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ONE MORE NOTION OF RELATIVE BOOLEANNESS

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Abstract. We investigate a condition on geometric morphisms which is strictly intermediate between relative Booleanness conditions previously studied by Kock and Reyes.

Introduction

Intuitively, the concept of relative Booleanness of a geometric morphism $F : \mathbf{E} \to \mathbf{B}$ expresses the property of the $\mathbf{B}$-topos determined by F "being no less Boolean than $\mathbf{B}$ itself". One such condition on F (see the condition (Reg) below) has been introduced by Anders Kock in [3]. Later in a correspondence between him, Peter Johnstone and the author several other possible conditions emerged. At least one of them (see (Clp) below) is not equivalent to (Reg) – in fact, strictly stronger than it. These two conditions have been investigated in [5].

In this note we consider a third condition which lies strictly between these two. We will give several equivalent forms of it, trying to show that it is also a natural one to consider.

All these notions have been discussed by the author several times with Peter Johnstone, Anders Kock, Gonzalo Reyes, Marta Bunge, Jonathon Funk and Richard Squire. The author is grateful to them for sharing their insights and ideas.

Most clear motivation for considering all these conditions comes from a particular case of a geometric morphism $\mathrm{Shv}(X) \to \mathrm{Shv}(Y)$ induced by a continuous map $f : X \to Y$ between topological spaces. In this case relative Booleanness of the geometric morphism is closely related to fibrewise discreteness of f; thus if a continuous map like f can be viewed as "a family of spaces continuously parametrized by Y", the fibrewise discrete maps correspond to families of *discrete* spaces.

In the theory of locales, such families of spaces are systematically studied by replacing them with the corresponding internal locales in $\mathrm{Shv}(Y)$. Thus one might say that fibrewise discrete maps $f : X \to Y$ must give rise to discrete internal locales in $\mathrm{Shv}(Y)$.

One can hardly think of a question as simple as "when is a space discrete"? Nevertheless it turns out that in the intuitionistic universes like $\mathrm{Shv}(Y)$ this question becomes quite subtle.

For a logician, discreteness of a space means in the first place that the lattice of its open sets is a Boolean algebra (although this condition is sufficient only under some separation conditions). But this is certainly not the right notion when the logic one relies upon is non-classical, since then one cannot even prove that a one-point space is discrete in the above sense!

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At the first thought, the most sensible approach to this problem is this: investigate those properties of geometric morphisms $F : \mathbf{E} \to \mathbf{B}$ which

a) reduce to Booleanness of $\mathbf{E}$ when $\mathbf{B}$ is Boolean;
b) are pullback stable.

While the first condition is clear (it just says that our notion extends the classical one conservatively), the second seems a natural requirement in light of the word "fibrewise". However, as Peter Johnstone revealed in the aforementioned correspondence with Anders Kock and the author, that there are no properties meeting both of these requirements: he recalled the remark by Dona Strauss that for a complete Boolean algebra B, in the pullback $\mathrm{Shv}(B \otimes B) \to \mathrm{Shv}(B)$ of $\mathrm{Shv}(B) \to \mathbf{Set}$ along itself, the topos $\mathrm{Shv}(B \otimes B)$ (equivalently, the frame $B \otimes B$) is not Boolean as soon as B is not atomic.

Here is one other approach to defining at least some possible non-classical notions of discrete space. Classically, discreteness of a space X can be expressed in either of two equivalent ways: "the only dense subspace of X is X itself" or, "every subspace of X is closed". One can then relativize these conditions by using relativized notions of dense and closed subspace – notions of *strongly dense* and *weakly closed* subspace introduced by Peter Johnstone in [2].

As shown in [5], a geometric morphism $F : \mathbf{E} \to \mathbf{B}$ satisfies the condition (Reg) if and only if the internally defined object of those subtoposes of $\mathbf{E}$ which are strongly dense relative to F reduces to the terminal object (its only element corresponding to $\mathbf{E}$ itself, which is of course strongly dense by trivial reasons).

We are going to investigate those F for which every subtopos of $\mathbf{E}$ is weakly closed. We will denote this condition by (Clop). As (Reg) above, it must be stated internally. We first note that it suffices to require weak closedness of *open* subtoposes only. Next the object of open subtoposes of $\mathbf{E}$ can be naturally identified with its subobject classifier Ω; those subtoposes which are weakly closed relative to F correspond to a certain subobject of Ω, and the condition (Clop) means that this subobject is in fact the whole Ω.

We will give some equivalent forms of (Clop). It will be shown that (Clop) implies (Reg) and is implied by (Clp). We will then give examples showing that none of these implications are reversible.

Notational conventions. we will fix throughout a geometric morphism $F : \mathbf{E} \to \mathbf{B}$. Subobject classifier of $\mathbf{E}$ will be denoted by Ω, that of $\mathbf{B}$ – by $\mathscr{Q}$. The map $F^*\mathscr{Q} \to \Omega$ classifying $F^*(\mathbf{true}) : F^*1 \to F^*\mathscr{Q}$ will be denoted by τ; its image will be denoted by $\mathscr{Q}_F$. Occasionally we will identify entities of type $\mathscr{Q}$ with their images in $\mathscr{Q}_F$ under τ. Thus for example given $\beta : I \to \mathscr{Q}$ in $\mathbf{B}$ and $\varphi : F^*I \to \Omega$ in $\mathbf{E}$, we might write $\beta \wedge \varphi$ as a shorthand for the composite

$$F^*I \xrightarrow{(\tau F^*\beta, \varphi)} \Omega \times \Omega \xrightarrow{\wedge} \Omega.$$

To facilitate parsing of complicated compound expressions in Heyting algebras, we will interchangeably denote implication by $a \Rightarrow b$, $a \to b$, b^a and $\neg_b a$.

