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THERMO-ELASTIC AND THERMO-PIEZO-ELASTIC INTERACTION CRACK TYPE BOUNDARY-TRANSMISSION PROBLEMS WITH REGARD TO THE MICROROTATION

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Abstract. The paper studies three-dimensional interaction crack type boundary-transmission problems of pseudo-oscillations between thermo-elastic and thermo-piezo-elastic bodies taking microrotations into account. The model under consideration is based on the Green–Naghdi theory of thermo-piezo-electricity without energy dissipation. This theory permits the thermal waves to propagate only with a finite speed. The system of partial differential equations of pseudo-oscillations is obtained from the corresponding dynamical model by the Laplace transform. Using the potential theory and the method of boundary pseudodifferential equations, we prove the existence, uniqueness and regularity of solutions.

1. Introduction

In the present paper, we consider a boundary-transmission problem for a composed elastic structure consisting of two contacting bodies occupying two three-dimensional adjacent regions $\overline{\Omega^{(1)}}$ and $\overline{\Omega^{(2)}}$ with a common contacting interface, being a proper part of the boundaries $\partial\Omega^{(1)}$ and $\partial\Omega^{(2)}$ (see Figure 1). We analyze the case in which contacting elastic bodies are subject to different mathematical models. In particular, we consider *Green–Naghdi's model of thermo-piezo-electricity without energy dissipation* in $\Omega^{(1)}$ and *the model of isotropic homogeneous couple-stress thermo-elasticity* in $\Omega^{(2)}$. Theoretical study of such problems attracts great attention due to the widespread application of modern sensing and actuating devices based on the ability to transform mechanical, electric and thermal energies from one form to another. Therefore the mathematical models that take into account coupling effects between thermo-mechanical and electric fields in elastic composites became very popular over the last decades (see, e.g., [1, 26, 27, 30] and references therein).

A remarkable feature of the Green–Naghdi model is a finite speed of heat propagation in contrast to an infinite speed of heat transfer occurring in the classical heat equation theory. Complete historical and bibliographical notes in this direction can be found in [19], where the dynamical equations of the thermo-piezo-electricity without energy dissipation are derived on the basis of the Green–Naghdi theory established in [15, 16] and obtained by Eringen in [12, 13].

We investigate a general boundary-transmission problem for the above described two-component elastic structure with the appropriate boundary and transmission conditions which cover the conditions arising in the case of interfacial cracks. In each region we consider the corresponding system of partial differential equations of pseudo-oscillations containing a complex parameter τ. These systems are obtained from the corresponding dynamical models by the Laplace transform.

Using the potential method and the theory of pseudodifferential equations on manifolds with a boundary, we study the interface crack boundary-transmission problems and prove the uniqueness and existence of solutions in appropriate function spaces. Further, we analyze regularity of solutions and characterize singularities of the corresponding thermo-mechanical and electric fields near the exceptional curves (crack edges, lines, where the different type boundary conditions collide, and

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interface edges). In the upcoming papers, we are doing to apply the obtained results to the study of asymptotic properties of solutions of the corresponding dynamical problems.

Note that in [8] we have investigated the mixed type boundary value problems of the theory of thermo-piezo-electricity without energy dissipation with interior cracks.

The present investigation can be considered as a continuation of papers [4,7,10,11,22] and [24], but it turned out to be more difficult as far as it refers to the interaction between different dimensional physical fields (for the 9-dimensional field in $\Omega^{(1)}$ and 7-dimensional field in $\Omega^{(2)}$ see the problem setting in Subsection 2.4).

