Transactions of A. Razmadze
Mathematical Institute
Vol. **178** (2024), issue 1, 69–76

ON I_3-CONVERGENCE, $I_3^\star$-CONVERGENCE AND I_3-CAUCHY SEQUENCES IN FUZZY NORMED SPACES

CARLOS GRANADOS[1], SUMAN DAS[2] AND BINOD CHANDRA TRIPATHY[3]

Abstract. In this paper, we introduce the notions of I_3-convergent, $I_3^\star$-convergent and I_3-Cauchy triple sequences in a fuzzy normed linear space. Besides, we establish some basic results related to those notions. Furthermore, we define the concepts of $I_3^\star$-Cauchy sequence in fuzzy normed space and some relations between I_3-Cauchy sequence in a fuzzy normed space are shown.

1. Introduction

The notion of ideal convergence was introduced as a generalization of statistical convergence, and any concept involving any of these notions have a vital role not only in pure mathematics, but also in other branches of science involving mathematics, especially in information theory, computer science, biological science, dynamical systems, geographic information systems, population modelling, and motion planning in robotics [13]. Among various developments of the theory of fuzzy sets (see [23]) a progressive development has been made to find the fuzzy analogues of the classical set theory [13]. Indeed, the fuzzy set theory has become an important area of research for the last 50 years. The notions of fuzzyness have been using by many researchers engaged in Cybernetics, Artificial Intelligence, Expert System and Fuzzy control, Pattern recognition, Operation research, Decision making, Image analysis, Projectiles, Probability theory, Agriculture, Weather forecasting. Besides, the fuzzy set theory has been used widely in many engineering applications such as bifurcation of non-linear dynamical systems, the control of chaos, the non-linear operator and population dynamics [13]. The fuzzyness of all subjects of mathematical sciences has been studied. It attracted many workers occupied with sequence spaces and summability theory to introduce different types of sequence spaces and study their different properties. In many situations, no means is available for determine the norm of a vector exactly, thus it seems that the concept of a fuzzy norm is more suitable than that of a crisp norm in these cases, namely, we can model the inexactness by fuzzy norm [13].

The principal notion of fuzzy norm was initiated by Katsaras [14] in studying fuzzy topological vector spaces. Furthermore, Alimohammady and Roohi [1] studied the compactness in fuzzy minimal spaces. Taking into account the notion of fuzzy number, Felbin [7] put forward the concept of fuzzy norm on a linear space, which is based on the treatment of a fuzzy metric introduced by Kaleva and Seikkala [11]. Some topological properties of fuzzy normed linear spaces were found in [3] and [22].

Otherwise, the notion of I-convergence was originally introduced by Kostyrko et al. [15]. Later, it was further investigated from a sequence space point of view and linked with the summability theory by Salát et al. [18, 19], and then was studied by Tripathy and Hazarika [20, 21], Hazarika [10, 16], Hazarika et al. [12], and Esi and Hazarika [6]. Moreover, I-convergence has been studied and extended in more general abstract spaces such as the fuzzy normed spaces [16], 2-normed linear spaces [8], n-normed linear spaces [9]. On the other hand, the notion of intuitionistic fuzzy sets were initially introduced by Anastassiou [13]. Then, Kumar and Kumar [17] studied I-convergence in intuitionistic fuzzy normed space. Dundar and Recai [4, 5] defined and studied I_2-convergent, $I_2^\star$-convergent and I_2-Cauchy sequences in fuzzy normed spaces.

2020 *Mathematics Subject Classification.* 34C41, 40A35.
Key words and phrases. Ideal spaces; I_3-convergence; I_3-Cauchy; Fuzzy normed space; I_3-limit point, I_3-cluster point.

This paper is organized as follows: In Section 2, we present some well-known definitions and results that will be useful for the developing of this paper. In Section 3, we introduce the notions of I_3-convergence and $I_3^\star$-convergence in a fuzzy normed space and establish some properties and relations between them. In Section 4, we introduce the notion of the I_3-Cauchy sequence in a fuzzy normed space and obtain some basic results. In the same section, the concepts of the $I_3^\star$-Cauchy sequence in a fuzzy normed space is defined and some relations between the I_3-Cauchy sequence are studied.

2. Preliminaries

In this section, we recall some well-known definitions related to fuzzy numbers and fuzzy normed space and ideal spaces.

Fuzzy sets are considered with respect to a non-empty base set X of elements of interest. The essential idea is that each element $x \in X$ is assigned a membership grade $\mu(x)$ taking values in $[0,1]$, with $\mu(x) = 0$ corresponding to non-membership, $0 < \mu(x) < 1$ to partial membership, and $\mu(x) = 1$ to full membership. Takin into account the notions introduced by [23], a fuzzy subset X is a non-empty subset $\{(x, \mu(x) : x \in X\}$ of $X \times [0,1]$ for some function $\mu : X \to [0,1]$. The function μ itself is sometimes used for the fuzzy set. A fuzzy set μ on $\mathbb{R}$ ($\mathbb{R}$ denotes the set of real numbers) is called a fuzzy number if it satisfies the following properties:

(1) μ is normal.
(2) μ is a fuzzy convex.
(3) μ is upper semi-continuous.
(4) $\sup \mu = Cl\{x \in \mathbb{R} : \mu(x) > 0\}$, denoted by $[\mu]_0$, is compact.