1. Recollections on strong density and weak closedness

Let us begin by recalling the well known correspondence between subtoposes of a topos $\mathbf{E}$ and *nuclei* – closure operators[1] $j : \Omega \to \Omega$. This correspondence enables one to consider the object of nuclei $\mathscr{N}_{\mathbf{E}} \rightarrowtail \Omega^{\Omega}$ as the *object of subtoposes* of $\mathbf{E}$: for any object X of $\mathbf{E}$, there is a one-to-one correspondence between morphisms $X \to \mathscr{N}_{\mathbf{E}}$ and subtoposes of the slice topos $\mathbf{E}/X$. Note however that this correspondence is order-reversing with respect to the subtopos inclusion and the natural internal order on Ω^{Ω}. It is well known that $\mathscr{N}_{\mathbf{E}}$ is actually a subframe in Ω^{Ω}, and in particular the lattice of subtoposes is a *coframe*. The unique frame homomorphism $! : \Omega \to \mathscr{N}_{\mathbf{E}}$ from the initial frame Ω can be characterized as follows: for a map $\varphi : X \to \Omega$ classifying a subobject $X' \rightarrowtail X$, the composite $!\varphi$ corresponds to the closed subtopos of $\mathbf{E}/X$ determined by X'. It is equally well known that as an internal frame $\mathscr{N}_{\mathbf{E}}$ is generated by the image of ! and by its negation, i. e. the composite $\neg!$; in fact $\mathscr{N}_{\mathbf{E}}$ can be characterized as the solution of a universal problem of making the image of ! consist of complemented elements. More precisely, $! : \Omega \to \mathscr{N}_{\mathbf{E}}$ is a frame epimorphism, and for another internal frame $\mathscr{F}$, the unique frame homomorphism $\mathscr{N}_{\mathbf{E}} \to \mathscr{F}$ exists if and only if the image of $! : \Omega \to \mathscr{F}$ "consists of complemented elements", that is, the composite

$$\Omega \xrightarrow{(!,\neg!)} \mathscr{F} \times \mathscr{F} \xrightarrow{\vee} \mathscr{F}$$

factors through the top singleton $\top : 1 \to \mathscr{F}$ of $\mathscr{F}$. The following generalization of this fact is straightforward:

1.1. **Proposition.** *For any topos* $\mathbf{B}$, *the canonical localic geometric morphism* $\mathrm{Shv}(\mathscr{N}_{\mathbf{B}}) \to \mathbf{B}$ *is a monomorphism in the 2-category of toposes, i. e. there is a unique isomorphism between any two lifts of a geometric morphism* $F : \mathbf{E} \to \mathbf{B}$ *to* $\mathrm{Shv}(\mathscr{N}_{\mathbf{B}})$. *Moreover such a lift exists iff the inverse image of any* $X' \rightarrowtail X$ *under* F *is a complemented subobject of* F^*X, *and iff* $F^*\mathbf{true} : F^*1 \rightarrowtail F^*\mathscr{Q}$ *is complemented.*

For the proof, it suffices to pass from $\mathbf{B}$ to $\mathbf{B}/X$, given that inverse image $X^*(\mathscr{N}_{\mathbf{B}})$ of $\mathscr{N}_{\mathbf{B}}$ along the local homeomorphism $\mathbf{B}/X \to \mathbf{B}$ is $\mathscr{N}_{\mathbf{B}/X}$. The last statement is also clear since every $F^*X' \rightarrowtail F^*X$ is a pullback of $F^*\mathbf{true}$.

1.2. **Definition.** (cf. Johnstone [2]). A nucleus j, the corresponding sublocale Ω^j of Ω, and the corresponding subtopos $\mathbf{E}^j$ of a topos $\mathbf{E}$ will be called *strongly dense* relative to a geometric morphism $F : \mathbf{E} \to \mathbf{B}$ if one of the following (evidently equivalent) conditions holds:

- the triangle

$$\begin{array}{ccc} & \mathscr{Q} & \\ {}^{!}\swarrow & & \searrow^{!} \\ F_*(\Omega^j) & \rightarrowtail & F_*(\Omega) \end{array}$$

[1] The initial definition by Lawvere and Tierney included the requirement of preserving binary meets; however it was shown in [4] that every closure operator on Ω preserves them

commutes, where the ! denotes the unique frame homomorphisms from the initial internal frame $\mathscr{Q}$ of $\mathbf{B}$;

- the triangle

$$\begin{array}{ccc} & F^*\mathscr{Q} & \\ {\scriptstyle\tau}\swarrow & & \searrow{\scriptstyle\tau} \\ \Omega & \xrightarrow{\ j\ } & \Omega \end{array}$$

commutes;

- $\mathscr{Q}_F \leqslant \Omega^j$ as subobjects of Ω.

We will also say F-dense, or just $\mathbf{B}$-dense, if F is understood. The *object of F-dense subtoposes* $\mathscr{D}_F \rightarrowtail \mathscr{N}_{\mathbf{E}}$ is thus most easily defined in the internal language as

$$\mathscr{D}_F = \{j \in \mathscr{N}_{\mathbf{E}} | j\tau = \tau\}. \qquad (1)$$

For any subtopos $\mathbf{E}^j \rightarrowtail \mathbf{E}$, we next define the *$F$-closure* of $\mathbf{E}^j$ as the largest among the subtoposes $\mathbf{E}' \rightarrowtail \mathbf{E}$ containing $\mathbf{E}^j$ in which the latter is strongly dense relative to the composite $\mathbf{E}' \rightarrowtail \mathbf{E} \to \mathbf{B}$. Accordingly we will speak about F-closures of nuclei and sublocales of Ω. The F-closure of a nucleus j will be denoted by j^F. A subtopos (nucleus, sublocale) is *F-closed* if it coincides with its own F-closure.

It follows immediately from results of [2] and [1] that one has

1.3. **Lemma.** *For a nucleus $j : \Omega \to \Omega$, the F-closure of the corresponding sublocale Ω^j is given by*

$$\left\{\varphi \in \Omega | \forall_{\beta\in\mathscr{Q}_F}\ \varphi^{j\beta} = \varphi^{\beta}\right\}.$$

Closed sublocales are F-closed. For sublocales and subtoposes, F-closure commutes with finite joins, and any meet of F-closeds is F-closed (so dually, F-closure of nuclei commutes with finite meets, and any join of F-closed nuclei is F-closed – i. e. F-closed nuclei form a subframe $\mathscr{N}_F$ of the frame of all nuclei $\mathscr{N}_{\mathbf{E}}$. In fact the locale of $\mathscr{N}_F$ coincides with the inverse image under F of the locale of $\mathscr{N}_{\mathbf{B}}$, the frame of nuclei on $\mathscr{Q}$; that is, $\mathscr{N}_F$ fits into a pullback square in the category of toposes and geometric morphisms

$$\begin{array}{ccc} \mathrm{Shv}(\mathscr{N}_F) & \longrightarrow & \mathrm{Shv}(\mathscr{N}_{\mathbf{B}}) \\ \downarrow & \lrcorner & \downarrow \\ \mathbf{E} & \xrightarrow{\ F\ } & \mathbf{B}. \end{array} \qquad (\dagger)$$

□

1.4. **Corollary.** *Under the (well known) lattice isomorphism between nuclei and frame congruences, the congruence corresponding to the F-closure of a nucleus j is the one generated by $\beta \sim j\beta$ for $\beta \in \mathscr{Q}_F$. In other words, j^F is the smallest among those nuclei which coincide with j when restricted to $\mathscr{Q}_F$.*

2. The conditions

2.1. **Definition.** The geometric morphism $F : \mathbf{E} \to \mathbf{B}$ is said to be *nowhere dense* if the subobject $\mathscr{D}_F \rightarrowtail \mathscr{N}_\mathbf{E}$ defined in 1.2 above coincides with the bottom singleton of $\mathscr{N}_\mathbf{E}$.