The paper is organized as follows. In Section 2, we describe the geometrical structure of the elastic composite body consisting of two interacting components, write down the governing pseudo-oscillation equations of Green–Naghdi's model of thermo-piezo-electricity without energy dissipation (PTEME model) and homogeneous isotropic couple-stress thermo-elasticity (CSTE model), formulate the interface crack type boundary-transmission problem and prove the uniqueness theorem in appropriate function spaces. In Section 3, we reduce equivalently the boundary-transmission problem to the system of boundary pseudo-differential equations, investigate the mapping properties of the corresponding pseudodifferential operator and prove the invertibility of the pseudodifferential operator in appropriate Bessel potential and Besov spaces. Further, we prove the theorem on the existence and some regularity results of solutions to the original interface crack boundary-transmission problem.

In Appendix, for the reader's convenience, we collected some auxiliary results used in the main text of the paper.

2. Formulation of the Interface Crack Boundary-Transmission Problem

2.1. **Geometrical configuration of the composite.** Let $\Omega^{(1)}$ and $\Omega^{(2)}$ be the bounded disjoint domains of the three-dimensional Euclidean space $\mathbb{R}^3$ with boundaries $\partial\Omega^{(1)}$ and $\partial\Omega^{(2)}$, respectively. Moreover, let $\partial\Omega^{(1)}$ and $\partial\Omega^{(2)}$ have a nonempty, simply connected intersection $\overline{\Gamma} := \partial\Omega^{(1)} \cap \partial\Omega^{(2)}$ of positive measure. From now on, Γ will be referred to as an *interface*. Throughout the paper, $n = n^{(1)}$ and $\nu = n^{(2)}$ stand for the outward unit normal vectors to $\partial\Omega^{(1)}$ and to $\partial\Omega^{(2)}$, respectively. Clearly, $n(x) = -\nu(x)$ for $x \in \Gamma$.

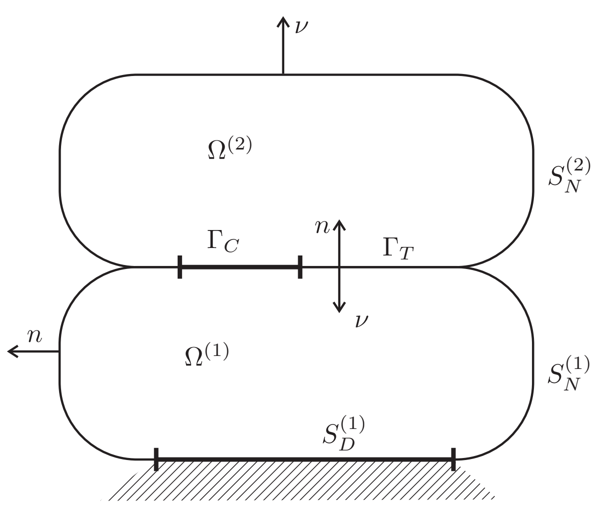

Figure 1. Composed body.

Further, let $\overline{\Gamma} = \overline{\Gamma_T} \cup \overline{\Gamma_C}$, where Γ_C is an open, simply connected proper part of Γ. Moreover, $\Gamma_T \cap \Gamma_C = \varnothing$ and $\partial\Gamma \cap \overline{\Gamma_C} = \varnothing$.

We set $S_N^{(2)} := \partial\Omega^{(2)} \setminus \overline{\Gamma}$ and $S^{(1)} := \partial\Omega^{(1)} \setminus \overline{\Gamma}$. Further, we denote by $S_D^{(1)}$ some open, nonempty, proper sub-manifold of $S^{(1)}$ and put $S_N^{(1)} := S^{(1)} \setminus \overline{S_D^{(1)}}$. Thus, we have the following dissections of the boundary surfaces (see Figure 1):

$$\partial\Omega^{(1)} = \overline{\Gamma_T} \cup \overline{\Gamma_C} \cup \overline{S_N^{(1)}} \cup \overline{S_D^{(1)}}, \quad \partial\Omega^{(2)} = \overline{\Gamma_T} \cup \overline{\Gamma_C} \cup \overline{S_N^{(2)}}.$$

In the sequel, for simplicity, we assume that $\partial\Omega^{(2)}$, $\partial\Omega^{(1)}$, $\partial S_N^{(2)}$, $\partial\Gamma_T$, $\partial\Gamma_C$, $\partial S_D^{(1)}$, $\partial S_N^{(1)}$ are C^∞-smooth and $\partial\Omega^{(2)} \cap \overline{S_D^{(1)}} = \varnothing$.