On the other hand, a partial order $\preceq$ on $L(\mathbb{R})$ ($L(\mathbb{R})$ denotes the set of all fuzzy numbers) is defined by $u \preceq v$ if $u_\alpha^- \leqslant v_\alpha^-$ and $u_\alpha^+ \leqslant v_\alpha^+$ for all $\alpha \in [0,1]$. Arithmetic operations for $t \in \mathbb{N}, \oplus, \ominus, \odot$ and $\oslash$ are defined and can be found in [16] (page 406).

Otherwise, let X be a vector space over $\mathbb{R}$, $\|\cdot\| : X \to L^\star(\mathbb{R})$ and the mapping $L; R$:(the left norm and the right norm, respectively) $[0,1] \times [0,1] \to [0,1]$ be symmetric, non-decreasing in both arguments and satisfy $L(0,0) = 0$ and $R(1,1) = 1$. Then the quadruple $(X, \|\cdot\|, L, R)$ is called a fuzzy normed liner space (simply, FNS) and $\|\cdot\|$ a fuzzy norm if the following statements are satisfied:

(1) $\|x\| = \tilde{0} \leftrightarrow x = 0$.
(2) $\|cx\| = |c| \odot \|x\|$, for $x \in X$ and $c \in \mathbb{R}$.
(3) For all $x, y \in X$,
  (a) $\|x+y\|(s+t) \geqslant L(\|x\|(s), \|y\|(t))$, whenever $s \leqslant \|x\|_1^-$, $t \leqslant \|y\|_1^-$ and $s+t \leqslant \|x+y\|_1^-$;
  (b) $\|x+y\|(s+t) \leqslant L(\|x\|(s), \|y\|(t))$, whenever $s \geqslant \|x\|_1^-$, $t \leqslant \|y\|_1^-$ and $s+t \geqslant \|x+y\|_1^-$.

On the other hand, an ideal I is a non-empty collection of subsets of X which satisfies the following properties:

(1) If $A \subset B$ and $B \in I$, then $A \in I$.
(2) If $A, B \in I$, then $A \cup B \in I$.

I is called a non-trivial ideal if $X \in I$. Besides, a non-trivial ideal I on X is called admissible if $\{x\} \in I$ for each $x \in X$. A non-trivial ideal I_3 of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ ($\mathbb{N}$ denoted the set of natural numbers) is called strongly admissible if $\{i\} \times \mathbb{N} \times \mathbb{N}$, $\mathbb{N} \times \{i\}\mathbb{N}$ and $\mathbb{N} \times \mathbb{N} \times \{i\}$ belongs to I_3 for each $i \in \mathbb{N}$. It is evident that a strongly admissible ideal is also admissible. Throughout the paper, we denote I_3 as a strongly admissible ideal in $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$. Moreover, let $X \neq \emptyset$. A non-empty class F of subsets of X is said to be a filter in X provided:

(1) $\emptyset \notin F$.
(2) If $A, B \in F$, then $A \cap B \in F$.
(3) $A \in F$ and $A \subset B$, then $B \in F$.

Let F be a non-trivial ideal on X, and $X \neq \emptyset$, then the class $F(I) = \{M \subset X :$ exists $A \in F$ such that $M = X - A\}$ is a filter on X and it is called the filter associated with F.

3. I_3-convergence and $I_3^{\star}$-convergence in Fuzzy Normed Spaces

In this section, we introduce and study some properties of $_3$-convergence in fuzzy normed spaces. After that, we show some relationships between I_3-convergence and $I_3^{\star}$-convergence of triple sequences in fuzzy normed spaces.

Definition 3.1. Let (X,ρ) be a fuzzy normed space and $I_3 \subset 2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. Then a triple sequence $x=(x_{nmj})$ is said to be I_3-convergent to $L \in X$, if for any $\epsilon > 0$, we have $A(\epsilon)=\{(n,m,j)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N} : \rho(x_{nmj},L)\geqslant\epsilon\}\in I_3$ and we denote this as I_3-$\lim\limits_{n,m,j\to\infty} x_{nmj}=L$.

Remark 3.1. If $I_3 \subset 2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$ is a strongly admissible ideal, then usual convergence implies I_3-convergence.

Example 3.1. If we take $I_3 = I_3(f) = \{A \subset \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \text{A is a finite subset }\}$, then $I_3(f)$ is a non-trivial admissible ideal of $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ and the corresponding convergence coincide with ordinary convergence with respect to a fuzzy norm on $(X,\|\cdot\|)$.

