{m} \to \mathbb{1},$$

which give the object $\mathbb{m}$ the structure of a product of n copies of the object $\mathbb{1}$, for all $n \in \mathbb{N}$. Morphisms of algebraic theories are functors that are the identity on objects and preserve finite products (those distinguished functors, in more brief terms, that preserve the morphisms p_i^n, $1 \leqslant i \leqslant n$). The category of algebraic theories is denoted $\mathcal{T}\!\mathit{heories}$.

A model of the theory $\mathbb{A}$ in a category $\mathscr{C}$ is a functor from $\mathbb{A}$ to $\mathscr{C}$ that preserves finite products. The category of set-valued models of a theory $\mathbb{A}$ is denoted $\mathbb{A}^b$. The assignment

$$M \mapsto M(\mathbb{1}), \qquad M \in |\mathbb{A}^b|$$

defines the usual "forgetful" functor

$$U_{\mathbb{A}} : \mathbb{A}^b \to \mathcal{S}\!\mathit{ets}. \tag{4.1}$$

This functor has a left adjoint $L_{\mathbb{A}} : \mathcal{S}\!\mathit{ets} \to \mathbb{A}^b$; a model M is called a free model of $\mathbb{A}$ on the set X if there is an isomorphism $M \cong L_{\mathbb{A}}(X)$.

For every $n \in \mathbb{N}$, the functor $\mathbb{A}(\mathbb{m}, -)$: $\mathbb{A} \to \mathcal{S}\!\mathit{ets}$ is a model of $\mathbb{A}$. Putting $\mathbb{m} \mapsto \mathbb{A}(\mathbb{m}, -)$ defines a functor

$$I_{\mathbb{A}} : \mathbb{A}^{op} \to \mathbb{A}^b.$$

It is known that $I_{\mathbb{A}}$ is a full embedding establishing an equivalence of $\mathbb{A}^{op}$ with the full subcategory of $\mathbb{A}^b$ consisting of finitely generated free models (i.e., free on finite sets) [21, 25].

A morphism of theories f: $\mathbb{A} \to \mathbb{B}$ induces a pair of functors

$$f_* : \mathbb{A}^b \to \mathbb{B}^b, \qquad f^b : \mathbb{B}^b \to \mathbb{A}^b,$$

where $f^b(M) = M \circ f$, $M \in |\mathbb{B}^b|$, while f_* is left adjoint to f^b, and moreover the diagram

$$\begin{array}{ccc} \mathbb{A}^{op} & \xrightarrow{I_{\mathbb{A}}} & \mathbb{A}^b \\ {\scriptstyle f^{op}}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathbb{B}^{op} & \xrightarrow{I_{\mathbb{B}}} & \mathbb{B}^b \end{array}$$

commutes [21].

There is a functor $\mathcal{R}\!\mathit{ings} \to \mathcal{T}\!\mathit{heories}$ assigning to a ring R the theory of left modules over R. We denote this theory by R; in other words, R^b is the category of left R-modules. This causes no confusion, as that functor is known to be a full embedding.

Recall also that for theories $\mathbb{A}$, $\mathbb{B}$ their tensor or Kronecker product $\mathbb{A} \otimes \mathbb{B}$ is defined (see [21, 25]). This is a theory whose set-valued models are the same as models of $\mathbb{A}$ in the category $\mathbb{B}^b$, or, equivalently, models of $\mathbb{B}$ in $\mathbb{A}^b$. By denoting by $\mathbb{N}$ the initial object of $\mathscr{T}\!\mathit{heories}$, we obtain $\mathbb{A} \otimes \mathbb{N} \cong \mathbb{A}$ for any theory $\mathbb{A}$, as $\mathbb{N}^b = \mathscr{S}\!\mathit{ets}$ [21]. The unique morphism $\mathbb{N} \to \mathbb{Z}$ induces a morphism of theories

$$i: \mathbb{A} = \mathbb{A} \otimes \mathbb{N} \to \mathbb{A} \otimes \mathbb{Z}.$$

Here $\mathbb{Z}$ denotes, simultaneously, the ring of integers and the theory of $\mathbb{Z}$-modules, i.e., of abelian groups. The category $(\mathbb{A} \otimes \mathbb{Z})^b$ is equivalent to the category of internal abelian groups of $\mathbb{A}^b$. $\mathbb{A} \otimes \mathbb{Z}$ is known to be representable by a ring for any $\mathbb{A}$ (see [25]). The functor $i^b: (\mathbb{A} \otimes \mathbb{Z})^b \to \mathbb{A}^b$ can be identified as the forgetful functor $\mathscr{A}\!\mathit{b}(\mathbb{A}^b) \to \mathbb{A}^b$. It has a left adjoint i_*, called abelianization, is written

$$(-)_{\mathrm{ab}}: \mathbb{A}^b \to \mathscr{A}\!\mathit{b}(\mathbb{A}^b).$$

DEFINITION 4.2. Let $\mathbb{A}$ be a theory. For functors

$$T: \mathbb{A}^{op} \to \mathscr{A}\!\mathit{b}(\mathbb{A}^b),$$

the cohomology of $\mathbb{A}$ with coefficients in T is defined by the equality

$$H^*(\mathbb{A}; T) = \mathrm{Ext}^*_{\mathscr{F}(\mathbb{A})}((I_{\mathbb{A}})_{\mathrm{ab}}, T),$$

where $\mathscr{F}(\mathbb{A})$ is the category of all functors from $\mathbb{A}^{op}$ to $\mathscr{A}\!\mathit{b}(\mathbb{A}^b)$ while $(I_{\mathbb{A}})_{\mathrm{ab}}$ is the composition

$$\mathbb{A}^{op} \xrightarrow{I_{\mathbb{A}}} \mathbb{A}^b \xrightarrow{(\ )_{\mathrm{ab}}} \mathscr{A}\!\mathit{b}(\mathbb{A}^b).$$

Remark. As mentioned, $\mathbb{A} \otimes \mathbb{Z}$ may be identified with some ring for any theory $\mathbb{A}$, which means that

$$\mathscr{A}\!\mathit{b}(\mathbb{A}^b) = (\mathbb{A} \otimes \mathbb{Z})^b$$

is an abelian category with enough projectives and injectives so that the Ext groups in Definition 4.2 can be understood in the standard sense. Also, since $\mathbb{A}^{op}$ is equivalent to the category of finitely generated free models of $\mathbb{A}$, for rings this definition coincides with Definition 1.2.

LEMMA 4.3. *For any theory $\mathbb{A}$ and any $n \in \mathbb{N}$ there are isomorphisms*

$$\mathrm{Hom}_{\mathscr{A}\!\mathit{b}(\mathbb{A}^b)}((I_{\mathbb{A}})_{\mathrm{ab}}(\mathbb{n}), A) \cong A(\mathbb{n})$$

for any A from $\mathscr{A}\!\mathit{b}(\mathbb{A}^b)$. In particular, $(I_{\mathbb{A}})_{\mathrm{ab}}(\mathbb{n})$ is projective in $\mathscr{A}\!\mathit{b}(\mathbb{A}^b)$.

Proof. Since $(-)_{\mathrm{ab}}$ is left adjoint to the forgetful functor, we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{Ab}(\mathbb{A}^b)}((I_{\mathbb{A}})_{\mathrm{ab}}(\mathbb{m}), A) &= \mathrm{Hom}_{\mathbb{A}^b}(I_{\mathbb{A}}(\mathbb{m}), A)\\ &= \mathrm{Hom}_{\mathbb{A}^b}(\mathbb{A}(\mathbb{m}, -), A) = \mathbb{A}(\mathbb{m}),\end{aligned}$$

the last equality by the Yoneda Lemma.

LEMMA 4.4. *The category $\mathcal{F}(\mathbb{A})$ of functors $\mathbb{A}^{op} \to \mathcal{Ab}(\mathbb{A}^b)$ is equivalent to the category of those bifunctors $\mathbb{A} \times \mathbb{A}^{op} \to \mathcal{Ab}$ which preserve finite products in the first variable. Moreover, under that equivalence the bifunctor*

$$\tilde{T}: \mathbb{A} \times \mathbb{A}^{op} \to \mathcal{Ab}, \tag{4.5}$$

determined by the equality $\tilde{T}(\mathbb{n}, \mathbb{m}) = T(\mathbb{m})(\mathbb{n})$, corresponds to the functor $T: \mathbb{A}^{op} \to \mathcal{Ab}(\mathbb{A}^b)$. It can be also expressed, in our previous notations, as

$$\tilde{T} = \mathcal{Hom}_{\mathcal{Ab}(\mathbb{A}^b)}((I_{\mathbb{A}})_{\mathrm{ab}}, T).$$

Proof. Since $\mathcal{Ab}(\mathbb{A}^b) \simeq (\mathbb{A} \otimes \mathbb{Z})^b$, the category $\mathcal{Ab}(\mathbb{A}^b)$ is equivalent to the category of models of $\mathbb{A}$ in $\mathcal{Ab}$, the category of abelian groups. This implies the first part of the assertion. The second follows from the first, using Lemma 4.3.