It is easy to deduce from results of [5] the following

2.2. **Proposition.** *A geometric morphism F is nowhere dense if and only if the following statement in the internal language of* $\mathbf{E}$ *holds:*

$$\forall_{\varphi\in\Omega} \bigwedge_{\beta\in\mathscr{Q}_F} \neg_\beta\neg_\beta\varphi = \varphi \tag{Reg}$$

Proof. One might rephrase definition of F-denseness as follows: a sublocale of Ω is F-dense iff it contains $\mathscr{Q}_F$. Now recall that for any subobject B of Ω there is a smallest sublocale containing it; the corresponding nucleus is given by

$$j_B\varphi = \bigwedge_{b\in B} \neg_b\neg_b\varphi.$$

In particular, for any $F : \mathbf{E} \to \mathbf{B}$ there always exists a smallest F-dense subtopos of $\mathbf{E}$, and the corresponding nucleus is the $j_{\mathscr{Q}_F}$ above. Then (Reg) means precisely that this smallest subtopos is the whole $\mathbf{E}$, so there are no other F-dense subtoposes. □

2.3. **Definition.** (cf. [5]) The $\mathbf{B}$-topos determined by the geometric morphism $F : \mathbf{E} \to \mathbf{B}$ will be called $\mathbf{B}$-*valued* if the inclusion $\mathscr{Q}_F \rightarrowtail \Omega$ is an isomorphism, i. e. $\tau : F^*\mathscr{Q} \to \Omega$ is epi.

Again it is straightforward to deduce from results of [5] the following

2.4. **Proposition.** *The* $\mathbf{B}$*-topos* $F : \mathbf{E} \to \mathbf{B}$ *is* $\mathbf{B}$*-valued iff the following statement in the internal language of* $\mathbf{E}$ *holds:*

$$\forall_{\varphi\in\Omega} \bigvee_{\beta\in\mathscr{Q}_F} \varphi \Leftrightarrow \beta = \mathbf{true} \tag{Clp}$$

2.5. **Remark.** It is easy to prove that (Clp) is also equivalent to

$$\forall_{\varphi,\psi\in\Omega}\varphi \to \psi = \bigvee_{\beta\in\mathscr{Q}_F} \varphi \to \beta \wedge \beta \to \psi \tag{Clp$'$}$$

It is now clear that (Clp) implies (Reg): if $\mathscr{Q}_F$ coincides with the whole Ω, a fortiori the smallest sublocale of Ω containing $\mathscr{Q}_F$ also does.

Our next condition is intermediate in strength between these two:

2.6. **Definition.** The $\mathbf{B}$-topos determined by $F : \mathbf{E} \to \mathbf{B}$, and F itself, is called *totally weakly closed* (twc for short) if the subframe $\mathscr{N}_F$ of $\mathscr{N}_{\mathbf{E}}$ is in fact the whole $\mathscr{N}_{\mathbf{E}}$ – that is, if it holds internally in $\mathbf{E}$ that every subtopos of it is weakly closed relative to F.

Alternatively, twc might be formulated as follows: F is twc iff the commutative square of geometric morphisms

$$\begin{array}{ccc} \mathrm{Shv}(\mathscr{N}_{\mathbf{E}}) & \longrightarrow & \mathrm{Shv}(\mathscr{N}_{\mathbf{B}}) \\ \downarrow & & \downarrow \\ \mathbf{E} & \xrightarrow{F} & \mathbf{B} \end{array} \qquad (*)$$

is pullback. In view of 1.1 and (†) this implies easily

2.7. **Proposition.** *A geometric morphism $F : \mathbf{E} \to \mathbf{B}$ is twc iff for any $G : \mathbf{E}' \to \mathbf{E}$, complementedness of the subobject $G^*F^*\mathbf{true} : G^*F^*1 \rightarrowtail G^*F^*\mathscr{2}$ implies complementedness of all $G^*(m)$ for all monos m in $\mathbf{E}$.*

In the internal language, twc can be formulated in the following way:

$$\forall_{\varphi\in\Omega}\forall_{j\in\mathscr{N}_{\mathbf{E}}}\left(\forall_{\beta\in\mathscr{2}_F}\varphi^{j\beta} = \varphi^{\beta}\right) \Rightarrow (j\varphi = \varphi)\,.$$

In fact one might as well quantify over open nuclei only, that is, over the j of the form $(\_)^{\psi}$:

2.8. **Proposition.** *A geometric morphism $F : \mathbf{E} \to \mathbf{B}$ is twc iff it holds internally in $\mathbf{E}$ that every* open *subtopos of it is weakly closed, i. e. iff the following statement in the internal language holds true:*

$$\forall_{\varphi\in\Omega}\forall_{\psi\in\Omega}\left(\forall_{\beta\in\mathscr{2}_F}\varphi^{\beta^{\psi}} = \varphi^{\beta}\right) \Rightarrow \left(\varphi^{\psi} = \varphi\right)\,. \qquad \text{(Clo)}$$

Proof. This is clear, given that $\mathscr{N}_{\mathbf{E}}$ is generated by open nuclei and their negations, i. e. closed nuclei, which are a fortiori weakly closed. More precisely, there is a well known representation for any nucleus,

$$j = \bigvee_{\psi\in\Omega} (\_)^{\psi} \wedge (j\psi \vee (\_)),$$

so if all nuclei of the form $(\_)^{\psi}$ are weakly closed, then so are their meets with closed nuclei $j\psi \vee (\_)$, and then also any joins of these. Obviously this argument can be carried out internally. □

It is then clear that (Clo) also implies (Reg) – any subtopos of $\mathbf{E}$ which is both strongly dense and weakly closed must be the whole $\mathbf{E}$.

Note that the external version of the above is far from being true unless F is localic: for example, let $\mathbf{B}$ be the topos of sets and $\mathbf{E}$ that of M-sets for some monoid M, then $\mathbf{E}$ will have only two open subtoposes, both of them closed. On the other hand weak closedness in this case is the same as ordinary closedness, so that F will be twc iff M is a group.