Definition 3.2. Let $(X,\|\cdot\|)$ be a fuzzy normed space. A triple sequence $x=(x_{nmj})_{(n,m,j)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$ in X is said to be I_3-convergent to $L_1\in X$ with respect to a fuzzy norm on X if for each $\epsilon>0$, the set $A(\epsilon)=\{(m,n,j)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N} : \|x_{nmj}-L_1\|_0^+\geqslant\epsilon\}\in I_3$. In this case, we denote this $x_{nmj}\to^{FI_3} L_1$, or $x_{nmj}\to L_1(FI_3)$, or FI_3-$\lim\limits_{n,m,j\to\infty} x_{nmj}=L_1$. L_1 is called FI_3-limit of (x_{nmj}) in X.

Remark 3.2. In terms of neighbourhoods, we have $x_{nmj}\to^{FI_3} L_1$, provided that for each $\epsilon>0$, $\{(n,m,j)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N} : x_{nmj}\notin V_{L_1}(\epsilon,0)\}\in I_3$. A useful interpretation of the last definition is $x_{nmj}\to^{FI_3} L_1 \Leftrightarrow FI_3$-$\lim\limits_{n,m,j\to\infty}\|x_{nmj}-L_1\|_0^+=0$.

Remark 3.3. Note that FI_3-$\lim\limits_{n,m,j\to\infty}\|x_{nmj}-L_1\|_0^+=0$ implies

$$FI_3-\lim\|x_{nmj}-L_1\|_\beta^- = FI_3\ \lim\|x_{nmj}-L_1\|_\beta^+=0,$$

for each $\beta\in[0,1]$, since $0\leqslant\|x_{nmj}-L_1\|_\beta^-\leqslant\|x_{nmj}-L_1\|_\beta^+\leqslant\|x_{nmj}-L_1\|_0^+$ holds for every $n,m,j\in\mathbb{N}$ and for each $\beta\in[0,1]$.

Definition 3.3. Let $(X,\|\cdot\|)$ be a fuzzy normed space. A triple sequence (x_{nmj}) in X is said to be $I_3^{\star}$-convergent to L with respect to the fuzzy norm of X if there exits a set $N\in F(I_3)$, $M=\{m_1<\cdots<m_k\cdots;n_1<\cdots<n_l<\cdots;w_1<\cdots<w_i<\cdots\}\subset\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ such that $\lim\limits_{k,l,i\to\infty}\|x_{m_kn_lw_i}-L\|$. For this, we denote $x_{nmj}\to^{FI_3^{\star}} L$, or $x_{nmj}\to L(FI_3^{\star})$, or $FI_3^{\star}$-$\lim\limits_{n,m,j\to\infty} x_{nmj}=L$.

Remark 3.4. We say that an admissible ideal $I_3\subset 2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$ satisfies the property (AP3), if for every countable family of mutually disjoint sets $\{A_1,A_2,\ldots\}$ belonging to I_3, there exits a countable family of sets $\{B_1,B_2,\ldots\}$ such that $A_i\cap B_i\in I_3^0$, this means that $A_i\cap B_i$ is included in the finite union of rows and columns in $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ for each $i\in\mathbb{N}$ and $B=\bigcup_{i=1}^{\infty}B_i\in I_3$, therefore $B_i\in I_3$ for each $i\in\mathbb{N}$.

Remark 3.5. Let (x_{nmj}) and (y_{nmj}) be two triple sequences in a fuzzy normed space $(X,\|\cdot\|)$ such that $x_{nmj}\to^{FI_3} L_x$ and $y_{nmj}\to^{FI_3} L_y$, where $L_x,L_y\in X$. Then, the following statements hold:

(1) $(x_{nmj}+y_{nmj})\to^{FI_3}(L_x+L_y)$.
(2) $(x_{nmj}y_{nmj})\to^{FI_3}(L_xL_y)$.
(3) $(\alpha x_{nmj})\to^{FI_3}\alpha L_x$, for $\alpha\in\mathbb{R}$.

The proofs of the above statements follow by Definition 3.2.

Theorem 3.1. *Let $(X,\|\cdot\|)$ be a fuzzy normed space and I_3 be an admissible ideal. Then $x_{nmj}\to^{FI_3^{\star}} L$ implies $x_{nmj}\to^{FI_3} L$.*

Proof. Let (X_{nmj}) be a sequence in a fuzzy normed space $(X,\|\cdot\|)$. Consider that $x_{nmj}\to^{FI_3^{\star}} L$. Then there exits a set $M=\{m_1<\cdots<m_k\cdots;n_1<\cdots<n_l<\cdots;w_1<\cdots<w_i<\cdots\}\subset\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ with $M\in F(I_3)$ such that

$$FI_3-\lim_{n,m,j\to\infty}\{\|x_{nmj}-L\|=0.$$

Let $\epsilon > 0$ be given. By (1), there exit integers $m_0, n_0 w_0 \in \mathbb{N}$ such that $\|x_{nmj} - L\|_0^+ < \epsilon$ for every $m_k, n_l, w_i \in M$, $k > m_0$, $l > n_0$ and $i > w_0$. Now, let $A = \{m_1, \dots, m_{m_0}; n_1, \dots, n_{n_0}; w_1, \dots, w_{w_0}\}$. Since $M \in F(I_3)$, there exits a set $B \in I_3$ such that $M = \mathbb{N} \times \mathbb{N} \times \mathbb{N} - B$. It can be seen that $A_1(\epsilon)\{(n,m,j) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \|x_{nmj} - L\|_0^+ \geqslant \epsilon\} \subseteq A \cup B$.