Lemma 4.3 makes it possible to use Corollary 3.11. By taking into account the second part of Lemma 4.4 we obtain

PROPOSITION 4.6. *Let $\mathbb{A}$ be a theory, and consider $T \in \mathcal{F}(\mathbb{A})$. Let $\tilde{T}$ be the bifunctor corresponding to T as in (4.5). Then there are isomorphisms*

$$H^*(\mathbb{A}; T) \cong H^*(\mathbb{A}^{op}; \tilde{T}).$$

Here the groups on the right are the Hochschild–Mitchell cohomologies of the category $\mathbb{A}^{op}$ of finitely generated free models of the theory $\mathbb{A}$ with coefficients in the bifunctor $\tilde{T}$.

By relying on Proposition 4.6, we can give more intricate descriptions of low-dimensional cohomology groups.

Let $\mathbb{A}$ be a theory and take some $M \in \mathcal{Ab}(\mathbb{A}^b)$. Then the underlying set UM, as in (4.1), has the structure of an abelian group. Recall that the structure projections $p_1, ..., p_n: \mathbb{n} \to \mathbb{1}$ induce isomorphisms

$$(Mp_1, ..., Mp_n): M(\mathbb{n}) \to \prod_{i=1}^{n} M(\mathbb{1})$$

so that $M(\mathbb{n})$ can be identified with $(UM)^n$. In particular, every morphism $\omega: \mathbb{n} - \mathbb{1}$ of the theory induces an operation $\omega_*: (UM)^n \to UM$ (an n-ary operation on the model M).

For a functor $T\colon \mathbb{A}^{op} \to \mathcal{Ab}(\mathbb{A}^b)$, let T_n denote the abelian group $U(T(\mathbb{n}))$. Actions of the functor T on morphisms $\phi\colon \mathbb{n} \to \mathbb{m}$ induce homomorphisms $\phi^*\colon T_m \to T_n$.

PROPOSITION 4.7. *Let $\mathbb{A}$ be a theory and let T be a functor from $\mathcal{F}(\mathbb{A})$. Then, in the above notations,*

$$H^0(\mathbb{A}; T) \cong \{a \in T_1\,;\, \forall \omega \in \mathbb{A}(\mathbb{n}, \mathbb{1}),\ \omega^* a = \omega_*(p_1^* a, ..., p_n^* a)\}.$$

Proof. By Proposition 4.6, $H^0(\mathbb{A}; T) = H^0(\mathbb{A}^{op}; \tilde{T})$. But the group $H^0(\mathbb{A}^{op}; \tilde{T})$ coincides with the end of the bifunctor $\tilde{T}$. Hence it consists of sequences $(a_0, a_1, ...)$ with $a_n \in \tilde{T}(\mathbb{n}, \mathbb{n})$, such that the equalities

$$\tilde{T}(\alpha, 1_{\mathbb{n}})(a_n) = \tilde{T}(1_{\mathbb{m}}, \alpha)(a_m) \tag{4.8}$$

hold for any $\alpha \in \mathbb{A}(\mathbb{n}, \mathbb{m})$. By Lemma 4.4, $\tilde{T}$ preserves products in the first variable. Hence

$$(\tilde{T}(p_1, 1_{\mathbb{n}}), ..., \tilde{T}(p_n, 1_{\mathbb{n}}))\colon \tilde{T}(\mathbb{n}, \mathbb{n}) \to \prod_{i=1}^{n} \tilde{T}(\mathbb{1}, \mathbb{n})$$

are isomorphisms. By using t_n to denote the inverse to this isomorphisms we obtain

$$a_n = t^{-1}(\tilde{T}(1_{\mathbb{1}}, p_1)(a_1), ..., \tilde{T}(1_{\mathbb{1}}, p_n)(a_1)),$$

since $\tilde{T}(p_i, 1_{\mathbb{n}})(a_n) = \tilde{T}(1_{\mathbb{1}}, p_i)(a_n)$. Hence in the sequence $(a_0, a_1, a_2, ...)$ all entries are expressible in terms of a_1 alone. So obviously Lemma 4.4 is equivalent to the condition of our proposition.

Keeping the above notations we now turn to an analogous description of H^1. To this end we introduce the following

DEFINITION 4.9. For a theory $\mathbb{A}$ and a functor $T \in \mathcal{F}(A)$, the abelian group $\mathrm{Der}(\mathbb{A}; T)$, of derivations of $\mathbb{A}$ with values in T, consists of sequences

$$d = (d_n\colon \mathbb{A}(\mathbb{n}, \mathbb{1}) \to T_n)_{n \in \mathbb{N}}$$

satisfying the equalities

$$d_m(\omega(\omega_1, ..., \omega_n)) = (\omega_1, ..., \omega_n)^*\,(d_n \omega) + \omega_*(d_m \omega_1, ..., d_m \omega_n)$$

for all $\omega \in \mathbb{A}(\mathbb{n}, \mathbb{1})$, $\omega_1, ..., \omega_n \in \mathbb{A}(\mathbb{m}, \mathbb{1})$, where $(\omega_1, ..., \omega_n)\colon \mathbb{m} \to \mathbb{n}$ is the unique morphism with the property $p_i(\omega_1, ..., \omega_n) = \omega_i$, $1 \leqslant i \leqslant n$, which exists thanks to $\mathbb{m}=\mathbb{1}^n$. The subgroup $\mathrm{Ider}(\mathbb{A};T)\subseteq\mathrm{Der}(\mathbb{A};T)$ of trivial derivations contains those sequences representable in the form

$$\mathbb{A}(\mathbb{m},\mathbb{1})\ni\omega\mapsto\omega^*a-\omega_*(p_1^*a,...,p_n^*a)$$

for some a from T_1. Addition in these groups is performed componentwise.

PROPOSITION 4.10. *For any theory* $\mathbb{A}$ *and any functor* $T\in\mathscr{F}(A)$ *there exists an isomorphism*

$$H^1(\mathbb{A};T)\cong\mathrm{Der}(\mathbb{A};T)/\mathrm{Ider}(\mathbb{A};T).$$

Proof. Again by Proposition 4.6, $H^1(\mathbb{A};T)=H^1(\mathbb{A}^{op};\tilde{T})$. Applying Definition 3.1 for dimension 1 in our case gives that any element of $H^1(\mathbb{A};T)$ can be represented by a family of maps

$$\tilde{d}=(\tilde{d}_{n,n'}:\mathbb{A}(\mathbb{m},\mathbb{m}')\to\tilde{T}(\mathbb{m}',\mathbb{m}))_{n,n'\in\mathbb{N}}$$

satisfying the cocycle condition

$$\psi^*\tilde{d}_{n,n'}(\phi)-\tilde{d}_{m,n'}(\phi\psi)+\phi_*\tilde{d}_{m,n}(\psi)=0$$

for any $\phi:\mathbb{m}\to\mathbb{m}'$, $\psi:\mathbb{m}\mathbb{m}\to\mathbb{m}$ in $\mathbb{A}$. Taking the structure projections $p_i:\mathbb{m}\to 1$, $1\leqslant i\leqslant n$, in place of ϕ gives

$$p_{i*}\tilde{d}_{m,n}(\psi)=\tilde{d}_{m,1}(p_i\psi)-\psi^*\tilde{d}_{n,1}(p_i).$$

Now recall that $\tilde{T}$ preserves products in the first variable; hence $\tilde{d}_{m,n}(\psi)$ is completely determined by its images under p_{i*} for $1\leqslant i\leqslant n$, so it suffices to know the $(\tilde{d}_{n,1})_{n\in\mathbb{N}}$ sequence. Furthermore, the element of $H^1(\mathbb{A};T)$ represented by the family $\tilde{d}$ will remain unchanged after adding to $\tilde{d}$ a coboundary, i.e., a family of type

$$(\delta\tilde{a})_{n,n'}(\phi)=\phi^*(\tilde{a}_{n'})-\phi_*(\tilde{a}_n)$$

for some sequence $(\tilde{a}_n)_{n\in N}\in\prod_{n\in N}T_{(\mathbb{m},\mathbb{m})}$. Once again using that $\tilde{T}$ preserves products on the covariant side, one can choose a sequence $(\tilde{a}_n)_{n\in\mathbb{N}}$ with $p_{i*}(\tilde{a}_n)=\tilde{d}_{n,1}(p_i)$, and, by adding $\delta\tilde{a}$ to $\tilde{d}$, obtain a new family $(\tilde{d}'_{n,n'})_{n,n'}$ with the property $\tilde{d}'_{n,1}(p_i)=0$ for $1\leqslant i\leqslant n\in\mathbb{N}$. Now define

$$d_n(\omega)=\tilde{d}'_{n,1}(\omega),\qquad\omega\in\mathbb{A}(\mathbb{m},\mathbb{1}).$$

It is easy to show that the cocycle condition for $\tilde{d}'$ is equivalent to d being a derivation in the sense of Definition 4.9. Finally, for a sequence $\tilde{a}\in\prod_n\tilde{T}(\mathbb{m},\mathbb{m})$, the condition $(\delta\tilde{a})_{n,1}(p_i)=0$ means that

$$p_{i*}(\tilde{a}_n)=p_i^*(\tilde{a}_1),$$

and since $\tilde{T}$ preserves suitable products, $\tilde{a}_n$ is determined by the elements $\tilde{a}_1 = a$ and, moreover, the derivation corresponding to the family $\delta\tilde{a}$ in this case will be exactly the trivial derivation corresponding to a.