The superscript $(\cdot)^\top$ denotes transposition operation.

Throughout the paper, the summation over the repeated indices is meant from 1 to 3, unless otherwise stated.

2.2. **CSTE model.** Suppose the domain Ω_2 is filled with a homogeneous isotropic thermo-elastic material. The corresponding system of differential equations of pseudo-oscillations with respect to the sought vector function $U^{(2)}$ obtained from the dynamical equations of the linear model of thermo-elasticity with microrotation has the following form (see [20]):

$$\begin{aligned}&(\mu^{(2)}+\varkappa^{(2)})\partial_j\partial_j u_i^{(2)} + (\lambda^{(2)}+\mu^{(2)})\partial_i\partial_j u_j^{(2)} - \rho_2\tau^2 u_i^{(2)} + \varkappa^{(2)}\varepsilon_{ijk}\partial_j\phi_k^{(2)} \\ &\qquad - \tau\beta_0^{(2)}\partial_i\vartheta^{(2)} = -\rho_2 f_i^{(2)}, \quad i=1,2,3, \end{aligned} \tag{2.1}$$

$$\begin{aligned}&\gamma^{(2)}\partial_j\partial_j\phi_i^{(2)} + (\alpha^{(2)}+\beta^{(2)})\partial_j\partial_i\phi_j^{(2)} - \tau^2 I_0^{(2)}\phi_i^{(2)} + \varkappa^{(2)}\varepsilon_{ijk}\partial_j u_k^{(2)} \\ &\qquad - 2\varkappa^{(2)}\phi_i^{(2)} = -\rho_2 X_i^{(2)}, \quad i=1,2,3, \end{aligned} \tag{2.2}$$

$$k^{(2)}\partial_j\partial_j\vartheta^{(2)} - \tau^2 a^{(2)}\vartheta^{(2)} - \tau\beta_0^{(2)}\partial_j u_j^{(2)} = -\frac{1}{T_0}\,\rho_2 Q^{(2)}, \tag{2.3}$$

where $\tau = \sigma + i\omega$ is a complex parameter, $U^{(2)} = (u_1^{(2)}, u_2^{(2)}, u_3^{(2)}, \phi_1^{(2)}, \phi_2^{(2)}, \phi_3^{(2)}, \vartheta^{(2)})^\top$, $u^{(1)} = (u_1^{(1)}, u_2^{(1)}, u_3^{(1)})^\top$ is the displacement vector, $\phi^{(2)} = (\phi_1^{(2)}, \phi_2^{(2)}, \phi_3^{(2)})^\top$ is the vector of microrotation, $\vartheta^{(2)}$ is the temperature and $(f_1^{(2)}, f_2^{(2)}, f_3^{(2)})$ is the external body force per unit mass, $Q^{(2)}$ is the external rate of supply of heat per unit mass, $X_i^{(2)}$ is the external body couple per unit mass, T_0 is the initial reference temperature. We employ the notation $\partial = \partial_x = (\partial_1, \partial_2, \partial_3)$, $\partial_j = \partial/\partial x_j$.

The coefficients $\lambda^{(2)}$, $\mu^{(2)}$, $\varkappa^{(2)}$, $\alpha^{(2)}$, $\beta^{(2)}$, $\gamma^{(2)}$ are the elastic constants, $\beta_0^{(2)}$, $a^{(2)}$, $k^{(2)}$ are the thermal constants and $I_0^{(2)}$ is the coefficient of inertia, ε_{ijk} is the Levi–Civita symbol (see [20]).