It is well known that I_3 is an admissible ideal, hence $A \in I_3$. Thus $A \cup B \in I_3$ and then $A_1(\epsilon) \in I_3$. Therefore we have $x_{nmj} \to^{FI_3} L$. □

Theorem 3.2. *Let I_3 be an admissible ideal with the property (AP3) and $(X, \|\cdot\|)$ be a fuzzy normed space and (x_{nmj}) be a sequence in X. Then $x_{nmj} \to^{I_3} L$ if and only if $x_{nmj} \to^{I_3^\star} L$.*

Proof. If $x_{nmj} \to^{FI_3^\star} L$, then $x_{nmj} \to^{FI_3} L$, this is by Theorem 3.1, where I_3 need not have the property (AP3). Now, it is enough if we prove that $x_{nmj} \to^{FI_3^\star} L$ under the assumption that $x_{nmj} \to^{FI_3} L$.

Let $x_{nmj} \to^{FI_3} L$. Then for every $\epsilon > 0$, there exit integers $n = n(\epsilon)$, $m = m(\epsilon)$ and

$$B(\epsilon) = \{(n,m,j) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \|x_{nmj} - L\|_0^+ \geqslant \epsilon\} \in I_3.$$

For $h \in \mathbb{N}$, we define the set P_h as follows: $P_1 = \{(n,m,j) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \|x_{nmj} - L\|_0^+ \geqslant 1\}$ and $P_h = \{(n,m,j) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \frac{1}{h} \leqslant \|x_{nmj} - L\|_0^+ < \frac{1}{h-1}\}$ for $h \geqslant 2 \in \mathbb{N}$.

It is clear that $\{P_h\}$ is a countable family of mutually disjoint sets belonging to I_3, then by (AP3) property of I_3, there exits a countable family of sets $\{Q_h\}$ in I_3 such that $P_h \Delta Q_h$ is a finite set for each $h \in \mathbb{N}$ and $Q = \bigcup_{h=1}^{\infty} Q_h \in I_3$. Since $Q \in I_3$, there exits a set $A \in F(I_3)$ such that $A = \mathbb{N} \times \mathbb{N} \times \mathbb{N} - Q$. To prove the result, it is enough to show that $x_{nmj} \to^{A} L$. Now, let $\gamma > 0$ be given, take an integer $l \in \mathbb{N}$ such that $\gamma > \frac{1}{l+1}$. As a consequence, we have

$$\begin{aligned} &\{(n,m,j) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \|x_{nmj} - L\|_0^+ \geqslant \gamma\} \\ &\subset \left\{(n,m,j) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \|x_{nmj} - L\|_0^+ \geqslant \frac{1}{l+1}\right\} \\ &= \bigcup_{h=1}^{l+1} P_h. \end{aligned}$$

Since $P_h \Delta Q_h$ is a finite set for each $h = 1, 2, 3, \dots, l+1$, there exit integers $m_0, n_0, w_0 \in \mathbb{N}$ such that

$$\Big(\bigcup_{h=1}^{l+1} Q_h\Big) \cap \{(n,m,j) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : n \geqslant m_0, m \geqslant n_0 \quad \text{and} \quad j \geqslant w_0\}$$
$$\Big(\bigcup_{h=1}^{l+1} P_h\Big) \cap \{(n,m,j) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : n \geqslant m_0, m \geqslant n_0 \quad \text{and} \quad j \geqslant w_0\}.$$

If $n \geqslant m_0, m \geqslant n_0$ and $j \geqslant w_0$, and $(n,m,j) \in A$, then $(n,m,j) \notin Q$. Therefore $(n,m,j) \notin \bigcup_{h=1}^{l+1} Q_h$ and then $(n,m,j) \notin \bigcup_{h=1}^{l+1} P_h$. Hence for every $n \geqslant m_0, m \geqslant n_0$ and $j \geqslant w_0$ and $(n,m,j) \in A$, we have $\|x_{nmj} - L\|_0^+ < \gamma$. Thus we have $x_{nmj} \to^{A} L$. □

Theorem 3.3. *Let $(X, \|\cdot\|)$ be a fuzzy normed space. If a triple sequence (x_{nmj}) in X is I_3-convergent to L_1, then L_1 is determined uniquely.*