Let us deduce some equivalent forms of twc by transforming (Clo) a little. One has

$$(x^y)^z = (x^z)^y$$

and

$$x^{x^{x^y}} = x^y$$

in any Heyting algebra. Hence

$$(\psi^{\psi^\beta})^{\psi^{\psi^{\beta^\varphi}}} = (\psi^{\psi^{\psi^{\beta^\varphi}}})^{\psi^\beta} = (\psi^{\beta^\varphi})^{\psi^\beta}.$$

Moreover

$$(x^y)^z = x^{y \wedge z}$$

and

$$\bigwedge_i x^{y_i} = x^{\bigvee_i y_i},$$

so one might rewrite (Clo) as follows:

$$\forall_{\varphi \in \Omega} \forall_{\psi \in \Omega} \psi^{\bigvee_{\beta \in \mathscr{Q}} \psi^\beta \wedge \beta^\varphi} = \psi^{\psi^\varphi}. \tag{Clo$'$}$$

Also another form of it is possible; using that

(a) rhs of (Clo)$'$ always implies lhs, so only left to right implication can be required; and

(b) : $x \leqslant z^y$ iff $x \wedge y \leqslant z$,

one sees that (Clo)$'$ can be rewritten in the form

$$\psi^{\varphi \vee \bigvee_{\beta \in \mathscr{Q}} \psi^\beta \wedge \beta^\varphi} = \psi. \tag{Clo$''$}$$

Although the latter condition seems to be the least enlightening one of all, it in fact leads to a significant simplification: it will turn out that it is equivalent to its own particular case with $\varphi = \psi$,

$$\forall_{\psi \in \Omega} \psi^{\bigvee_{\beta \in \mathscr{Q}} \psi \Leftrightarrow \beta} = \psi,$$

i. e. to

$$\forall_{\psi \in \Omega} \left((\psi \in \mathscr{Q}_F) \Rightarrow \psi \right) \Rightarrow \psi. \tag{Clop}$$

Now this one is quite intuitive: in essence it says that if one wants to prove some statement ψ in **E**, one may always freely use the assumption that ψ "comes from **B**", i. e. its truth value lies in the image of $F^*\mathscr{Q} \to \Omega$. This very principle we are going to use in the proof of

2.9. **Theorem.** *A geometric morphism $F : \mathbf{E} \to \mathbf{B}$ is twc iff it satisfies the condition* (Clop) *above.*

Proof. Given (Clop), for any φ and ψ we must infer ψ from the assumptions $\varphi \Rightarrow \psi$ and $\left(\bigvee_{\beta \in \mathscr{Q}} (\varphi \Rightarrow \beta) \wedge (\beta \Rightarrow \psi) \right) \Rightarrow \psi$. Now as we said above, (Clop) enables us to assume in doing this that ψ is some $\beta_0 \in \mathscr{Q}_F$. But then $\varphi \Rightarrow \beta_0$ and $\left(\bigvee_{\beta \in \mathscr{Q}} (\varphi \Rightarrow \beta) \wedge (\beta \Rightarrow \beta_0) \right) \Rightarrow \beta_0$ together easily imply β_0, and we are done. □

Let us give an equivalent reformulation of this, in terms of the slice topos $\mathbf{E}/\Omega$. Consider the closed subtopos $\mathbf{E}_\bullet \rightarrowtail \mathbf{E}/\Omega$ of the latter corresponding to $\mathbf{true} : 1 \rightarrowtail \Omega$. Since the subobject $\mathscr{Q}_F \rightarrowtail \Omega$ contains $\mathbf{true} : 1 \rightarrowtail \Omega$, it determines a subobject of the terminal 1_Ω in $\mathbf{E}_\bullet$. Let us denote this subobject with $\mathscr{Q}_F$ again. One then has

2.10. **Proposition.** *A geometric morphism $F : \mathbf{E} \to \mathbf{B}$ is twc iff the above subobject $\mathscr{Q}_F \rightarrowtail 1_\Omega$ is $\neg\neg$-dense in the topos $\mathbf{E}_\bullet$, i. e. has nonempty meet with each nonempty subobject of 1_Ω in $\mathbf{E}_\bullet$.*

Proof. The topos $\mathbf{E}/\Omega$ is equivalent to the topos $\mathrm{Shv}(\Omega^\Omega)$ of sheaves in $\mathbf{E}$ on the internal frame Ω^Ω; $\mathbf{true} : 1 \rightarrowtail \Omega$, considered as a subobject of 1_Ω in $\mathrm{Shv}(\Omega^\Omega)$, gives rise to the element of Ω^Ω which is the identity map $1_\Omega : \Omega \to \Omega$. Thus the subtopos $\mathbf{E}_\bullet$ corresponds to its quotient frame which as an internal lattice is isomorphic to the lattice of those elements of Ω^Ω which lie above 1_Ω. Thus the latter is the bottom element of this quotient frame. Moreover implication in this frame is pointwise, i. e. for $\Phi, \Psi : \Omega \to \Omega$ one has

$$(\Phi \Rightarrow \Psi)(\varphi) = (\Phi(\varphi) \Rightarrow \Psi(\varphi)).$$

In particular one has

$$(\neg\Phi)(\varphi) = (\Phi(\varphi) \Rightarrow 1_\Omega(\varphi)) = (\Phi(\varphi) \Rightarrow \varphi).$$

Thus Φ is $\neg\neg$-dense in this quotient frame, i. e. $\neg\Phi = 1_\Omega$, iff

$$\forall_\varphi (\Phi(\varphi) \Rightarrow \varphi) = \varphi.$$

Thus (Clop) says precisely that $((\_) \in \mathscr{Q}_F) : \Omega \to \Omega$ is a $\neg\neg$-dense element of this quotient frame. □

Let us also mention one more form of (Clop) making it appear especially similar to (Reg). We will use the identity

$$((x \leftrightarrow y) \to x) = ((x \leftrightarrow y) \to y)$$

valid in any Heyting algebra; using it, we can rewrite (Clop), i. e.

$$\forall_{\varphi\in\Omega} \left(\bigwedge_{\beta\in\mathscr{Q}_F} ((\varphi \leftrightarrow \beta) \to \varphi) \right) = \varphi,$$

as follows:

$$\forall_{\varphi\in\Omega} \left(\bigwedge_{\beta\in\mathscr{Q}_F} (\varphi \leftrightarrow \beta) \to \beta \right) = \varphi,$$

which can be produced from (Reg), i. e. from

$$\forall_{\varphi\in\Omega} \left(\bigwedge_{\beta\in\mathscr{Q}_F} (\varphi \to \beta) \to \beta \right) = \varphi,$$

just by replacing the innermost "$\to$" with "$\leftrightarrow$".