Our next task is to define extensions of algebraic theories. To this end we note that if $f\colon \mathbb{B} \to \mathbb{A}$ is a morphism of algebraic theories, then the functor $f^b\colon \mathbb{A}^b \to \mathbb{B}^b$ preserves products, so it carries abelian group objects to abelian group objects, and thus the functor $f^b\colon \mathscr{Ab}(\mathbb{A}^b) \to \mathscr{Ab}(\mathbb{B}^b)$ is defined.

Definition 4.11. Let $\mathbb{A}$ be a theory and T any functor from $\mathbb{A}^{op}$ to $\mathscr{Ab}(\mathbb{A}^b)$. An extension of $\mathbb{A}$ by T,

$$T \rightarrowtail \mathbb{B} \xrightarrow{f} \!\!\!\!\rightarrow \mathbb{A},$$

consists of a morphism of theories $f\colon \mathbb{B} \to \mathbb{A}$ which is surjective on morphisms together with an action $\mu_{\mathbb{B}}\colon f^bTf^{op} \times I_{\mathbb{B}} \to I_{\mathbb{B}}$ of the internal abelian group $f^bTf^{op}\colon \mathbb{B}^{op} \to \mathscr{Ab}(\mathbb{B}^b)$ in the category of all functors from $\mathbb{B}^{op}$ to $\mathbb{B}^b$, on the object $I_{\mathbb{B}}$ of this category, satisfying

(a) $\varepsilon \circ \mu_{\mathbb{B}} = \varepsilon \circ p_{\mathbb{B}}$, where $p_{\mathbb{B}}\colon f^bTf^{op} \times I_{\mathbb{B}} \to I_{\mathbb{B}}$ is the projection, while $\varepsilon\colon I_{\mathbb{B}} \to f^bI_{\mathbb{A}}f^{op}$ is obtained by applying the functor $I_{\mathbb{B}}$ to the unit $1_{\mathbb{B}^b} \to f^bf_*$ of the adjunction $f_* \to f^b$; and

(b) the natural transformation

$$(\mu_{\mathbb{B}}, p_{\mathbb{B}})\colon f^bTf^{op} \times I_{\mathbb{B}} \to I_{\mathbb{B}} \times_\varepsilon I_{\mathbb{B}}$$

is an isomorphism, where $I_{\mathbb{B}} \times_\varepsilon I_{\mathbb{B}}$ is the pullback

$$\begin{array}{ccc} I_{\mathbb{B}} \times_\varepsilon I_{\mathbb{B}} & \longrightarrow & I_{\mathbb{B}} \\ \downarrow & & \downarrow{\scriptstyle \varepsilon} \\ I_{\mathbb{B}} & \xrightarrow[\varepsilon]{} & f^bI_{\mathbb{A}}f^{op} \end{array}$$

A morphism from the extension $T \rightarrowtail \mathbb{B} \twoheadrightarrow^f \mathbb{A}$ to another one, $T \rightarrowtail \mathbb{B}_1 \twoheadrightarrow^{f_1} \mathbb{A}$, consists of a morphism of theories, $l\colon \mathbb{B} \to \mathbb{B}_1$ with $f_1 l = f$ and $l_* \mu_{\mathbb{B}} = \mu_{\mathbb{B}_1} l^{op}$.

Proposition 4.12. *The category of all extensions of a theory $\mathbb{A}$ by the functor $T \in \mathscr{F}(A)$ is a groupoid, whose set of components is naturally bijective to $H^2(\mathbb{A}; T)$:*

$$N(\mathbb{A}; T) \approx H^2(\mathbb{A}; T).$$

Proof. The first assertion is a fairly standard one, so we turn to the second. By Proposition 4.6 we have

$$H^2(\mathbb{A}; T) \cong H^2(\mathbb{A}^{op}; \tilde{T}),$$

where $\tilde{T}$: $\mathbb{A} \times \mathbb{A}^{op} \to \mathscr{Ab}$ is the bifunctor obtained from T as in Lemma 4.4. Hence there is a bijection

$$H^2(\mathbb{A}; T) \approx M(\mathbb{A}^{op}; \tilde{T}),$$

where on the right (as in Section 3) we have the set of equivalence classes of linear extensions of the category $\mathbb{A}^{op}$ by the bifunctor $\tilde{T}$. Take one such linear extension

$$\tilde{T} + \longrightarrow \underline{E} \xrightarrow{p} \mathbb{A}^{op}.$$

By Lemma 4.4, $\tilde{T}$ preserves products in the first variable; this enables us to use Proposition 3.4 to conclude that $\underline{E}^{op}$ can be given a structure of algebraic theory in such a way that p^{op}: $\underline{E}^{op} \to \mathbb{A}$ will be a morphism of theories. Moreover this structure is obviously unique up to isomorphism.

Now suppose a morphism of theories f: $\mathbb{B} \to \mathbb{A}$ is given with $\mathbb{B}^{op} = \mathbb{E}$ and $p = f^{op}$. By Lemma 4.4 we have isomorphisms

$$\tilde{T}(\mathbb{n}, \mathbb{m}) \cong \operatorname{Hom}_{\mathbb{A}^b}(I_{\mathbb{A}}(\mathbb{n}), T(\mathbb{m})), \qquad \mathbb{n}, \mathbb{m} \in |\mathbb{A}|.$$

By the definition of linear extensions there is an action

$$\tilde{T}(f^{op}\mathbb{n}, f^{op}\mathbb{m}) \times \mathbb{B}^{op}(\mathbb{n}, \mathbb{m}) \to \mathbb{B}^{op}(\mathbb{n}, \mathbb{m}), \qquad \mathbb{n}, \mathbb{m} \in |\mathbb{B}|.$$

Since $I_{\mathbb{B}}$: $\mathbb{B}^{op} \to \mathbb{B}^b$ is a full embedding and

$$\begin{aligned}\tilde{T}(f^{op}\mathbb{n}, f^{op}\mathbb{m}) &\cong \operatorname{Hom}_{\mathbb{A}^b}(I_{\mathbb{A}} f^{op}\mathbb{n}, Tf^{op}\mathbb{m}) \\ &= \operatorname{Hom}_{\mathbb{A}^b}(f_* I_{\mathbb{B}}\mathbb{n}, Tf^{op}\mathbb{m}) = \operatorname{Hom}_{\mathbb{B}^b}(I_{\mathbb{B}}\mathbb{n}, f^b T f^{op}\mathbb{m}),\end{aligned}$$

we also obtain actions

$$\operatorname{Hom}_{\mathbb{B}^b}(I_{\mathbb{B}}\mathbb{n}, f^b T f^{op}\mathbb{m}) \times \operatorname{Hom}_{\mathbb{B}^b}(I_{\mathbb{B}}\mathbb{n}, I_{\mathbb{B}}\mathbb{m}) \to \operatorname{Hom}_{\mathbb{B}^b}(I_{\mathbb{B}}\mathbb{n}, I_{\mathbb{B}}\mathbb{m})$$

for $\mathbb{n}, \mathbb{m} \in |\mathbb{B}|$. In particular, for $\mathbb{n} = \mathbb{1}$ this gives actions

$$\mu_{\mathbb{m}}: f^b T f^{op}\mathbb{m} \times I_{\mathbb{B}}\mathbb{m} \to I_{\mathbb{B}}\mathbb{m};$$

then varying $\mathbb{m}$ over $\mathbb{B}^{op}$ gives a natural transformation

$$\mu: f^b T f^{op} \times I_{\mathbb{B}} \to I_{\mathbb{B}}.$$

It is easy to show that the pair (f, μ) determines an element in $N(\mathbb{A}; T)$.

Conversely, suppose we are given a surjective morphism of theories $f\colon \mathbb{B} \to \mathbb{A}$ and an action $\mu\colon f^b T f^{op} \times I_{\mathbb{B}} \to I_{\mathbb{B}}$, such that $(f, \mu) \in N(\mathbb{A}, T)$. Take $a \in \tilde{T}(\mathfrak{n}, \mathfrak{m})$ and $\lambda \in \mathbb{B}^{op}(\mathfrak{n}, \mathfrak{m})$, and let

$$\tilde{a}\colon I_{\mathbb{B}}\mathfrak{n} \to f^b T f^{op}\mathfrak{m}$$

be the morphism corresponding to a under the isomorphism

$$\tilde{T}(\mathfrak{n}, \mathfrak{m}) \cong \mathrm{Hom}_{\mathbb{B}^b}\,(I_{\mathbb{B}}\mathfrak{n}, f^b T f^{op}\mathfrak{m}).$$

Then $\lambda + a = \mu \circ (\tilde{a}, I_{\mathbb{B}}\lambda)$ defines an action

$$\tilde{T}(\mathfrak{n}, \mathfrak{m}) \times \mathrm{Hom}_{\mathbb{B}^{op}}(\mathfrak{n}, \mathfrak{m}) \to \mathrm{Hom}_{\mathbb{B}^{op}}(\mathfrak{n}, \mathfrak{m}),$$

for $\mathfrak{n}, \mathfrak{m} \in |\mathbb{B}|$. It can be shown that this gives a linear extension

$$\tilde{T} + \longrightarrow \mathbb{B}^{op} \xrightarrow{f^{op}} \mathbb{A}^{op}$$

and hence an element in $M(\mathbb{A}^{op}; \tilde{T})$, and that the maps

$$N(\mathbb{A}; T) \to M(\mathbb{A}^{op}; \tilde{T}), \qquad M(\mathbb{A}^{op}; \tilde{T}) \to N(\mathbb{A}; T)$$

so defined are mutually inverse bijections.