Proof. Let (x_{nmj}) be any triple sequence and consider $FI_3\text{-}\lim_{n,m,j\to\infty} x_{nmj} = L_1$ and $FI_3\text{-}\lim_{n,m,j\to\infty} x_{nmj} = L_2$, where $L_1 \neq L_2$. Since $L_1 \neq L_2$, we might suppose that $L_1 > L_2$. Now, choose $\epsilon = \frac{L_1 - L_2}{3}$, thus the neighbourhoods $(L_1 - \epsilon, L_1 + \epsilon)$ and $(L_2 - \epsilon, L_2 + \epsilon)$ of L_1 and L_2, respectively. Since L_1 and L_2 both are the I_3-limits of the sequence (x_{nmj}), therefore both sets $A(\epsilon) = \{(n,m,j) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \|x_{nmj} - L_1\|_0^+ \geqslant \epsilon\}$ and $B(\epsilon) = \{(n,m,j) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \|x_{nmj} - L_2\|_0^+ \geqslant \epsilon\}$ belong to $F(I_3)$. Since $F(I_3)$ is a filter on $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$, hence $A^c(\epsilon) \cap B^c(\epsilon)$ is a non-empty set in $F(I_3)$. As a consequence, we have a contradiction due to the neighbourhoods $(L_1 - \epsilon, L_1 + \epsilon)$ and $(L_2 - \epsilon, L_2 + \epsilon)$ of L_1 and L_2, respectively are disjoint. Therefore we have that $L_1 = L_2$. □

Theorem 3.4. *Let $(X, \|\cdot\|)$ be a fuzzy normed space. If X has no accumulation point, then FI_3-convergence and $FI_3^\star$-convergence coincide for each strongly admissible ideal I_3.*

Proof. Let $x = (x_{nmj})$ be a triple sequence in X and $L \in X$. By Theorem 3.1, $x_{nmj} \to^{FI_3^\star} L$ implies $x_{nmj} \to^{FI_3} L$. Now, consider $FI_3\text{-}\lim_{n,m,j\to\infty} x_{nmj} = L$. Since X has no accumulation point, thus there exits $\epsilon > 0$ such that $B_L(\epsilon, 0) = \{x \in X : \|x - L\|_0^+ < \epsilon\} = \{L\}$. Since $FI_3\text{-}\lim_{n,m,j\to\infty} x_{nmj} = L$, thus $\{(n, m, j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \|x_{nmj} - L\|_0^+ \geqslant \epsilon\} \in I_3$. Therefore we have

$$\{(n, m, j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \|x_{nmj} - L\|_0^+ < \epsilon\} = \{(n, m, j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \|x_{nmj} - L\|_0^+ = 0\} \in F(I_3).$$

Hence $FI_3^\star\text{-}\lim_{n,m,j\to\infty} x_{nmj} = L$. □

Theorem 3.5. *Let $(X, \|\cdot\|)$ be a fuzzy normed space. If X has at least one accumulation point and for any arbitrary triple sequence (x_{nmj}) of elements of X and for each $L \in X$, $FI_3\text{-}\lim_{n,m,j\to\infty} x_{nmj} = L$ implies $I_3^\star\text{-}\lim_{n,m,j\to\infty} x_{nmj} = L$, then I_3 has (AP3) property.*

Proof. Let $L \in X$ be an accumulation point of X, so there exits a sequence $(l_k)_{k\in\mathbb{N}}$ of distinct elements of X such that $l_k \neq L$ for any k, $\lim_{k\to\infty} l_k = L$ and the sequence $\{\|l_k - L\|_0^+\}_{k\in\mathbb{N}}$ is decreasing to 0. Take $s_k = \{\|l_k - L\|_0^+\}$, for $k \in \mathbb{N}$. Now, let $\{A_i\}_{i\in\mathbb{N}}$ be a disjoint family of non-empty sets from I_3. Define the sequence (x_{nmj}) as follows:

$$x_{nmj} = \begin{cases} l_i & \text{if } (n, m, j) \in A_i \\ L & \text{if } (n, m, j) \notin A_i \end{cases}$$

for fixed i. Now, let $\delta > 0$. Choose $k \in \mathbb{N}$ such that $s_k < \delta$. Then $A(\delta) = \{(m, n, j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \|x_{nmj} - L\|_0^+ \geqslant \delta\} \subset A_1 \cup \cdots \cup A_k$. Therefore $A(\delta) \in I_3$ and then $FI_3\text{-}\lim_{n,m,j\to\infty} x_{nmj} = L$, by virtue of assumption, we have that $FI_3^\star\text{-}\lim_{n,m,j\to\infty} x_{nmj} = L$. Thus there exits a set $R \in I_3$ such that $M = \mathbb{N}\times\mathbb{N}\times\mathbb{N} - R \in F(I_3)$ and

$$\lim_{n,m,j\to\infty} x_{nmj} = L, \text{ where } (n, m, j) \in M.$$

Now, take $R_i = A_i \cap H$, for $i \in \mathbb{N}$. Then $R_i \in I_3$, for each $i \in \mathbb{N}$. Besides,

$$\bigcup_{i=1}^{\infty} R_i = R \cap \bigcup_{i=1}^{\infty} A_i \subset R \text{ and so, } \bigcup_{i=1}^{\infty} R_i \in I_3.$$

Now, fix $i \in \mathbb{N}$, if $A_i \cap M$ is not included in the finite union of rows and columns in $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$, then M should contain an infinite sequence of elements $\{(n_k, m_k, j_k)\}$, where both $n_k, m_k, j_k \to \infty$ and $x_{n_k m_k j_k} = l_k \neq L$, for all $k \in \mathbb{N}$ which contradicts (4). Therefore $A_i \cap M$ should be contained in the finite union of rows and columns $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$. Hence $A_i \Delta R_i = A_i - R_i = A_i - R = A_i \cap M$ is also included in the finite union of rows and columns. As a consequence, this implies that I_3 has property (AP3). □

4. I_3-Cauchy Sequences in Fuzzy Normed Spaces

In this section, we introduce the notion of I_3-Cauchy sequences. Besides, we show some relationship between I_3-convergence and I_3-Cauchy triple sequences in fuzzy normed spaces. After that, we show other relationships between I_3-Cauchy and $I_3^\star$-Cauchy triple sequences in fuzzy normed spaces.