3. Special cases, examples

First of all let us note that for any $F : \mathbf{E} \to \mathbf{B}$ with Boolean $\mathbf{B}$, all three conditions (Reg), (Clop), (Clp) are equivalent to Booleanness of $\mathbf{E}$.

Let us now consider the case when $F : \mathbf{E} \to \mathbf{B}$ is localic, i. e. has the form $\mathrm{Shv}(E) \to \mathbf{B}$ for some internal frame E in $\mathbf{B}$ (which is then isomorphic to $F_*(\Omega)$). It was shown in [5] how to reformulate (Reg) and (Clp) in terms of the unique internal frame homomorphism $! : \mathscr{Q} \to E$ from the initial internal frame $\mathscr{Q}$ of $\mathbf{B}$: (Reg) is then equivalent to

$$\forall_{e \in E} \left(\bigwedge_{\beta \in \mathscr{Q}} (e \to !\beta) \to !\beta \right) = e,$$

whereas (Clp) holds iff

$$\forall_{e \in E} \left(\bigvee_{\beta \in \mathscr{Q}} (e \leftrightarrow !\beta) \right) = \top_E.$$

Quite similarly one has

3.1. **Proposition.** *The localic geometric morphism* $\mathrm{Shv}(E) \to \mathbf{B}$ *is twc iff the internal frame* E *of* $\mathbf{B}$ *satisfies*

$$\forall_{e \in E} \left(\bigwedge_{\beta \in \mathscr{Q}} (e \leftrightarrow !\beta) \to !\beta \right) = e.$$

Proof. As mentioned before, twc is equivalent to the requirement that the above square of toposes (*) is pullback, which in our case is equivalent to requiring the square of frames

$$\begin{array}{ccc} \mathscr{N}_E & \longleftarrow & \mathscr{N}_{\mathbf{B}} \\ \uparrow & & \uparrow \\ E & \longleftarrow & \mathscr{Q} \end{array}$$

to be pushout, i. e. the induced frame homomorphism $E \otimes \mathscr{N}_{\mathbf{B}} \to \mathscr{N}_E$ to be isomorphism. Now it is proved in [1] that $E \otimes \mathscr{N}_{\mathbf{B}}$ is precisely the frame of weakly closed sublocales of the locale determined by E, i. e. twc in this case means that every sublocale of this locale is weakly closed. Then arguing with $! : \mathscr{Q} \to E$ exactly as for $\tau : F^*\mathscr{Q} \to \Omega$ above, we obtain easily that twc is equivalent to the above analog of (Clop). □

Now let us consider the particular case of the above when $\mathbf{B}$ is the Sierpiŝki topos $\mathbf{Set}^{\to}$, i. e. the topos of sheaves on the three element frame $0 < s < 1$. Let $F : \mathbf{E} \to \mathbf{B}$ be induced by a frame homomorphism $F : (0 < s < 1) \to E$, determined by an element $e = F(s)$ which might be arbitrary. As is well known, E is reconstructed uniquely up to isomorphism by the corresponding open sublocale (quotient frame) $U = [\bot, e]$, closed sublocale $C = [e, \top]$ and the "fringe map"

$l = e \vee (\_)^e : U \to C$ which might be arbitrary finite meet preserving map. Namely, E is isomorphic to the frame

$$U \times_l C = \{(u, c) \in U \times C | c \leqslant l(u)\}$$

via the homomorphism $E \ni x \mapsto (e \wedge x, e \vee x)$ whose inverse is given by $(u, c) \mapsto u^e \wedge c$. Joins and meets in $U \times_l C$ are componentwise, implication is given by

$$(u, c)^{(u', c')} = (u^{u'}, l(u^{u'}) \wedge c^{c'})$$

and the element corresponding to e is $(\top, \bot)$, i. e. the above frame homomorphism corresponds to $F : (0 < s < 1) \to U \times_l C$ given by $(0 < s < 1) \mapsto ((\bot, \bot) \leqslant (\top, \bot) \leqslant (\top, \top))$. It is then straightforward to calculate

$$\begin{aligned}
\neg(u, c) &= (\neg u, l(\neg u) \wedge \neg c)\\
\neg\neg(u, c) &= (\neg\neg u, l(\neg\neg u) \wedge \neg(l(\neg u) \wedge \neg c))\\
{}_{(\top,\bot)}\neg(u, c) &= (\top, \neg c)\\
{}_{(\top,\bot)}\neg{}_{(\top,\bot)}\neg(u, c) &= (\top, \neg\neg c)\\
(u, c)^{(\top,\bot)} &= (u, l(u))\\
(u, c) \leftrightarrow (\top, \bot) &= (u, l(u) \wedge \neg c)\\
{}_{(\top,\bot)}\neg((u, c) \leftrightarrow (\top, \bot)) &= (\top, \neg(l(u) \wedge \neg c))\\
\bigwedge_{\beta \in (0<s<1)} ((u, c) \to F(\beta)) \to F(\beta) &= \neg\neg(u, c) \wedge {}_{(\top,\bot)}\neg{}_{(\top,\bot)}\neg(u, c)\\
&= (\neg\neg u, l(\neg\neg u) \wedge \neg(l(\neg u) \wedge \neg c) \wedge \neg\neg c)\\
&= (\neg\neg u, l(\neg\neg u) \wedge \neg((l(\neg u) \wedge \neg c) \vee \neg c))\\
&= (\neg\neg u, l(\neg\neg u) \wedge \neg\neg c)\\
\bigwedge_{\beta \in (0<s<1)} ((u, c) \leftrightarrow F(\beta)) \to F(\beta) &= \neg\neg(u, c) \wedge {}_{(\top,\bot)}\neg((u, c) \leftrightarrow (\top, \bot))\\
&= (\neg\neg u, l(\neg\neg u) \wedge \neg(l(\neg u) \wedge \neg c) \wedge \neg(l(u) \wedge \neg c))\\
&= (\neg\neg u, l(\neg\neg u) \wedge \neg((l(\neg u) \wedge \neg c) \vee (l(u) \wedge \neg c)))\\
&= (\neg\neg u, l(\neg\neg u) \wedge \neg((l(\neg u) \vee l(u)) \wedge \neg c))\\
\bigvee_{\beta \in (0<s<1)} ((u, c) \leftrightarrow F(\beta)) &= \neg(u, c) \vee (u, c) \leftrightarrow (\top, \bot) \vee (u, c)\\
&= (\neg u \vee u, (l(\neg u) \wedge \neg c) \vee (l(u) \wedge \neg c) \vee c)\\
&= (\neg u \vee u, ((l(\neg u) \vee l(u)) \wedge \neg c) \vee c).
\end{aligned}$$