EXAMPLES 4.13. For a surjective morphism of algebraic theories

$$f\colon \mathbb{B} \to \mathbb{A}$$

the induced functor $f^b\colon \mathbb{A}^b \to \mathbb{B}^b$ is a full embedding, so if we identify $\mathbb{A}^b$ with its image in $\mathbb{B}^b$ under f^b, identifying also the category $\mathbb{A}^{op}$ with the full subcategory in $\mathbb{A}^b$ consisting of free finitely generated models, then in an informal way we may picture extensions of the theory $\mathbb{A}$ by the functor $T\colon \mathbb{A}^{op} \to \mathcal{A}b(\mathbb{A}^b)$ as surjective morphisms of theories $f\colon \mathbb{B} \to \mathbb{A}$ together with a family of "central extensions"

$$0 \to T(f_* X) \to X \to f_* X \to 1$$

functorial in $X \in |\mathbb{B}^b|$.

Let

$$0 \longrightarrow M \longrightarrow R \xrightarrow{f} S \longrightarrow 0 \tag{4.14}$$

be a singular extension of rings and let T be the functor defined by $T(Y) = M \otimes_S Y$ for $Y \in |S\text{-}\underline{\mathrm{mod}}|$. Since for every free R-module X there is a short exact sequence

$$0 \to T(f_* X) \to X \to f_* X \to 0,$$

where $f_*\colon R\text{-}\underline{\mathrm{mod}} \to S\text{-}\underline{\mathrm{mod}}$ is the functor with $f_*(X) = S \otimes_R X$, we obtain the extension

$$T \rightarrowtail \begin{pmatrix}\text{theory of}\\ R\text{-modules}\end{pmatrix} \twoheadrightarrow \begin{pmatrix}\text{theory of}\\ S\text{-modules}\end{pmatrix} \tag{4.15}$$

of the theory of S-modules by T. In this way one can define a homomorphism from the second Mac Lane cohomology group $H^2(S; M)$ to $H^2(S; T)$, carrying (4.14) to (4.15). By Theorem A and Definition 4.2 this is an isomorphism.

Consider a free group G with the lower central series $\cdots < \Gamma_3 G < \Gamma_2 G < G$. By the classical result of Witt [24] there is an exact sequence

$$0 \to L_n(G_{\mathrm{ab}}) \to G/\Gamma_{n+1} G \to G/\Gamma_n G \to 1,$$

where $L_n(G_{\mathrm{ab}})$ is the n-dimensional homogeneous component of the free Lie ring generated by the abelian group G_{ab}. From this one deduces that

$$L_n((\cdot)_{\mathrm{ab}}) \rightarrowtail \begin{pmatrix}\text{theory of class } n\\ \text{nilpotent groups}\end{pmatrix} \twoheadrightarrow \begin{pmatrix}\text{theory of class } n-1\\ \text{nilpotent groups}\end{pmatrix} \tag*{$(4.16)_n$}$$

is an extension of algebraic theories for $n \geqslant 2$.

Similarly there are extensions

$$L_n((\cdot)_{\mathrm{ab}}) \rightarrowtail \begin{pmatrix}\text{theory of class } n\\ \text{nilpotent Lie rings}\end{pmatrix} \twoheadrightarrow \begin{pmatrix}\text{theory of class } n-1\\ \text{nilpotent Lie rings}\end{pmatrix}. \tag*{$(4.17)_n$}$$

For $n = 2$, Proposition 4.12, $(4.16)_2$ and $(4.17)_2$ determine elements of $H^2(\mathbb{Z}; \Lambda^2)$. At the end of Section 2 it was shown that $H^2(\mathbb{Z}; \Lambda^2) = \mathbb{Z}/2\mathbb{Z}$. It can be easily shown that $(4.17)_2$ determines the trivial element, while $(4.16)_2$ gives the nontrivial one.

Similar to $(4.17)_2$, for any commutative ring R there exist extensions of theories

$$S_R^2 \rightarrowtail \begin{pmatrix}\text{theory of commutative } R\text{-algebras}\\ \text{with the identity } xyz = 0\end{pmatrix} \xrightarrow{(\cdot)_{\mathrm{ab}}} \begin{pmatrix}\text{theory of}\\ R\text{-modules}\end{pmatrix}, \tag{4.18}$$

$$\overset{2}{\underset{R}{\otimes}} \rightarrowtail \begin{pmatrix}\text{theory of (associative) } R\text{-algebras}\\ \text{with the identity } xyz = 0\end{pmatrix} \twoheadrightarrow \begin{pmatrix}\text{theory of}\\ R\text{-modules}\end{pmatrix}, \tag{4.19}$$

$$\Lambda_R^2 \rightarrowtail \begin{pmatrix}\text{theory of anticommutative}\\ R\text{-algebras with the identity } xyz = 0\end{pmatrix} \xrightarrow{(\cdot)_{\mathrm{ab}}} \begin{pmatrix}\text{theory of}\\ R\text{-modules}\end{pmatrix}. \tag{4.20}$$

These extensions determine trivial elements of the groups $H^2(R; S^2_R)$, $H^2(R; \otimes^2_R)$, and $H^2(R; \Lambda^2_R)$, respectively. By Corollary 2.16, $H^2(R; \otimes^2_R) = 0$, so (4.19) is the only possible extension of the theory of R-modules by $\otimes^2_R$. On the other hand, Proposition 2.21 says that the groups $H^2(R; S^2_R)$ and $H^2(R; \Lambda^2_R)$ are, in general, non-zero; they are relatively easily calculable when ${}_2R = 0$. Consequently there exist some nontrivial extensions of the theory of R-modules by S^2_R and Λ^2_R. We know nothing about the corresponding theories, although for $R = \mathbb{Z}$, $H^2(\mathbb{Z}; S^2) = 0$, so (4.18) is the only possible extension of the theory of abelian groups by the functor $S^2_{\mathbb{Z}}$. We also know nothing about the group

$$H^2\begin{pmatrix}\text{theory of class} & \\ n-1 \text{ nilpotent groups}; & L_n((\cdot)_{\text{ab}})\end{pmatrix}$$

for $n > 2$, whose nontrivial elements are represented by the extensions $(4.17)_n$.

Definition 4.2 shows that the cohomologies $H^*(\mathbb{A}; -): \mathscr{F}(\mathbb{A}) \to \mathscr{A}\ell$ of a theory $\mathbb{A}$ constitute an exact connected sequence of functors. Let us consider functorial properties of the cohomology in the first variable.

For a morphism of theories $f: \mathbb{B} \to \mathbb{A}$ and a functor $T: \mathbb{A}^{op} \to \mathscr{A}\ell(\mathbb{A}^b)$, denote by f^*T the composition

$$\mathbb{B}^{op} \xrightarrow{f^{op}} \mathbb{A}^{op} \xrightarrow{T} \mathscr{A}\ell(\mathbb{A}^b) \cong (\mathbb{A} \otimes \mathbb{Z})^b \xrightarrow{(f \otimes \mathbb{Z})^b} (\mathbb{B} \otimes \mathbb{Z})^b \cong (\mathbb{B}^b).$$

Assigning $T \mapsto f^*T$ defines an exact functor $f^*: \mathscr{F}(\mathbb{A}) \to \mathscr{F}(\mathbb{B})$; moreover, the diagram

$$\begin{tikzcd} \mathbb{B} \times \mathbb{B}^{op} \arrow[rr, "f \times f^{op}"] \arrow[dr, "f^*T"'] & & \mathbb{A} \times \mathbb{A}^{op} \arrow[dl, "\tilde{T}"] \\ & \mathscr{A}\ell & \end{tikzcd}$$

commutes. As in Section 3 this defines a homomorphism in the cohomology of categories,

$$H^*(\mathbb{A}^{op}; \tilde{T}) \to H^*(\mathbb{B}^{op}; \widetilde{f^*T}),$$

and Proposition 4.6 enables us to transform this to the homomorphism $H^*(\mathbb{A}; T) \to H^*(\mathbb{B}; f^*T)$. Then Proposition 4.7 easily implies

PROPOSITION 4.21. *For a surjective morphism of theories $f: \mathbb{B} \to \mathbb{A}$, the induced homomorphism*

$$H^0(\mathbb{A}; T) \to H^0(\mathbb{B}; f^*T)$$

is an isomorphism for any $T \in \mathscr{F}(\mathbb{A})$.

Consider the functor

$$\Omega\colon \mathcal{T}\!\mathit{heories} \to \mathcal{S}\!\mathit{ets}^{\mathbb{N}}$$

defined for a theory $\mathbb{A}$ *by* $\Omega(\mathbb{A}) = (\mathbb{A}(\mathbb{m}, \mathbb{1}))_{n \in \mathbb{N}}$. *It is known (see* [21, 25]) *that* Ω *has a left adjoint* L. *The theory* $\mathbb{A}$ *is called free if there is an object* P *of* $\mathcal{S}\!\mathit{ets}^{\mathbb{N}}$ *with* $\mathbb{A} \cong L(P)$

PROPOSITION 4.22. *For a free theory* $\mathbb{A}$, $H^n(\mathbb{A}; -) = 0$ *for* $n \geqslant 2$.