Definition 4.1. Let $(X, \|\cdot\|)$ be a fuzzy normed space. A tripe sequence $x = (x_{nmj})$ in X is said to be I_3-Cauchy or FI_3-Cauchy triple sequence with respect to the fuzzy norm on X if for each $\epsilon > 0$, there exit integers $p = p(\epsilon)$, $q = q(\epsilon)$ and $g = g(\epsilon)$ such that the set $\{(n, m, j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \|x_{nmj} - x_{pqg}\|_0^+ \geqslant \epsilon\}$ belongs to I_3.

Theorem 4.1. *Let I_3 be an admissible ideal and let $(X, \|\cdot\|)$ be a fuzzy normed space. Then every FI_3-convergent sequence is FI_3-Cauchy sequence.*

Proof. Let $x_{nmj} \to^{FI_3} L$, then for a given $\epsilon > 0$, we have $A(\epsilon) = \{(n,m,j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \|x_{nmj} - L\|_0^+ \geqslant \epsilon\} \in I_3$. Since I_3 is an admissible ideal, there exit integers $n_0, m_0, j_0 \in \mathbb{N}$ such that $n_0, m_0, j_0 \notin A(\epsilon)$. Now, let $A_1(\epsilon) = \{(n,m,j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \|x_{nmj} - x_{n_0m_0j_0}\|_0^+ \geqslant 2\epsilon\}$. Since $\|\cdot\|_0^+$ being a norm in the usual sense, we have $\|x_{nmj} - L\|_0^+ + \|x_{n_0m_0j_0}\| \geqslant \|x_{nmj} - x_{n_0m_0j_0}\|_0^+$. We can see that if $n, m, j \in A_1(\epsilon)$, then $\|x_{nmj} - L\|_0^+ + \|x_{n_0m_0j_0} - L\|_0^+ \geqslant 2\epsilon$.

Otherwise, since $n_0, m_0, j_0 \notin A(\epsilon)$, we have $\|x_{n_0m_0j_0} - L\|_0^+ < \epsilon$. Thus we can imply that $\|x_{nmj} - L\| < \epsilon$, therefore $n, m, j \in A(\epsilon)$. This implies that $A_1(\epsilon) \subset A(\epsilon)$ for each $\epsilon > 0$, this is that $A_1(\epsilon) \in I_3$. This proves that (x_{nmj}) is a FI_3-Cauchy sequence. □

Theorem 4.2. *Let $(X, \|\cdot\|)$ be a fuzzy normed space. Then a triple sequence (x_{nmj}) is FI_3-convergent if and only if it is the FI_3-Cauchy triple sequence.*

Proof. In Theorem 4.1, we have proved that every FI_3-convergent is FI_3-Cauchy. Now, establish the converse proof. Let (x_{nmj}) be a FI_3-Cauchy triple sequence and (ϵ_{pqg}) be a strictly decreasing sequence of numbers converging to zero. Since (x_{nmj}) is the FI_3-Cauchy triple sequence, there exist three strictly increasing sequences n_p, m_q, j_g of positive integers such that the set $A(\epsilon_{pqg}) = \{(n,m,j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \|x_{nmj} - x_{n_pm_qj_g}\|_0^+ \geqslant \epsilon_{pqg}\}$ belongs to I_3, where $p, q, g \in \mathbb{N}$. This implies that

$$\{(n,m,j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \|x_{nmj} - x_{n_pm_qj_g}\|_0^+ < \epsilon_{pqg}\} \neq \emptyset$$

and belongs to $F(I_3)$. Now, let s, t, r be three positive integers such that $s \neq t \neq r$. By (5), the both sets $D(\epsilon_{pqg}) = \{(n,m,j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \|x_{nmj} - x_{n_pm_qj_g}\|_0^+ < \epsilon_{pqg}\}$ and $C(\epsilon_{str}) = \{(n,m,j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \|x_{nmj} - x_{n_sm_tj_r}\|_0^+ < \epsilon_{str}\}$ are non-empty in $F(I_3)$. Since $F(I_3)$ is a filter on $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$, thus $D(\epsilon_{pqg}) \cap C(\epsilon_{str}) \neq \emptyset \in F(I_3)$. Therefore for each triplet (p,q,g) and (s,t,r) of positive integers with $p \neq q \neq g$ and $s \neq t \neq r$, we can choose a triplet $(n_{(p,q,g),(s,t,r)}, m_{(p,q,g),(s,t,r)}, j_{(p,q,g),(s,t,r)}) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ such that $\|x_{m_{pqgstr}n_{pqgrst}j_{pqgrst}} - x_{n_pm_qj_g}\|_0^+ < \epsilon_{pqg}$ and $\|x_{m_{pqgstr}n_{pqgrst}j_{pqgrst}} - x_{n_sm_tj_r}\|_0^+ < \epsilon_{str}$. It follows that