Using these calculations one then proves easily

3.2. **Proposition.** *A geometric morphism* $\mathrm{Shv}(E) \to \mathbf{Set}^{\to}$ *corresponding to an element* $e = (\top, \bot)$ *of a frame* $E = U \times_l C$ *for a finite meet preserving map* $l : U \to C$ *between frames is either nowhere dense or twc iff both* $U = [\bot, e]$ *and* $C = [e, \top]$ *are Boolean, or equivalently, the conditions*

$$\forall_{u \leqslant e} e \leqslant u \vee \neg u \ \text{ and } \ \forall_{c \geqslant e} c \vee c \to e = \top$$

hold; it is $\mathbf{Set}^{\rightarrow}$*-valued iff moreover*

$$\forall_{u\in U} l(u) \vee l(\neg u) = \top, \tag{‡}$$

or equivalently

$$\forall_{u\leqslant e}(e \to u) \vee \neg u = \top$$

holds.

Proof. Above calculations imply immediately that all of the (Reg), (Clop) and (Clp) imply Booleanness of U. Also, taking $u = \top$, Booleanness of C follows. Conversely if both U and C are Boolean then (Reg) holds, (Clop) is equivalent to

$$l(u) \wedge \neg((l(\neg u) \vee l(u)) \wedge \neg c) = c$$

and (Clp) is equivalent to

$$((l(\neg u) \vee l(u)) \wedge \neg c) \vee c = \top,$$

for all $c \leqslant l(u)$. Then using Booleanness of C the first of these identities can be rewritten as

$$l(u) \wedge (\neg(l(\neg u) \vee l(u)) \vee c) = c$$

or

$$(l(u) \wedge \neg l(\neg u) \wedge \neg l(u)) \vee c = c$$

which always holds since $l(u) \wedge \neg l(u) = \bot$. As for the second identity, it is equivalent to

$$(l(\neg u) \vee l(u) \vee c) \wedge (\neg c \vee c) = \top,$$

i. e. to

$$l(\neg u) \vee l(u) = \top.$$

In terms of E, $l(u) = e \vee (e \to u)$, so this condition means

$$e \vee (e \to (e \wedge \neg u)) \vee e \vee (e \to u) = \top,$$

i. e.

$$e \vee (e \to \neg u) \vee (e \to u) = \top.$$

But $u \leqslant e$ implies $e \to \neg u = \neg u$ while Booleanness of U, i. e. $e \leqslant \neg u \vee u$, implies $e \leqslant \neg u \vee (e \to u)$, hence the last condition is equivalent to

$$\neg u \vee (e \to u) = \top.$$

□

Since there are obviously lots of finite meet preserving maps l between Boolean algebras which do not satisfy (‡) above, we see in particular that neither (Reg) nor (Clop) does imply (Clp). Note also that openness of $\mathrm{Shv}(U \times_l C) \to \mathbf{Set}^{\rightarrow}$ is equivalent to $l(\bot) = \bot$, so that under this additional assumption (Reg) and (Clop) will hold iff l preserves $\bot$ and finite meets, whereas (Clp) will hold iff l is a Boolean algebra homomorphism. So also for open geometric morphisms (Clp) is strictly stronger than (Clop) and (Reg).

Let us now investigate the case when the geometric morphism $\mathbf{E} \to \mathbf{B}$ is of the form $(F^* \dashv F_*) : \mathbf{Set}^{\mathbb{C}} \to \mathbf{Set}^{\mathbb{D}}$, induced by a functor $F : \mathbb{C} \to \mathbb{D}$ between small categories. In this case, one has

3.3. **Proposition.** *The geometric morphism* $\mathbf{Set}^{\mathbb{C}} \to \mathbf{Set}^{\mathbb{D}}$ *induced by a functor* $F : \mathbb{C} \to \mathbb{D}$ *satisfies* (Reg) *if and only if for every object* C *of* $\mathbb{C}$ *there is a morphism* $i : C \to C_1$ *such that for any* $j : C_1 \to X$ *with* $F(j)$ *split mono,* ji *is a split mono too.*

Proof. The condition (Reg) can be formulated as follows: the composite

$$\Omega \xrightarrow{\Phi} \Omega^{F^*\mathscr{Q}} \xrightarrow{\forall_{\mathscr{Q}}} \Omega,$$

is the identity map, where Φ is given by $\Phi(\varphi) = ((\varphi \to \beta) \to \beta)_{\beta \in \mathscr{Q}}$. By the universal property of Ω, this is the same as requiring that the inverse image of **true** under this composite is **true** again. Moreover since inverse image of **true** under $\forall_{\mathscr{Q}}$ is $\mathbf{true}^{\mathscr{Q}} : 1^{\mathscr{Q}} \to \Omega^{\mathscr{Q}}$, (Reg) holds iff the square

$$\begin{array}{ccc} 1 & \longrightarrow & 1^{\mathscr{Q}} \\ {\scriptstyle \mathbf{true}}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{true}^{\mathscr{Q}}} \\ \Omega & \xrightarrow{\Phi} & \Omega^{\mathscr{Q}} \end{array}$$

is pullback.

Now in $\mathbf{Set}^{\mathbb{C}}$, one may use for Ω the functor assigning to $C \in \mathbb{C}$ the set $\Omega(C)$ of *cosieves* on C, i. e. subfunctors of the representable functor $h^C = \hom_{\mathbb{C}}(C, \_)$. Moreover cosieves on C can be viewed as sets $\mathscr{C}$ of morphisms $c : C \to C'$ originating in C which satisfy

$$\forall_{C \xrightarrow{c} C' \xrightarrow{c'} C''} c \in \mathscr{C} \Rightarrow c'c \in \mathscr{C}.$$

Similarly for $\mathscr{Q}$, of course; in particular $F^*\mathscr{Q}$, i. e. the composite of $\mathscr{Q} : \mathbb{D} \to \mathbf{Set}$ with $F : \mathbb{C} \to \mathbb{D}$, assigns to C the set of cosieves on FC, whereas $\tau : F^*\mathscr{Q} \to \Omega$ sends a cosieve $\mathscr{D}$ on FC to the cosieve $F^{-1}\mathscr{D}$ on C given by $F^{-1}\mathscr{D} = \{c : C \to C' | Fc \in \mathscr{D}\}$.