Due to the positiveness of internal energy, the coefficients of system (2.1)–(2.3) must satisfy the following conditions:

$$\begin{gathered}\varkappa^{(2)} > 0, \quad \varkappa^{(2)} + 2\mu^{(2)} > 0, \quad \varkappa^{(2)} + 2\mu^{(2)} + 3\lambda^{(2)} > 0, \\ \gamma^{(2)} > |\beta^{(2)}|, \quad \beta^{(2)} + \gamma^{(2)} + 3\alpha^{(2)} > 0, \\ a^{(2)} > 0, \quad k^{(2)} > 0, \quad \rho_2 > 0, \quad I_0^{(2)} > 0, \quad \beta_0^{(2)} > 0, \end{gathered} \tag{2.4}$$

where ρ_2 is the mass density of $\Omega^{(2)}$.

Denote by

$$A^{(2)}(\partial_x, \tau) = [A_{ij}^{(2)}(\partial_x, \tau)]_{7\times 7}$$

the matrix differential operator generated by the left-hand side expressions in (2.1)–(2.3),

$$\begin{gathered}A_{ij}^{(2)}(\partial,\tau) = \delta_{ij}(\mu^{(2)}+\varkappa^{(2)})\partial_l\partial_l + (\lambda^{(2)}+\mu^{(2)})\partial_i\partial_j - \tau^2\rho_2\delta_{ij}, \quad A_{i,j+3}^{(2)}(\partial,\tau) = -\varkappa^{(2)}\varepsilon_{ijl}\partial_l, \\ A_{i7}^{(2)}(\partial,\tau) = -\tau\beta_0^{(2)}\partial_i, \quad A_{i+3,j}^{(2)}(\partial,\tau) = -\varkappa^{(2)}\varepsilon_{ijl}\partial_l, \\ A_{i+3,j+3}^{(2)}(\partial,\tau) = \delta_{ij}\gamma^{(2)}\partial_l\partial_l + (\alpha^{(2)}+\beta^{(2)})\partial_i\partial_j - (2\varkappa^{(2)}+\tau^2 I_0^{(2)})\delta_{ij}, \quad A_{i+3,7}^{(2)}(\partial,\tau) = 0, \\ A_{7j}^{(2)}(\partial,\tau) = -\tau\beta_0^{(2)}\partial_j, \quad A_{7,j+3}^{(2)}(\partial,\tau) = 0, \quad A_{77}^{(2)}(\partial,\tau) = k^{(2)}\partial_l\partial_l - \tau^2 a^{(2)}, \quad i,j=1,2,3.\end{gathered}$$

The system of equations (2.1)–(2.3) can be written in the matrix form

$$A^{(2)}(\partial_x,\tau)U^{(2)} = \mathcal{F}^{(2)},$$

where

$$U^{(2)} = (u_1^{(2)}, u_2^{(2)}, u_3^{(2)}, \phi_1^{(2)}, \phi_2^{(2)}, \phi_3^{(2)}, \vartheta^{(2)})^\top,$$

$$\mathcal{F}^{(2)} = -\Big(\rho_2 f_1^{(2)}, \rho_2 f_2^{(2)}, \rho_2 f_3^{(2)}, \rho_2 X_1^{(2)}, \rho_2 X_2^{(2)}, \rho_2 X_3^{(2)}, \frac{1}{T_0}\rho_2 Q^{(2)}\Big)^\top$$

and $A^{(2)}(\partial_x,\tau)$ is the 7-dimensional matrix differential operator corresponding to system (2.1)–(2.3).