$$\begin{aligned} &\|x_{n_pm_qj_g} - x_{n_sm_tj_r}\|_0^+ \\ &\leqslant \|x_{m_{pqgstr}n_{pqgrst}j_{pqgrst}} - x_{n_pm_qj_g}\|_0^+ + \|x_{m_{pqgstr}n_{pqgrst}j_{pqgrst}} - x_{n_sm_tj_r}\|_0^+ \\ &< \epsilon_{pqg} + \epsilon_{str} \to 0, \text{ as } p,q,g,s,r,t \to \infty. \end{aligned}$$

This implies that $(x_{n_pm_q,j_g})$ is a FI_3-Cauchy triple sequence in a fuzzy normed space, hence it satisfies the Cauchy convergence criterion, thus the sequence $(x_{n_pm_q,j_g})$ converges to a finite limit L. Besides, as we have that $\epsilon_{pqg} \to 0$ as $p, q, g \to \infty$, for each $\epsilon > 0$ we can take positive integers p_0, q_0, g_0 such that for $p \geqslant p_0$, $q \geqslant q_0$ and $g \geqslant g_0$, so,

$$\epsilon_{p_0q_0g_0} < \frac{\epsilon}{2} \text{ and } \|x_{n_pm_qj_g} - L\|_0^+ < \frac{\epsilon}{2}.$$

Now, define the set $A(\epsilon) = \{(n,m,j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \|x_{nmj} - L\|_0^+ \geqslant \epsilon\}$. We prove that $A(\epsilon) \subset A(\epsilon_{p_0q_0g_0})$. Let $(m,n,j) \in A(\epsilon)$, then by the second half of (6), we have

$$\begin{aligned} \epsilon \leqslant \|x_{nmj} - L\|_0^+ &\leqslant \|x_{nmj} - x_{n_{p_0}m_{q_0}j_{g_0}}\| + \|x_{n_{p_0}m_{q_0}j_{g_0}} - L\|_0^+ \\ &\leqslant \|x_{nmj} - x_{n_{p_0}m_{q_0}j_{g_0}}\|_0^+ + \frac{\epsilon}{2}. \end{aligned}$$

This implies $\frac{\epsilon}{2} \leqslant \|x_{nmj} - x_{n_{p_0}m_{q_0}j_{g_0}}\|_0^+$ and hence by the first half of (6), we have $\epsilon_{p_0q_0g_0} \leqslant \|x_{nmj} - x_{n_{p_0}m_{q_0}j_{g_0}}\|_0^+$. This implies that $(m,n,j) \in A(\epsilon_{p_0q_0g_0})$ and therefore $A(\epsilon)$ is contained in $A(\epsilon_{p_0q_0g_0})$. Since $A(\epsilon_{p_0q_0g_0})$ belongs to I_3, $A(\epsilon)$ belongs to I_3. This proves that (x_{nmj}) is FI_3-convergent to L. □

Definition 4.2. Let $(X, \|\cdot\|)$ be a fuzzy normed space. A triple sequence $x = (x_{nmj})$ in X is said to be $I_3^\star$-Cauchy or $FI_3^\star$-Cauchy triple sequence with respect to a fuzzy norm on X, if there exists a set $M \in F(I_3)$ (i.e., $R = \mathbb{N}\times\mathbb{N}\times\mathbb{N} - M \in I_3$) and $k_0 = k_0(\epsilon)$ such that for every $\epsilon > 0$ and for $(n,m,j),(s,t,r) \in M$, $\|x_{nmj} - x_{str}\|_0^+ < \epsilon$, whenever $m,n,j,s,t,r > k_0$. We denote this as $\lim\limits_{m,n,j,s,t,r\to\infty} \|x_{nmj} - x_{str}\|_0^+ = 0$.

Theorem 4.3. *Let I_3 be an admissible ideal of $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$. If a triple sequence (x_{nmj}) in X is a $FI_3^\star$-Cauchy sequence, then it is a FI_3-Cauchy sequence.*

Proof. Let (x_{nmj}) be a $FI_3^\star$-Cauchy sequence. Then, there exits a set $M \in F(I_3)$ (i.e., $R = \mathbb{N}\times\mathbb{N}\times\mathbb{N} - M \in I_3$) and $k_0 = k_0(\epsilon)$ such that for every $\epsilon > 0$ and for $(n,m,j),(s,t,r) \in M$, $\|x_{nmj} - x_{str}\|_0^+ < \epsilon$, whenever $n,m,j,s,t,r \geqslant k_0$. Then

$$A(\epsilon) = \{(n,m,j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \|x_{nmj} - x_{str}\|_0^+ \geqslant \epsilon\}$$
$$\subset R \cup [M \cap ((\{1,\dots,k_0\}\times\mathbb{N}) \cup (\{1,\dots,k_0\}\times\mathbb{N}) \cup (\{1,\dots,k_0\}\times\mathbb{N}))].$$