In these terms, we can view the Φ above as the map

$$\begin{aligned} \Omega(C) \to \hom_{\mathbf{Set}^{\mathbb{C}}}(h^C, \Omega^{F^*\mathscr{Q}}) &\approx \hom_{\mathbf{Set}^{\mathbb{C}}}(h^C \times F^*\mathscr{Q}, \Omega) \\ &\approx \big(\text{subfunctors of } h^C \times F^*\mathscr{Q}\big) \end{aligned}$$

given by

$$\Phi(\mathscr{C} \subseteq h^C) = \left\{(C \xrightarrow{c} C', \mathscr{D}' \subseteq h^{FC'}) | (F^{-1}\mathscr{D}')^{c\mathscr{C}} \subseteq F^{-1}\mathscr{D}'\right\},$$

where $c\mathscr{C}$ is the image of $\mathscr{C}$ under $\Omega(c) : \Omega(C) \to \Omega(C')$, with $c' \in c\mathscr{C}$ iff $c'c \in \mathscr{C}$ for any $c' : C' \to C''$, while "$(F^{-1}\mathscr{D}')^{c\mathscr{C}}$" refers to the implication in the Heyting algebra of cosieves. The latter is given by

$$\mathscr{C}_2^{\mathscr{C}_1} = \left\{C \xrightarrow{c} C' | c\mathscr{C}_1 \subseteq c\mathscr{C}_2\right\}.$$

We thus conclude that (Reg) in this case means

$$\forall_{C \in \mathbb{C}} \forall_{\mathscr{C} \in \Omega(C)} \left(\forall_{C \xrightarrow{c} C'} \forall_{\mathscr{D}' \in \mathscr{Q}(FC')} (F^{-1}\mathscr{D}')^{c\mathscr{C}} \subseteq F^{-1}\mathscr{D}'\right) \Leftrightarrow \mathscr{C} = h^C,$$

or equivalently

$$\forall_{C \in \mathbb{C}} \forall_{\mathscr{C} \subsetneq h^C} \exists_{C \xrightarrow{c} C'} \exists_{\mathscr{D}' \in \mathscr{Q}(FC')} \exists_{\left(C' \xrightarrow{c'} C''\right) \notin F^{-1}\mathscr{D}'} c' \in (F^{-1}\mathscr{D}')^{c\mathscr{C}},$$

or, expanding further,

$$\forall_{C\in\mathbb{C}}\forall_{\mathscr{C}\subsetneqq h^C}\exists_{C\xrightarrow{c}C'}\exists_{\mathscr{D}'\in\mathscr{Q}(FC')}\exists_{\left(C'\xrightarrow{c'}C''\right)\notin F^{-1}\mathscr{D}'}\forall_{C''\xrightarrow{c''}C'''}F(c''c')\notin\mathscr{D}'$$
$$\Rightarrow c''c'c\notin\mathscr{C}.$$

Observe now that if this condition is satisfied for some cosieve $\mathscr{C}$, then it will also hold for all smaller ones (with the same choice of c, $\mathscr{D}'$ and c'). Thus it suffices to require it for the *largest* cosieve $\mathscr{C}$ with $\mathscr{C}\neq h^C$, which always exists and is the set of all morphisms $C\to C'$ which are not split monos. Thus (Reg) is equivalent to

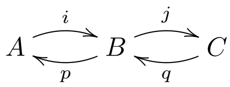
$$\forall_{C\in\mathbb{C}}\exists_{C\xrightarrow{c}C'}\exists_{\mathscr{D}'\in\mathscr{Q}(FC')}\exists_{\left(C'\xrightarrow{c'}C''\right)\notin F^{-1}\mathscr{D}'}\forall_{C''\xrightarrow{c''}C'''}F(c''c')\notin\mathscr{D}'$$
$$\Rightarrow\exists_{C'''\xrightarrow{p}C}pc''c'c=1_C.$$

Observe further that, for given C, the required properties are satisfied by some c, c' and $\mathscr{D}'$ if and only if they are satisfied by c, c' and, instead of $\mathscr{D}'$, the largest cosieve on FC' among those not containing Fc'. Such one always exists; it is $\{d:FC'\to D|\forall_{d':D\to FC''}d'd\neq Fc'\}$. Hence (Reg) is equivalent to the condition

$$\forall_{C\in\mathbb{C}}\exists_{C\xrightarrow{c}C'\xrightarrow{c'}C''}\forall_{C''\xrightarrow{c''}C'''}\forall_{FC'''\xrightarrow{d'}FC''}d'F(c''c')=Fc'\Rightarrow\exists_{C'''\xrightarrow{p}C}pc''c'c=1_C. \tag{Reg$_0$}$$

It is now clear that once F satisfies the condition in our proposition, this (Reg)$_0$ will be satisfied: for given C one may just take $C'=C''=C_1$, $c=i$ and $c'=1_{C_1}$. Conversely, suppose F satisfies (Reg)$_0$; then take $C_1=C''$, $i=c'c$. Suppose given $c'':C_1\to C'''$ such that Fc'' is a split mono, i. e. there is a $d':FC'''\to FC_1$ with $d'Fc''=1_{FC_1}$. Then also $d'F(c''c')=Fc'$, so by (Reg)$_0$ there is a p with $pc''c'c=1_C$, i. e. $c''c'c$ is a split mono too. □

3.4. **Remark.** We see in particular that (Reg) holds for geometric morphisms induced by any functors $F:\mathbb{C}\to\mathbb{D}$ which reflect split monos, i. e. satisfy $F(c)$ split mono $\Rightarrow c$ split mono for any $c:C\to C'$ in $\mathbb{C}$. Converse is not true however. Let, for example, $\mathbb{D}$ be the category freely generated by the graph



subject to the relations $pi=1_A$ and $qj=1_B$ ($\mathbb{D}$ is finite), and let $F:\mathbb{C}\to\mathbb{D}$ be the inclusion of the subcategory of $\mathbb{D}$ generated by i, j and pq. Then one checks easily that F satisfies (Reg), although j becomes split mono in $\mathbb{D}$ without being so in $\mathbb{C}$.