The stress differential operator of thermo-elasticity is defined as follows:

$$\mathcal{T}^{(2)} = \mathcal{T}^{(2)}(\partial_x,\nu,\tau) := [\mathcal{T}_{ij}^{(2)}(\partial_x,\nu,\tau)]_{7\times 7},$$

where

$$\mathcal{T}_{ij}^{(2)}(\partial,\nu,\tau) = \lambda^{(2)}\nu_i\partial_j + \mu^{(2)}\nu_j\partial_i + \delta_{ij}(\mu^{(2)}+\varkappa^{(2)})\nu_k\partial_k, \quad \mathcal{T}_{i,j+3}^{(2)}(\partial,\nu,\tau) = -\varkappa^{(2)}\varepsilon_{ijk}n_k,$$
$$\mathcal{T}_{i7}^{(2)}(\partial,\nu,\tau) = -\tau\beta_0^{(2)}\nu_i, \quad \mathcal{T}_{i+3,j}^{(2)}(\partial,\nu,\tau) = 0,$$
$$\mathcal{T}_{i+3,j+3}^{(2)}(\partial,\nu,\tau) = \alpha^{(2)}\nu_i\partial_j + \beta^{(2)}\nu_j\partial_i + \delta_{ij}\gamma^{(2)}\nu_k\partial_k, \quad \mathcal{T}_{i+3,7}^{(2)}(\partial,\nu,\tau) = \nu_2^{(2)}\varepsilon_{lik}\nu_l\partial_k,$$
$$\mathcal{T}_{7j}^{(2)}(\partial,\nu,\tau) = 0, \quad \mathcal{T}_{7,j+3}^{(2)}(\partial,\nu,\tau) = -\nu_2^{(2)}\varepsilon_{ljk}\nu_l\partial_k,$$
$$\mathcal{T}_{77}^{(2)}(\partial,\nu,\tau) = k^{(2)}\nu_l\partial_l, \ \ i,j=1,2,3.$$

By $A^{(2,0)}(-i\xi)$ with $\xi\in\mathbb{R}^3$ we denote the principal homogeneous symbol matrix of the operator $A^{(2)}(\partial_x,\tau)$,

$$A^{(2,0)}(-i\xi) = -A^{(2,0)}(\xi) = -[A_{ij}^{(2,0)}(\xi)]_{3\times 3}$$

$$= -\begin{bmatrix} [\delta_{ij}(\mu^{(2)}+\kappa^{(2)})|\xi|^2 + (\lambda^{(2)}+\mu^{(2)})\xi_i\xi_j]_{3\times 3} & [0]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & [\delta_{ij}\gamma^{(2)}|\xi|^2 + (\alpha^{(2)}+\beta^{(2)})\xi_i\xi_j]_{3\times 3} & \\ [0]_{1\times 3} & [0]_{3\times 3} & k^{(2)}|\xi|^2 \end{bmatrix}_{7\times 7}.$$

Inequalities (2.4) imply that the matrix $A^{(2,0)}(\xi)$ is positive definite, i.e., there is a positive constant C depending only on the material parameters such that

$$\big(A^{(2,0)}(\xi)\zeta\cdot\zeta\big) = \big(-A^{(2,0)}(-i\xi)\zeta\cdot\zeta\big) \geq C|\xi|^2|\zeta|^2 \ \text{ for all } \ \xi\in\mathbb{R}^3 \ \text{ and for all } \ \zeta\in\mathbb{C}^7.$$

Here and in what follows, the central dot denotes the scalar product in the space of complex-valued vectors $\mathbb{C}^m$ and the overline denotes complex conjugation.

2.3. **PTEME model.** The domain Ω_1 is filled with a thermo-electro-elastic material. The corresponding system of differential equations of pseudo-oscillations with respect to the sought vector function $U^{(1)}$ has the following form (see [19]):

$$(\mu^{(1)}+\varkappa^{(1)})\partial_j\partial_j u_i^{(1)} + (\lambda^{(1)}+\mu^{(1)})\partial_i\partial_j u_j^{(1)} - \rho_1\tau^2 u_i^{(1)} + \varkappa^{(1)}\varepsilon_{ijk}\partial_j\phi_k^{(1)}$$
$$+ \lambda_0^{(1)}\partial_i\varphi^{(1)} - \tau\beta_0^{(1)}\partial_i\vartheta^{(1)} = -\rho_1 g_i^{(1)}, \ \ i=1,2,3, \tag{2.5}$$