Proof. Since

$$H^n(\mathbb{A}; -) = \operatorname{Ext}^n_{\mathcal{F}(\mathbb{A})}(I_{\mathrm{ab}}, -),$$

it suffices to consider the case $n = 2$. For $T \in \mathcal{F}(\mathbb{A})$, Proposition 4.12 shows that $H^2(\mathbb{A}; T) \approx N(\mathbb{A}; T)$, where $N(\mathbb{A}; T)$ is the group of extensions

$$T \rightarrowtail \mathbb{B} \overset{}{\underset{f}{\twoheadrightarrow}} \mathbb{A}.$$

Since f is surjective, $\Omega(f)$ has a section in $\mathcal{S}\!\mathit{ets}^{\mathbb{N}}$, and since $\mathbb{A}$ is free, this section determines a section of f in $\mathcal{T}\!\mathit{heories}$. This shows that $N(\mathbb{A}; T) = 0$.

Now Proposition 4.22 with Proposition 3.7 enables us to prove Theorem C, stating that the cohomology of algebraic theories can be expressed by the cotriple cohomology of Barr and Beck [2]. First recall the definition.

Let $\mathbb{G}$ be a cotriple [2] (comonad [15]) in a category $\mathscr{C}$. For $X \in |\mathscr{C}|$, the cotriple resolution $\mathbb{G}_* X \to X$ is the augmented simplicial object of $\mathscr{C}$ whose nth component is $G^{n+1}X$ [2]. Denote by $\mathscr{C}/X$ the comma category [15] whose objects are morphisms of $\mathscr{C}$ over X, $f\colon Y \to X$. Let

$$T\colon (\mathscr{C}/X)^{op} \to \mathcal{A}\mathit{b}$$

be any functor. Then the $\mathbb{G}$-cotriple cohomology of the object X with coefficients in T is defined by the equality

$$H^n_{\mathbb{G}}(X; T) = \pi^n(T(\mathbb{G}_* X)),$$

where $T(\mathbb{G}_* X)$ is the cosimplicial abelian group obtained from $\mathbb{G}_* X$ by the componentwise application of T, while π^* are the cohomotopy groups of that cosimplicial abelian group (by definition, for cosimplicial abelian groups A^*, $\pi^*(A^*) = H^*(\operatorname{Ch} A^*)$, where $\operatorname{Ch} A^*$ is the associated cochain complex with the same components as A^*, with the coboundary operator given by the alternating sum of cofaces of A^*).

Also recall that for $\mathscr{F}: \mathscr{A} \to \mathscr{C}$, $U: \mathscr{C} \to \mathscr{A}$ with F left adjoint to U, there is a canonical cotriple structure on $G = FU: \mathscr{C} \to \mathscr{C}$ [15]. In that case $U\mathbb{G}_* X \to UX$ is a contractible augmented simplicial object of $\mathbb{A}$, for any $X \in |\mathscr{C}|$.

For a theory $\mathbb{A}$ and $T \in \mathscr{F}(\mathbb{A})$, define functors $H^*(-, T)$ and $\mathrm{Der}(-, T)$ from $(\mathscr{T}\!\mathit{heories}/A)^{op}$ to $\mathscr{A}\!\mathit{b}$ by

$$H^*(\mathbb{B} \xrightarrow{f} \mathbb{A}; T) = H^*(\mathbb{B}; f^*T),$$

$$\mathrm{Der}(\mathbb{B} \xrightarrow{f} \mathbb{A}; T) = \mathrm{Der}(\mathbb{B}; f^*T),$$

where $\mathbb{B} \to^f \mathbb{A}$ is an object of $\mathscr{T}\!\mathit{heories}/\mathbb{A}$.

THEOREM C. *Let $\mathbb{G}$ be the cotriple on $\mathscr{T}\!\mathit{heories}$ induced by the adjunction*

$$\mathscr{T}\!\mathit{heories} \underset{L}{\overset{\Omega}{\rightleftarrows}} (\mathscr{S}\!\mathit{ets})^{\mathbb{N}}.$$

For a theory $\mathbb{A}$ and a functor $T \in \mathscr{F}(\mathbb{A})$, there are isomorphisms

$$H^n_{\mathbb{G}}(\mathbb{A}; H^1(-; T)) \cong H^{n+1}(\mathbb{A}; T), \qquad n \geqslant 0,$$

$$H^n_{\mathbb{G}}(\mathbb{A}; \mathrm{Der}(-; T)) \cong \begin{cases} H^{n+1}(\mathbb{A}; T), & n > 0, \\ \mathrm{Der}(\mathbb{A}; T), & n = 0. \end{cases}$$

Proof. Let $\varepsilon: \mathbb{G}_* \mathbb{A} \to \mathbb{A}$ be the cotriple resolution of $\mathbb{A}$. Since $\Omega\mathbb{G}_*\mathbb{A} \to \Omega\mathbb{A}$ is a contractible augmented simplicial object of $\mathscr{S}\!\mathit{ets}^{\mathbb{N}}$, we have a map

$$\mathbb{G}_*\mathbb{A}(\mathbb{n}, \mathbb{1}) \to \mathbb{A}(\mathbb{n}, \mathbb{1})$$

of contractible augmented simplicial sets, for $n \in \mathbb{N}$. Since

$$\mathbb{B}(\mathbb{n}, m) \approx \mathbb{B}(\mathbb{n}, \mathbb{1})^m$$

for every $n, m \in \mathbb{N}$, $\mathbb{B} \in |\mathscr{T}\!\mathit{heories}|$, the augmented simplicial category $\varepsilon: \mathbb{G}_*\mathbb{A}^{op} \to \mathbb{A}^{op}$ satisfies the conditions of Proposition 3.7. Hence there is a spectral sequence with

$$E_1^{pq} = H^q(\mathbb{G}_p\mathbb{A}^{op}; \tilde{T}) \Rightarrow H^{p+q}(\mathbb{A}^{op}; \tilde{T}),$$

where $\tilde{T}$ corresponds to T as in Lemma 4.4. Since $\mathbb{G}_p\mathbb{A}$ is a free theory, Propositions 4.6 and 4.22 imply that

$$E_1^{pq} = 0 \qquad \text{for} \quad q \geqslant 2, p \geqslant 0.$$

By Proposition 4.21 the cosimplicial abelian group E_1^{*0} is constant. Hence $E_2^{pq}=0$ for $p>0$. The available information on the terms E_*^{**} now implies that

$$E_2^{n,1}=H^{n+1}(\mathbb{A}^{op};\tilde{T})=H^{n+1}(\mathbb{A};T),\qquad n\geqslant 0.$$

But

$$E_2^{n,1}=\pi^n H^1(\mathbb{G}_*\mathbb{A}^{op};\tilde{T})=H^n_{\mathbb{G}}(\mathbb{A};H^1(-;T)).$$

Hence,

$$H^n_{\mathbb{G}}(\mathbb{A};H^1(-;T))\cong H^{n+1}(\mathbb{A};T),\qquad n\geqslant 0.$$

For a morphism of theories, $f\colon\mathbb{B}\to\mathbb{A}$, there is an exact sequence

$$0\to H^0(\mathbb{B};f^*T)\to Uf^*T(\mathbb{1})\to \mathrm{Der}(\mathbb{B};f^*T)\to H^1(\mathbb{B};f^*T)\to 0.$$

Since $U_{\mathbb{B}}f^*T(\mathbb{1})=U_{\mathbb{A}}T(\mathbb{1})$ and $H^0(\mathbb{B};f^*T)=H^0(\mathbb{A};T)$, this gives the exact sequence

$$0\to H^0(\mathbb{A};T)\to U_{\mathbb{A}}T(\mathbb{1})\to \mathrm{Der}(\mathbb{B};f^*T)\to H^1(\mathbb{B};f^*T)\to 0,$$

which in turn implies exactness of the sequence of cosimplicial abelian groups

$$0\to H^0(\mathbb{A};T)\to U_{\mathbb{A}}T(\mathbb{1})\to \mathrm{Der}(\mathbb{G}_*\mathbb{A};T)\to H^1(\mathbb{G}_*\mathbb{A};T)\to 0.$$

Since the first two entries are constant,

$$\pi^n\,\mathrm{Der}(\mathbb{G}_*\mathbb{A};T)\cong\pi^n H^1(\mathbb{G}_*\mathbb{A};T),\qquad n>0.$$

Whence the second part of theorem.

5. Towards Some Calculations

The cohomology of some "classical" theories, such as those of groups, monoids, nonassociative monoids, commutative nonassociative nonoids, and G-sets for a monoid G, turn out to be successfully calculable. Calculations rely on the fact that in [6] Eilenberg and Mac Lane have built very handy projective resolutions of functors $(I_{\mathbb{A}})_{\mathrm{ab}}$ from Section 4, under the name of freely acyclic constructions.