To consider another two conditions, let us first explicate which cosieves $\mathscr{C}\subseteq h^C$ are classified by $\mathscr{Q}_F\subseteq\Omega$, i. e. have the form $F^{-1}\mathscr{D}$ for some cosieve on FC. One has

3.5. **Lemma.** *For a functor $F:\mathbb{C}\to\mathbb{D}$, the classifying map $h^C\to\Omega$ of a cosieve $\mathscr{C}$ on $C\in\mathbb{C}$ factors through $\mathscr{Q}_F\rightarrowtail\Omega$ if and only if*

$$\forall_{\left(C\xrightarrow{c_1}C_1\right)\in\mathscr{C}}\forall_{C\xrightarrow{c_2}C_2}\forall_{FC_1\xrightarrow{d}FC_2}dFc_1=Fc_2\Rightarrow c_2\in\mathscr{C}.$$

Proof. If $\mathscr{C} = F^{-1}\mathscr{D}$, then $c_1 \in \mathscr{C}$ means $Fc_1 \in \mathscr{D}$, hence $dFc_1 \in \mathscr{D}$, so if $dFc_1 = Fc_2$, then $Fc_2 \in \mathscr{D}$, i. e. $c_2 \in \mathscr{C}$. Conversely, if $\mathscr{C}$ satisfies the condition of the lemma, then consider $\mathscr{D} = \{dFc|\,(c : C \to C') \in \mathscr{C}, d : FC' \to D\}$. Then for any $c : C \to C'$ one has $Fc \in \mathscr{D}$ iff $Fc = dFc'$ with $c' \in \mathscr{C}$, which by the condition is clearly equivalent to $c \in \mathscr{C}$. $\square$

Having this, it is easy to obtain

3.6. **Proposition.** *The geometric morphism* $\mathbf{Set}^{\mathbb{C}} \to \mathbf{Set}^{\mathbb{D}}$ *induced by a functor* $F : \mathbb{C} \to \mathbb{D}$ *is* $\mathbf{Set}^{\mathbb{D}}$*-valued if and only if* F *reflects right divisibility, i. e.*

$$\forall_{C \xrightarrow{c_1} C_1} \forall_{C \xrightarrow{c_2} C_2} \left(\exists_{FC_1 \xrightarrow{x} FC_2} xFc_1 = Fc_2\right) \Rightarrow \left(\exists_{y:C_1 \to C_2} yc_1 = c_2\right).$$

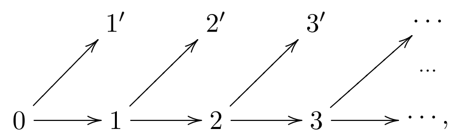
Proof. Clearly (Clp) holds iff *every* cosieve in $\mathbb{C}$ satisfies the condition of the lemma; in particular, for any $c_2 : C \to C_2$ it must be satisfied by the largest cosieve on C not containing c_2, i. e. by $\{c_1 : C \to C_1 | \forall_{c:C_1 \to C_2} cc_1 \neq c_2\}$. This readily implies the condition in our porposition. Conversely, if this condition holds, then obviously every cosieve will satisfy the condition of the lemma. $\square$

3.7. **Proposition.** *The geometric morphism* $\mathbf{Set}^{\mathbb{C}} \to \mathbf{Set}^{\mathbb{D}}$ *induced by a functor* $F : \mathbb{C} \to \mathbb{D}$ *is twc if and only if for any cosieve* $\mathscr{C} \subsetneqq h^C$ *in* $\mathbb{C}$ *there exists* $c : C \to C'$, $c \notin \mathscr{C}$ *with the property*

$$\forall_{C' \xrightarrow{c_1} C_1} \forall_{C' \xrightarrow{c_2} C_2} \forall_{FC_1 \xrightarrow{d} FC_2} dFc_1 = Fc_2 \Rightarrow (c_1 c \in \mathscr{C} \Rightarrow c_2 c \in \mathscr{C}).$$

Proof. In view of the proposition 2.10, our geometric morphism is twc iff $\mathscr{Q}_F \rightarrowtail \Omega$ is $\neg\neg$-dense as a subobject of 1_Ω in the closed subtopos of $\mathbf{Set}^{\mathbb{C}}/\Omega$ corresponding to $\mathbf{true} : 1 \rightarrowtail \Omega$. But $\mathbf{Set}^{\mathbb{C}}/\Omega$ is equivalent to $\mathbf{Set}^{\int \Omega}$, where $\int \Omega$ is the Grothendieck construction of the functor $\Omega : \mathbb{C} \to \mathbf{Set}$, i. e. the category whose objects are cosieves $\mathscr{C} \subseteq h^C$, whereas a morphism from $\mathscr{C} \subseteq h^C$ to $\mathscr{C}' \subseteq h^{C'}$ is given by a morphism $c : C \to C'$ such that $c\mathscr{C} = \mathscr{C}'$. Then $\mathbf{true} : 1 \rightarrowtail \Omega$ determines a subcategory of $\int \Omega$ consisting of cosieves which coincide with the whole representable functors h^C, and the corresponding closed subtopos is the topos of presheaves on the complement of this subcategory, i. e. on the subcategory of $\int \Omega$ whose objects are cosieves different from the h^C's. Moreover the subobject of the terminal determined by $\mathscr{Q}_F$ corresponds to the subcategory whose objects are cosieves of the form $F^{-1}\mathscr{D}$, for cosieves $\mathscr{D}$ on objects of $\mathbb{D}$, and this subobject is then $\neg\neg$-dense iff from each object there is a morphism to some $F^{-1}\mathscr{D}$ – in other words, for each cosieve $\mathscr{C} \subsetneqq h^C$ there is a morphism $c : C \to C'$ with $c\mathscr{C} \subsetneqq h^{C'}$ and $c\mathscr{C}$ equal to $F^{-1}\mathscr{D}'$ for some $\mathscr{D}' \subset h^{FC'}$. Now using the lemma we arrive at the required statement. $\square$

We finally give an example showing that (Clop) is strictly stronger than (Reg). Let $\mathbb{C}$ be the category looking like



let $\mathbb{D}$ be its full subcategory on $0, 1, 2, ...$, and let $F : \mathbb{C} \to \mathbb{D}$ be the functor which is identity on $\mathbb{D}$ and projects n' to n, $n = 1, 2, 3, ...$ Then it is clear that cosieves of the form $F^{-1}\mathscr{D}$ are the empty ones, the representables h^n, and the cosieves $\mathscr{C}_n$, $n = 1, 2, ...$ containing all the objects m, m' with $m \geqslant n$. Now consider the cosieve $\mathscr{C}$ (say, on 0) which contains precisely the objects $1', 2', 3', ...$ Then one has $\mathscr{C} \leftrightarrow \mathscr{C}_n = \mathscr{C} \cap \mathscr{C}_n$, hence

$$\bigcap_{\mathscr{D} \in \mathscr{Q}} \left(\mathscr{C} \leftrightarrow F^{-1}\mathscr{D}\right) \to F^{-1}\mathscr{D} = h^0 \neq \mathscr{C},$$

so that $\mathscr{C}$ violates (Clop) for F. On the other hand the remark 3.4 makes it clear that any geometric morphism induced by a monotone map between posets with discrete point inverses satisfies (Reg).

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