$$\gamma^{(1)}\partial_j\partial_j\phi_i^{(1)} + (\alpha^{(1)}+\beta^{(1)})\partial_j\partial_i\phi_j^{(1)} - \tau^2 I_0^{(1)}\phi_i^{(1)} + \varkappa^{(1)}\varepsilon_{ijk}\partial_j u_k^{(1)}$$
$$- 2\varkappa^{(1)}\phi_i^{(1)} = -\rho_1 X_i^{(1)}, \ \ i=1,2,3, \tag{2.6}$$

$$(a_0^{(1)}\partial_j\partial_j - \xi_0^{(1)})\varphi^{(1)} - j_0^{(1)}\tau^2\varphi^{(1)} - \lambda_2^{(1)}\partial_j\partial_j\psi^{(1)} + \nu_1^{(1)}\partial_j\partial_j\vartheta^{(1)}$$
$$+ c_0^{(1)}\tau\vartheta^{(1)} - \lambda_0^{(1)}\partial_j u_j^{(1)} = -\rho_1 F^{(1)}, \tag{2.7}$$

$$\lambda_0^{(1)}\partial_j\partial_j\varphi^{(1)} + \chi^{(1)}\partial_j\partial_j\psi^{(1)} + \nu_3^{(1)}\partial_j\partial_j\vartheta^{(1)} = -g^{(1)}, \tag{2.8}$$

$$k^{(1)}\partial_j\partial_j\vartheta^{(1)} - \tau^2 a^{(1)}\vartheta^{(1)} - \tau\beta_0^{(1)}\partial_j u_j^{(1)} - \tau c_0^{(1)}\varphi^{(1)} + \nu_1^{(1)}\partial_j\partial_j\varphi^{(1)}$$
$$- \nu_3^{(1)}\partial_j\partial_j\psi^{(1)} = -\frac{1}{T_0}\,\rho_1 Q^{(1)}, \tag{2.9}$$

where $U^{(1)} = (u_1^{(1)}, u_2^{(1)}, u_3^{(1)}, \phi_1^{(1)}, \phi_2^{(1)}, \phi_3^{(1)}, \varphi^{(1)}, \psi^{(1)}, \vartheta^{(1)})^\top$, $u^{(1)} = (u_1^{(1)}, u_2^{(1)}, u_3^{(1)})^\top$ is the displacement vector, $\phi^{(1)} = (\phi_1^{(1)}, \phi_2^{(1)}, \phi_3^{(1)})^\top$ is the vector of microrotation, $\varphi^{(1)}$ is the microstretch, $\psi^{(1)}$ is the electric field potential, $\vartheta^{(1)}$ is the temperature and $(g_1^{(1)}, g_2^{(1)}, g_3^{(1)})$ is the external body force per unit mass, $Q^{(1)}$ is the external rate of supply of heat per unit mass, $X_i^{(1)}$ is the external body couple per unit mass, $F^{(1)}$ is the microstretch body force, $g^{(1)}$ is the density of free charge.

Denote by

$$A^{(1)}(\partial_x,\tau)=[A_{ij}^{(1)}(\partial_x,\tau)]_{9\times 9}$$

the matrix differential operator generated by the left-hand side expressions in (2.5)–(2.9),