A. *Cohomology of Theories of Groups and of Monoids*

Recall that for a group or monoid G, internal to a category $\mathbb{K}$, BG

denotes the simplicial object in $\mathbb{K}$ whose nth component is G^n and whose face and degeneracy operators are, in element notation,

$$d_i(x_1, ..., x_n) = \begin{cases} (x_2, ..., x_n), & i=0, \\ (x_1, ..., x_i x_{i+1}, ..., x_n), & 0<i<n, \\ (x_1, ..., x_{n-1}), & i=n, \end{cases}$$

$$s_i(x_1, ..., x_n) = (x_1, ..., x_i, 1, x_{i+1}, ..., x_n), \qquad 0 \leqslant i \leqslant n.$$

Let $\mathbb{A}$ be the theory of groups (resp. monoids). The identity functor $1_{\mathbb{A}}: \mathbb{A} \to \mathbb{A}$ preserves products, hence it defines a model of the theory $\mathbb{A}$ in the category $\mathbb{A}$, i.e., an internal group (resp. monoid) in $\mathbb{A}$. The underlying object of this internal group (resp. monoid) is, clearly, $\mathbb{1}$; consequently we obtain the simplicial object $B\mathbb{1}$ in $\mathbb{A}$. Obviously $\mathbb{A}^b$ is the category of groups (resp. monoids), $\mathscr{Ab}(\mathbb{A}^b)$ is the category of abelian groups, and $(\cdot)_{\mathrm{ab}}: \mathbb{A}^b \to \mathscr{Ab}(\mathbb{A}^b)$ coincides with the one-dimensional integral homology functor.

PROPOSITION 5.1. *Let $\mathbb{A}$ be the theory of groups or of monoids and let $T: \mathbb{A}^{op} \to \mathscr{Ab}$ be any functor. Then there are natural isomorphisms*

$$H^n(\mathbb{A}; T) \cong \pi^{n+1}(TB\mathbb{1}), \qquad n \geqslant 0,$$

where the cohomotopy groups of the cosimplicial abelian group obtained from $B\mathbb{1}$ by applying T componentwise are on the right.

Proof. Given a group or a monoid G, denote by $C_*(G)$ the complex of chains of the simplicial set BG. As is widely known, $H_*(C_*(G)) = H_*(G; \mathbb{Z})$ is the integral homology of G (see [14]). When G is free, $H_n(C_*(G)) = 0$ for $n > 1$. Let $C_*^+(G)$ be the nonnegative chain complex with $C_n^+(G) = C_{n+1}(G)$, $n \geqslant 0$, and whose boundary operator is that of $C_*(G)$. Then for free G we have

$$H_n C_*^+(G) = \begin{cases} 0, & n>0 \\ G_{\mathrm{ab}}, & n=0. \end{cases}$$

By varying G over $\mathbb{A}^{op}$ we obtain a resolution of the functor $(I_{\mathbb{A}})_{\mathrm{ab}}$ from $\mathscr{F}(\mathbb{A})$ (here we identify $\mathbb{A}^{op}$, as before, with the category of finitely generated free groups (resp. monoids)).

Since $C_n^+(G) = \mathbb{Z}[G^{n+1}]$ and $G^{n+1} = \mathbb{A}^{op}(\mathbb{m} + \mathbb{1}, G)$ for $G \in |\mathbb{A}^{op}|$, Proposition 2.5 implies that

$$C_*^+ \to (I_{\mathbb{A}})_{\mathrm{ab}}$$

is a projective resolution in the category $\mathscr{F}(\mathbb{A})$. Hence

$$H^n(\mathbb{A}; T) = \mathrm{Ext}^*_{\mathscr{F}(\mathbb{A})}((I_{\mathbb{A}})_{\mathrm{ab}}, T) = H^* \mathrm{Hom}_{\mathscr{F}}(C^+_*, T) = \pi^{n+1}(TB\mathbb{1})$$

for any $T: \mathbb{A}^{op} \to \mathscr{A}\ell$. The last equality follows from

$$\mathrm{Hom}_{\mathscr{F}}(C^+_i, T) = T(\mathbb{m} + \mathbb{1}), \qquad i \in \mathbb{N}.$$

PROPOSITION 5.2. *Let $\underline{\mathrm{gr}}_k$ (resp. $\underline{\mathrm{mon}}_k$) be the category of free groups (resp. monoids) of rank not exceeding k, and $\underline{\mathrm{gr}}_\infty$ (resp. $\underline{\mathrm{mon}}_\infty$) be the category of free groups (resp. monoids) of finite rank. For any functor $T: \underline{\mathrm{gr}}_\infty \to \mathscr{A}\ell$ (resp. $\underline{\mathrm{mon}}_\infty \to \mathscr{A}\ell$), the restriction natural transformations*

$$H^i(\underline{\mathrm{gr}}_\infty : \tilde{T}) \to H^i(\underline{\mathrm{gr}}_k; \tilde{T}),$$

$$(\textit{resp. } H^i(\underline{\mathrm{mon}}_\infty; \tilde{T}) \to H^i(\underline{\mathrm{mon}}_k; \tilde{T}))$$

are isomorphisms for $i \leqslant k-1$, where the bifunctor $\tilde{T}$ is defined by

$$\tilde{T}(X, Y) = \mathrm{Hom}(X, TY).$$

Proof. Consider the group case only, as the monoids are dealt with in absolutely the same way. By Corollary 3.11 we have isomorphisms

$$H^*(\underline{\mathrm{gr}}_\infty, \tilde{T}) \cong \mathrm{Ext}^*_{\mathscr{F}}((\cdot)_{\mathrm{ab}}, T)$$

$$H^*(\underline{\mathrm{gr}}_k; \tilde{T}) \cong \mathrm{Ext}^*_{\mathscr{F}(k)}((\cdot)_{\mathrm{ab}}, T),$$

where $\mathscr{F}$ (resp. $\mathscr{F}(k)$) is the category of functors from $\underline{\mathrm{gr}}_\infty$ (resp. $\underline{\mathrm{gr}}_k$) to $\mathscr{A}\ell$. From the proof of Proposition 5.1 one can deduce that C^+_* is a projective resolution of the functor $(\cdot)_{\mathrm{ab}}$ in $\mathscr{F}$ and is also a resolution in $\mathscr{F}(k)$ up to dimension $k-1$. Furthermore

$$\mathrm{Hom}_{\mathscr{F}}(C^+_i, T) = T(\mathbb{i} + \mathbb{1}), \qquad i \geqslant 0,$$

$$\mathrm{Hom}_{\mathscr{F}(k)}(C^+_i, T) = T(\mathbb{i} + \mathbb{1}), \qquad 0 \leqslant i \leqslant k-1.$$

This readily implies the proposition.

B. *The Theory of Nonassociative Monoids*

Let $\mathbb{H}$ denote the theory of nonassociative monoids, that is, the theory whose models are determined by universal algebras M with a binary operation (multiplication) with a two-sided unit $1 \in M$. Then $\mathscr{A}\ell(\mathbb{H}^b)$ is the category of abelian groups, while the abelianization functor assigns to a nonassociative monoid M the abelian group M_{ab} with generators $\langle a \rangle$, $a \in M$, and defining relations

$$\langle 1 \rangle = 0, \qquad \langle ab \rangle = \langle a \rangle + \langle b \rangle.$$

For any nonassociative monoid M, define the chain complex $C_*(M)$ by $C_n(M)=0$ for $n\neq 0, 1$, while the group $C_0(M)$ (resp. $C_1(M)$) is the abelian group generated by symbols $[a]$ (resp. $[a, b]$) for $a, b\in M$, with defining relations $[1]=0$ (resp. $[1, a]=0=[a, 1]$, $a\in M$); the boundary operator $\partial\colon C_1(M)\to C_0(M)$ is given by

$$\partial([a, b])=[a]-[ab]+[b], \qquad a, b\in M.$$

It is proved in [6] that if M is a free object in $\mathbb{H}^b$, then

$$H_i(C_*(M))=\begin{cases}0, & i\neq 0,\\ M_{\mathrm{ab}}, & i=0.\end{cases}$$

Let M vary over H^{op} to obtain a resolution of the functor $(I_{\mathbb{H}})_{\mathrm{ab}}$ in the category $\mathscr{F}(\mathbb{H})$:

$$C_*(\cdot)\to (I_{\mathbb{H}})_{\mathrm{ab}}.$$

Let us show that this is a projective resolution in $\mathscr{F}(H)$. To this end, note that the functors given by

$$M\mapsto \mathbb{Z}[M], \qquad M\mapsto \mathbb{Z}[M^2], \qquad M\in\mathbb{H}^\infty,$$

are projective objects in $\mathscr{F}(\mathbb{H})$; since

$$M=H^{op}(\mathbb{1}, M), \qquad M^2=H^{op}(\mathbb{2}, M)$$

and one can apply Proposition 2.5. From the definitions follows the existence of epimorphisms $\mathbb{Z}[M]\to C_0(M)$ and $\mathbb{Z}[M^2]\to C_1(M)$. These homomorphisms have functorial sections

$$s_0\colon C_0(M)\to \mathbb{Z}[M], \qquad s_1\colon C_1(M)\to\mathbb{Z}[M^2]$$

given by the equalities

$$s_0[a]=[a]-[1]$$
$$s_1[a, b]=[a, b]-[1, b]-[a, 1]+[1, 1],$$

$a, b\in M$. Hence the functors $C_0, C_1\colon \mathbb{H}^{op}\to\mathscr{Ab}$ are projective objects in $\mathscr{F}(\mathbb{H})$. These considerations imply the following propositions:

PROPOSITION 5.3. *Let $\mathbb{H}$ be the theory of nonassociative monoids and $T\colon \mathbb{H}^{op}\to\mathscr{Ab}$ be any functor. Denote by $\mu\colon \mathbb{2}\to\mathbb{1}$ the multiplication in $\mathbb{H}$ and by $e\colon \mathbb{0}\to\mathbb{1}$ the unit. Then*

$$H^i(\mathbb{H}; T)=\begin{cases}0, & i\geqslant 2,\\ \operatorname{Coker}\delta, & i=1,\\ \operatorname{Ker}\delta, & i=0,\end{cases}$$

where

$$\delta\colon \operatorname{Ker}(Te) \to \operatorname{Ker}(T\mathbb{2} \xrightarrow{\binom{Te_1}{Te_2}} T\mathbb{1} \oplus T\mathbb{1})$$

is induced by $T\mu$ *while* e_1 *and* e_2 *are the morphisms*

$$e_1 : \mathbb{1} = \mathbb{0} \times \mathbb{1} \xrightarrow{e \times \mathbb{1}} \mathbb{1} \times \mathbb{1} = \mathbb{2}$$

$$e_2 : \mathbb{1} = \mathbb{1} \times \mathbb{0} \xrightarrow{\mathbb{1} \times e} \mathbb{1} \times \mathbb{1} = \mathbb{2}.$$

PROPOSITION 5.4. *Let* $\mathbb{H}_k$ *be the category of free nonassociative monoids of rank not exceeding k and* $\mathbb{H}_\infty$ *the category of all free finitely generated nonassociative monoids. Then for any functor* $T\colon \mathbb{H}_\infty \to \mathcal{Ab}$, *the restriction homomorphisms in cohomology groups of categories*

$$H^*(\mathbb{H}_\infty; \tilde{T}) \to H^*(\mathbb{H}_k; \tilde{T})$$

are isomorphisms for any $k \geqslant 2$, *where the bifunctor* T *is defined by* $\tilde{T}(X, Y) = \operatorname{Hom}(X, TY)$, $X, Y \in \mathbb{H}_\infty$.

C. *Theory of Commutative Nonassociative Monoids*

PROPOSITION 5.5. *Let* $\mathbb{C}\mathrm{omm}$ *be the theory of commutative nonassociative monoids and* $T\colon \mathbb{C}\mathrm{omm}^{op} \to \mathcal{Ab}$ *be any functor. Then*

$$H^n(\mathbb{C}\mathrm{omm}; T) \cong H^{n+2}(\mathbb{C}\mathrm{omm}; T) \qquad \textit{for} \quad n \geqslant 4.$$

Proof. Let A be some commutative nonassociative monoid. Consider the chain complex $C_*(A)$ with $C_0(A)$ and $C_1(A)$ as in the previous example, while for $n \geqslant 2$,

$$C_2(A) = C_1(A), \qquad C_n(A) = C_1(A) \oplus C_0(A), n \geqslant 3.$$

The boundary homomorphism is defined by

$$\begin{aligned}
&\partial_0[x, y] = [x] - [xy] + [y] \\
&\partial_1[x, y] = [x, y] - [y, x] \\
&\partial_k[x, y] = [x, y] + [y, x], \qquad \partial_k[a] = [a, a], \qquad && n \geqslant 2 \text{ even} \\
&\partial_n[x, y] = [x, y] - [y, x], \qquad \partial_n[a] = [a, a] - 2[a], \qquad && n \geqslant 3 \text{ odd.}
\end{aligned}$$

In [6] it is proved that for free A's one has

$$H_i(C_*(A)) = \begin{cases} A_{\mathrm{ab}}, & i = 0 \\ 0, & i > 0. \end{cases}$$

Hence $C_*(\cdot)$ is a resolution of the functor $(I_{\mathbb{C}\mathrm{omm}})_{\mathrm{ab}}$ in the category $\mathscr{F}(\mathbb{C}\mathrm{omm})$. As in Proposition 5.3, one can prove that the functors $C_0(\cdot)$ and $C_1(\cdot)$ are projective. This gives the desired periodic resolution of the object $(I_{\mathbb{C}\mathrm{omm}})_{\mathrm{ab}}$.

Moreover an analogue of Proposition 5.4 holds with $\mathbb{H}$ replaced by $\mathbb{C}\mathrm{omm}$.

D. *Theories of G-Sets*

Let G be a monoid, and denote by $\mathbb{G}$ the theory of left G-sets, i.e., $\mathbb{G}^b \simeq G\text{-}\mathscr{S}et\mathscr{s}$. It is clear that

$$\mathscr{A}\ell(\mathbb{G}^b) \simeq G\text{-}\underline{\mathrm{mod}},$$

while the functor $(\cdot)_{\mathrm{ab}}\colon \mathbb{G}^b \to G\text{-}\underline{\mathrm{mod}}$ is given by $X \mapsto \mathbb{Z}[X]$, where the structure of a G-module on $\mathbb{Z}[X]$ is inherited from the action of G on X.

In the category of G-sets consider the augmented simplicial object $\varepsilon\colon B_*(G; X) \to X$, where $B_n(G, X) = G^{n+1} \times X$, $n \geqslant 0$, and

$$\varepsilon(g, x) = gx,$$

$$d_i(g_0, g_1, ..., g_n, x) = \begin{cases} (g_0, ..., g_i g_{i+1}, ..., g_n, x), & 0 \leqslant i < n, \\ (g_0, ..., g_{n-1}, g_n, x), & i = n, \end{cases}$$

$$s_i(g_0, ..., g_n, x) = (g_0, ..., g_i, 1, g_{i+1}, ..., g_n, x), \qquad 0 \leqslant i \leqslant n,$$

and the action of G on $B_n(G, X)$ is given by

$$g(g_0, g_1, ..., g_n, x) = (gg_0, g_1, ..., g_n, x).$$

It is easy to show that $\varepsilon\colon B_*(G, X) \to X$ is contractible in the category of sets; in fact the maps

$$h\colon X \to G \times X, \qquad h_n\colon B_n(G; X) \to B_{n+1}(G; X), n \geqslant 0,$$

given by

$$h(x) = (1, x), \qquad h_n(g_0, ..., g_n, x) = (1, g_0, ..., g_n, x)$$

give a contraction for it. Hence

$$\mathbb{Z}\varepsilon\colon C_*(B_*(G; X)) \to \mathbb{Z}X$$

is a contractible augmented chain complex, where

$$C_*\colon (\text{simplicial sets}) \to (\text{chain complexes})$$

is the functor giving chains with integer coefficients. Let $\Lambda = \mathbb{Z}[G]$ be the semigroup ring of G. Then one can show easily that there is an isomorphism of G-modules

$$C_n B_*(G; X) \cong \bigoplus_{G^n} \Lambda[X], \qquad \text{for every } n. \tag{5.6}$$

Since $X = \mathbb{G}^{op}(\mathbb{1}, X)$ for $X \in \mathbb{G}^{op}$, Proposition 2.5 implies that

$$Z\varepsilon_* : C_*(B_*(G; -)) \to (\cdot)_{\mathrm{ab}}$$

is the projective resolution in the category of functors from $\mathbb{G}^{op}$ to G-mod (i.e., in $\mathscr{F}(\mathbb{G})$). Let T: $\mathbb{G}^{op} \to G$-mod be an arbitrary functor. Then

$$\begin{aligned} H^*(\mathbb{G}; T) &= \operatorname{Ext}^*_{\mathscr{F}(\mathbb{G})}((\cdot)_{\mathrm{ab}}, T) \\ &= H^*(\operatorname{Hom}_{\mathscr{F}(\mathbb{G})}(C_n B_*(G; -), T)) \cong H^*(\mathbb{G}; T(\mathbb{1})), \end{aligned}$$

where the last group is that of Eilenberg–Mac Lane cohomology of the monoid G in the obvious G–G-bimodule $T(1)$, since (5.6) implies that

$$\operatorname{Hom}_{\mathscr{F}(G)}(C_n B_*(G; -), T) = \mathscr{S}et_s(G^n, T(\mathbb{1})).$$

Hence we have arrived at

PROPOSITION 5.7. *Let G be a monoid and $\mathbb{G}$ the theory of G-sets. For any functor T: $\mathbb{G}^{op} \to G$-mod there are isomorphisms*

$$H^*(\mathbb{G}; T) \cong H^*(G; T(\mathbb{1})),$$

where on the right are Eilenberg–Mac Lane cohomology groups of the monoid G with coefficients in the G–G-bimodule $T(\mathbb{1})$, with left action determined by T's taking values in G-mod, while the right action is T applied to the action of G on $\mathbb{1}$ via $\operatorname{Hom}_{\mathbb{G}}(\mathbb{1}, \mathbb{1}) \cong G$.