Since I_3 is an admissible ideal, we have

$$R \cup [M \cap ((\{1,\dots,k_0\}\times\mathbb{N}) \cup (\{1,\dots,k_0\}\times\mathbb{N}) \cup (\{1,\dots,k_0\}\times\mathbb{N}))] \in I_3.$$

Therefore $A(\epsilon) \in I_3$. This proves that (x_{nmj}) is the FI_3-Cauchy triple sequence in X. □

References

1. M. Alimohammady, M. Roohi, Compactness in fuzzy minimal spaces. *Chaos, Solitons and Fract.* **28** (2006), no. 4, 906–912.
2. G. Anastassiou, Fuzzy approximation by fuzzy convolution type operators. *Compt. Math. Appl.* **48** (2004), no. 9, 1369–1386.
3. N. Das, P. Das, Fuzzy topology generated by fuzzy norm. *Fuzzy Sets and Systems* **107** (1999), no. 3, 349–354.
4. E. Dundar, M. Recai, On I_2-convergence and I_2^*-convergence of double sequences in fuzzy normed spaces. *Konuralp J. Math.* **7** (2019), no. 2, 405–409.
5. E. Dundar, M. Recai, On I_2-Cauchy double sequences in fuzzy normed sets. *Communications in Advanced Mathematical Sciences* **2** (2019), no. 2, 154–160.
6. A. Esi, B. Hazarika, λ-ideal convergence in intuitionistic fuzzy 2-normed linear space. *J. Intell. Fuzzy Systems* **24** (2013), no. 4, 725–732.
7. C. Felbin, Finite-dimensional fuzzy normed linear space. *Fuzzy Sets and Systems* **48** (1992), no. 2, 239–248.
8. M. Gurdal, On ideal convergent sequences in 2-normed spaces. *Thai J. Math.* **4** (2006), no. 1, 85–91.
9. M. Gurdal, A. Sahiner, Ideal convergence in n-normal spaces and some new sequence spaces via n-norm. *J Fund Sci.* **4** (2008), no. 1, 233–244.
10. B. Hazarika, Some new sequence spaces of fuzzy numbers defined by Orlicz functions using a fuzzy metric. *Comput. Math. Appl.* **61** (2011), no. 9, 2762–2769.
11. B. Hazarika, Fuzzy real valued lacunary I-convergent sequences. *Appl. Math. Lett.* **2**5 (2012), no. 3, 466–470.
12. B. Hazarika, V. Kumar, Generalized ideal convergence in intuitionistic fuzzy normed linear spaces. *Filomat* **27** (2013), no. 5, 811–820.
13. B. Hazarika, V. Kumar, Fuzzy real valued I-convergent double sequences in fuzzy normed spaces. *J. Intell. Fuzzy Systems* **26** (2014), no. 5, 2323–2332.
14. A. Katsaras, Fuzzy topological vector spaces. II. *Fuzzy Sets and Systems* **12** (1984), no. 2, 143–154.
15. P. Kostyrko, T. Salát, W. Wilezyński, I-convergence. *Real Anal. Exchange* **26** (2000/01), no. 2, 669–685.
16. K. Kumar, V. Kumar, On the I and I^* convergence of sequences in fuzzy normed spaces. *Adv. Fuzzy Sets Syst.* **3** (2008), no. 3, 341–365.
17. V. Kumar, K. Kumar, On ideal convergence of sequences in intuitionistic fuzzy normed spaces. *Selçuk J. Appl. Math.* **10** (2009), no. 2, 27–41.
18. T. Salát, B. Tripathy, M. Ziman, On some properties of I-convergence. *Tatra Mt. Math. Publ.* **28** (2004), part II, 279–286.
19. T. Salát, B. Tripathy, M. Ziman, On I-convergence field. *Ital. J. Pure Appl. Math.* **17** (2005), 45–54.
20. B. C. Tripathy, B. Hazarika, I-convergent sequence spaces associated with multiplier sequences. *Math. Inequal. Appl.* **11** (2008), no. 3, 543–548.
21. B. C. Tripathy, B. Hazarika, I-monotonic and I-convergent sequences. *Kyungpook Math. J.* **51** (2011), no. 2, 233–239.
22. J. Xiao, X. Zhu, On linearly topological structure and property of fuzzy normed linear space. *Fuzzy Sets and Systems* **125** (2002), no. 2, 153–161.
23. L. Zadeh, Fuzzy sets. *Information and Control* **8** (1965), 338–353.

(Received 26.02.2021)

[1]Universidad del Atlántico Barranquilla, Colombia

[2]Department of Mathematics, Tripura University, Tripura, India

[3]Department of Mathematics, Tripura University, Tripura, India
Email address: carlosgranadosortiz@outlook.es
Email address: sumandas18842@gmail.com
Email address: tripathybc@gmail.com