$$A_{ij}^{(1)}(\partial,\tau)=\delta_{ij}(\mu^{(1)}+\varkappa^{(1)})\partial_l\partial_l+(\lambda^{(1)}+\mu^{(1)})\partial_i\partial_j-\tau^2\rho_1\delta_{ij},\quad A_{i,j+3}^{(1)}(\partial,\tau)=-\varkappa^{(1)}\varepsilon_{ijl}\partial_l,$$
$$A_{i7}^{(1)}(\partial,\tau)=\lambda_0^{(1)}\partial_i,\quad A_{i8}^{(1)}(\partial,\tau)=0,\quad A_{i9}^{(1)}(\partial,\tau)=-\tau\beta_0^{(1)}\partial_i,\quad A_{i+3,j}^{(1)}(\partial,\tau)=-\varkappa^{(1)}\varepsilon_{ijl}\partial_l,$$
$$A_{i+3,j+3}^{(1)}(\partial,\tau)=\delta_{ij}\gamma^{(1)}\partial_l\partial_l+(\alpha^{(1)}+\beta^{(1)})\partial_i\partial_j-(2\varkappa^{(1)}+\tau^2I_0^{(1)})\delta_{ij},\quad A_{i+3,j+6}^{(1)}(\partial,\tau)=0,$$
$$A_{7j}^{(1)}(\partial,\tau)=-\lambda_0^{(1)}\partial_j,\ \ A_{7,j+3}^{(1)}(\partial,\tau)=0,\ \ A_{77}^{(1)}(\partial,\tau)=a_0^{(1)}\partial_l\partial_l-(\xi_0^{(1)}+\tau^2j_0^{(1)}),$$
$$A_{78}^{(1)}(\partial,\tau)=-\lambda_2^{(1)}\partial_l\partial_l,\ \ A_{79}^{(1)}(\partial,\tau)=\nu_1^{(1)}\partial_l\partial_l+\tau c_0^{(1)},$$
$$A_{8j}^{(1)}(\partial,\tau)=0,\quad A_{8,j+3}^{(1)}(\partial,\tau)=0,\quad A_{87}^{(1)}(\partial,\tau)=\lambda_2^{(1)}\partial_l\partial_l,\quad A_{88}^{(1)}(\partial,\tau)=\chi^{(1)}\partial_l\partial_l,$$
$$A_{89}^{(1)}(\partial,\tau)=\nu_3^{(1)}\partial_l\partial_l,\quad A_{9j}^{(1)}(\partial,\tau)=-\tau\beta_0^{(1)}\partial_j,\quad A_{9,j+3}^{(1)}(\partial,\tau)=0,$$
$$A_{97}^{(1)}(\partial,\tau)=\nu_1^{(1)}\partial_l\partial_l-\tau c_0^{(1)},\quad A_{98}^{(1)}(\partial,\tau)=-\nu_3^{(1)}\partial_l\partial_l,\quad A_{99}^{(1)}(\partial,\tau)=k^{(1)}\partial_l\partial_l-\tau^2a^{(1)},\ \ i,j=1,2,3.$$

The system of equations (2.5)–(2.9) can be written in the matrix form

$$A^{(1)}(\partial_x,\tau)U^{(1)}=\mathcal{F}^{(1)},$$

where

$$U^{(1)}=(u_1^{(1)},u_2^{(1)},u_3^{(1)},\phi_1^{(1)},\phi_2^{(1)},\phi_3^{(1)},\varphi^{(1)},\psi^{(1)},\vartheta^{(1)})^\top,$$
$$\mathcal{F}^{(1)}=-\Big(\rho_2g_1^{(1)},\rho_2g_2^{(1)},\rho_2g_3^{(1)},\rho_2X_1^{(1)},\rho_2X_2^{(1)},\rho_2X_3^{(1)},\rho_2F^{(1)},g^{(1)},\frac{1}{T_0}\rho_2Q^{(1)}\Big)^\top$$

and $A^{(1)}(\partial_x,\tau)$ is the 9-dimensional matrix differential operator corresponding to system (2.5)–(2.9).