PROPOSITION 5.8. *For a monoid G, denote by $\mathbb{G}_\infty$ (resp. $\mathbb{G}_k$) the category of free finitely generated G-sets (resp. those of rank not exceeding k). Then for any functor T: $\mathbb{G}_\infty \to G$-mod the restriction homomorphism in Hochschild–Mitchell cohomology*

$$H^*(\mathbb{G}_\infty; \tilde{T}) \to H^*(\mathbb{G}_k; \tilde{T})$$

is an isomorphism for $k \geqslant 1$, where $\tilde{T}$ is the bifunctor

$$\tilde{T}(X, Y) = \operatorname{Hom}_G(X, TY), \qquad X, Y \in \mathbb{G}_\infty.$$

6. Further Areas of Investigation and Open Problems

1. In Section 2 (Theorem A) the isomorphisms

$$H^*(R; T(R)) \cong \operatorname{Ext}^*_{\mathscr{F}(R)}(I, T)$$

were established, where $\mathscr{F}(R)$ is the category of all functors from the category $\underline{M}_R$ of finitely generated free left R-modules to the category of all R-modules, $I\colon \underline{M}_R \to R\text{-}\underline{\text{mod}}$ is the embedding, and $T\colon \underline{M}_R \to R\text{-}\underline{\text{mod}}$ is an additive functor, while H^* denotes the Mac Lane cohomology. Since the category of additive functors from $\underline{M}_R$ to R-$\underline{\text{mod}}$ is equivalent to the category of R–R-bimodules, the embedding

$$(\text{additive functors}) \subset (\text{all functors})$$

induces homomorphisms

$$\operatorname{Ext}^i_{R\text{-}R}(R, T(R)) \to \operatorname{Ext}^i_{\mathscr{F}(R)}(I, T).$$

These are isomorphisms for $i = 0, 1$. And if the additive group of R is torsion-free, isomorphism holds also in dimension 2 (this follows from 3.6 of [19]).

But between additive functors and all functors there are the so-called quadratic, cubical, and other functors. It would be interesting to find out what the corresponding Ext groups will give. In [18], the following was proposed.

Conjecture. Let $\mathscr{P}(R, n)$ be the full subcategory of $\mathscr{F}(R)$ consisting of those functors $T\colon \underline{M}_R \to R\text{-}\underline{\text{mod}}$ with Eilenberg–Mac Lane degree $\leqslant n$ (cf. [7]), i.e., $T_{n+1} = 0$ for the $(n+1)$st cross-effects. Suppose that R has a torsion-free additive group. Then

$$\operatorname{Ext}^i_{\mathscr{P}(R,n)}(I, T) \to \operatorname{Ext}^i_{\mathscr{F}(R,n)}(I, T)$$

is an isomorphism for $n \leqslant 2i$ for any additive functor $T\colon \underline{M}_R \to R\text{-}\underline{\text{mod}}$.

This conjecture is true for $n = 1, 2, 3$ (see [18]).

2. One can define Mac Lane homology $H_*(R; M)$ of a ring R with coefficients in a bimodule M by replacing the functor Hom by $\otimes$ in Definition 2.1; for homology groups, a dual of Theorem A is valid. Hence according to [18] there exists a natural transformation

$$\theta_*\colon K^s_*(R) \to H_*(R; R)$$

from Waldhausen's stable K-theory to the Mac Lane homology. Is θ_* an isomorphism? This is so in all the cases when the values of stable K-theory are known to us.

3. There are many important algebraic theories $\mathbb{A}$ with trivial cohomology for coefficients in any functor $T: \mathbb{A}^{op} \to \mathcal{Ab}(\mathbb{A}^b)$ because the category $\mathcal{Ab}(\mathbb{A}^b)$ is trivial, for example, the theory of rings with unit. This signifies that it would be desirable to find still more general coefficients for our cohomology. One well-known general approach to this, in the spirit of Barr and Beck and Quillen [2, 19], suggests internal abelian groups of the comma category $\mathcal{Theories}/\mathbb{A}$ as coefficients. For the Hochschild–Mitchell cohomology, Baues and Wirsching generalize the $H^*(\mathbb{A}; D)$, the cohomology of a category $\mathbb{A}$ with coefficients D, by the so-called natural systems [3]. Namely, the natural system D consists of the family (D_f) indexed by morphisms f of $\mathbb{A}$ and the families of homomorphisms of abelian groups $(g^*: D_f \to D_{fg})_{(f,g) \in T_2(A)}$, $(f_*: D_g \to D_{fg})_{(f,g) \in T_2(\mathbb{A})}$ indexed by the set $T_2(\mathbb{A})$ of composable pairs of morphisms of $\mathbb{A}$. These are required to satisfy certain natural equalities (see [3]). Now it can be shown that for any theory $\mathbb{A}$ there is an equivalence

$$\mathcal{Ab}(\mathcal{Theories}/\mathbb{A}) \simeq (\text{natural systems } D \text{ on } \mathbb{A}^{op} \text{ satisfying (6.1)}),$$

where the condition (6.1) is

$$\begin{aligned}
&\text{for any morphism } f: x \to y_1 \times y_2 \times \cdots \times y_n \text{ the} \\
&\text{homomorphism } (p_1^*, ..., p_n^*): D_f \to D_{p_1 f} \times \cdots \times D_{p_n f} \\
&\text{is an isomorphism, where } p_i: y_1 \times \cdots \times y_n \to y_i, \\
&1 \leqslant i \leqslant n, \text{ are the projections.}
\end{aligned} \tag{6.1}$$

Any bifunctor $D: \mathbb{A} \times \mathbb{A}^{op} \to \mathcal{Ab}$ can be viewed as a natural system by setting $D_{f: X \to Y} = D(X, Y)$; in particular $\mathcal{F}(\mathbb{A})$ is a full subcategory of $\mathcal{Ab}(\mathcal{Theories}/\mathbb{A})$.

Accordingly there are two ways to define cohomology groups $H^*(\mathbb{A}; M)$ for $M \in |\mathcal{Ab}(\mathcal{Theories}/\mathbb{A})|$. The first one relies on the cotriple cohomology of Barr and Beck,

$$H^n(\mathbb{A}; M) = H^{n-1}_{\mathbb{G}}(\mathbb{A}; \mathrm{Der}(-, M)), \qquad n \geqslant 2,$$

where $\mathbb{G}$ is the comonad from Theorem C of Section 4. The second way follows Baues and Wirsching's approach [3] and gives $H^n(\mathbb{A}; M) = H^n(\mathbb{A}^{op}; \tilde{M})$, where $\tilde{M}$ is the natural system, satisfying (6.1), which corresponds to M under the equivalence mentioned there. Theorem C says that when coefficients are restricted to $\mathcal{F}(\mathbb{A})$ these approaches lead to the same result. Will this be the case for any coefficients from $\mathcal{Ab}(\mathcal{Theories}/\mathbb{A})$? It can be shown that this is so in dimensions $\leqslant 2$, and probably the answer is affirmative in all dimensions. By the way, another motivation for widening the area of coefficients is that, if one wants to consider extensions of type $(4.16)_n$ with nilpotent groups replaced by merely solvable ones, one encounters precisely these general coefficients which do not arise from any object of $\mathscr{F}$.

4. For algebraic theories $\mathbb{A}$ let us identify the category $\mathbb{A}^{op}$ with the category of finitely generated free models of $\mathbb{A}$, and let $\mathbb{A}_r^{op}$ be the full subcategory of $\mathbb{A}^{op}$ consisting of models that are free on sets of cardinality $\leqslant r$. Call the theory $\mathbb{A}$ stable if for every i there exists an r_0 such that the restriction homomorphisms

$$H^i(\mathbb{A}^{op}, D) \to H^i(\mathbb{A}_r^{op}; D_r) \tag{6.2}$$

are isomorphisms for every $r \geqslant r_0$ and every bifunctor $D: \mathbb{A} \times \mathbb{A}^{op} \to \mathscr{A}\ell$ which preserves products in the first variable (here D_r is the restriction of D to $\mathbb{A}_r$, and cohomology groups are those of the Hochschild–Mitchell type).

For a stable theory $\mathbb{A}$ denote by $f(\mathbb{A}, i)$ the smallest r_0 such that (6.2) are isomorphisms for $r \geqslant r_0$. By Propositions 3.12, 5.2, 5.4, and 5.8, theories of R-modules R for any ring R, of groups $\underline{\mathrm{gr}}$, of monoids $\underline{\mathrm{mon}}$, of nonassociative monoids $\mathbb{H}$, of commutative nonassociative monoids $\mathbb{C}\mathrm{omm}$, and of G-sets $\mathbb{G}$ for any monoid G, are stable. Moreover, we showed that

$$f(R, n) \leqslant 2^n, \qquad f(\underline{\mathrm{gr}}, n) \leqslant n+1, \qquad f(\underline{\mathrm{mon}}, n) \leqslant n+1,$$
$$f(\mathbb{H}, n) \leqslant 2, \qquad f(\mathbb{C}\mathrm{omm}, n) \leqslant 2, \qquad f(\mathbb{G}, n) \leqslant 1.$$

We know no answer to the following questions:

Is the estimate $f(R, n) \leqslant 2^n$ the best one? (It seems unlikely.)

Do there exist unstable theories?

It is also interesting to characterize those theories for which $f(\mathbb{A}, n)$ is a bounded function of n.

5. In Proposition 2.21 we have calculated the groups

$$H^2(R; \Lambda_R^2), \qquad H^2(R; S_R^2)$$

for commutative rings R, especially those with ${}_2R = 0$. Construct explicitly the extensions

$$\mathscr{H}om_R(I, \Lambda_R^2) + \to ? \to \underline{M}_R$$
$$\mathscr{H}om_R(I, S_R^2) + \to ? \to \underline{M}_R$$

corresponding to the elements of $H^2(R; \Lambda_R^2) \cong H^0(R; \overline{R/2R})$, $H^2R; S_R^2) \cong H^1(R; \overline{R/2R})$.

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