The stress differential operator of thermo-electro-elasticity is defined as follows:

$$\mathcal{T}^{(1)}=\mathcal{T}^{(1)}(\partial_x,\nu,\tau):=[\mathcal{T}_{ij}^{(1)}(\partial_x,\nu,\tau)]_{9\times 9},$$

where

$$\mathcal{T}_{ij}^{(1)}(\partial,n,\tau)=\lambda^{(1)}n_i\partial_j+\mu^{(1)}n_j\partial_i+\delta_{ij}(\mu^{(1)}+\varkappa^{(1)})n_k\partial_k,\quad \mathcal{T}_{i,j+3}^{(1)}(\partial,n,\tau)=-\varkappa^{(1)}\varepsilon_{ijk}n_k,$$
$$\mathcal{T}_{i7}^{(1)}(\partial,n,\tau)=\lambda_0^{(1)}n_i,\quad \mathcal{T}_{i8}^{(1)}(\partial,n,\tau)=0,\quad \mathcal{T}_{i9}^{(1)}(\partial,n,\tau)=-\tau\beta_0^{(1)}n_i,\quad \mathcal{T}_{i+3,j,}^{(1)}(\partial,n,\tau)=0,$$
$$\mathcal{T}_{i+3,j+3}^{(1)}(\partial,n,\tau)=\alpha^{(1)}n_i\partial_j+\beta^{(1)}n_j\partial_i+\delta_{ij}\gamma^{(1)}n_k\partial_k,\quad \mathcal{T}_{i+3,7}^{(1)}(\partial,n,\tau)=b_0^{(1)}\varepsilon_{lik}n_l\partial_k,$$
$$\mathcal{T}_{i+3,8}^{(1)}(\partial,n,\tau)=\lambda_1^{(1)}\varepsilon_{lik}n_l\partial_k,\quad \mathcal{T}_{i+3,9}^{(1)}(\partial,n,\tau)=\nu_2^{(1)}\varepsilon_{lik}n_l\partial_k,\quad \mathcal{T}_{7j}^{(1)}(\partial,n,\tau)=0,$$
$$\mathcal{T}_{7,j+3}^{(1)}(\partial,n,\tau)=-b_0^{(1)}\varepsilon_{ljk}n_l\partial_k,\quad \mathcal{T}_{77}^{(1)}(\partial,n,\tau)=a_0^{(1)}n_k\partial_k,\quad \mathcal{T}_{78}^{(1)}(\partial,n,\tau)=-\lambda_2^{(1)}n_k\partial_k,$$
$$\mathcal{T}_{79}^{(1)}(\partial,n,\tau)=\nu_1^{(1)}n_k\partial_k,\quad \mathcal{T}_{8j}^{(1)}(\partial,n,\tau)=0,\quad \mathcal{T}_{8,j+3}^{(1)}(\partial,n,\tau)=-\lambda_1^{(1)}\varepsilon_{ljk}n_l\partial_k,$$
$$\mathcal{T}_{87}^{(1)}(\partial,n,\tau)=\lambda_2^{(1)}n_k\partial_k,\quad \mathcal{T}_{88}^{(1)}(\partial,n,\tau)=\chi^{(1)}n_k\partial_k,\quad \mathcal{T}_{89}^{(1)}(\partial,n,\tau)=\nu_3^{(1)}n_k\partial_k,$$
$$\mathcal{T}_{9j}^{(1)}(\partial,n,\tau)=0,\quad \mathcal{T}_{9,j+3}^{(1)}(\partial,n,\tau)=-\nu_2^{(1)}\varepsilon_{ljk}n_l\partial_k,\quad \mathcal{T}_{97}^{(1)}(\partial,n,\tau)=\nu_1^{(1)}n_k\partial_k,$$
$$\mathcal{T}_{98}^{(1)}(\partial,n,\tau)=-\nu_3^{(1)}n_k\partial_k,\quad \mathcal{T}_{99}^{(1)}(\partial,n,\tau)=k^{(1)}n_l\partial_l,\ \ i,j=1,2,3.$$

The coefficients $\lambda^{(1)}$, $\mu^{(1)}$, $\varkappa^{(1)}$, $\lambda_0^{(1)}$, $\beta_0^{(1)}$, $\alpha^{(1)}$, $\beta^{(1)}$, $\gamma^{(1}$, $\lambda_1^{(1)}$, $\nu_1^{(1)}$, $a_0^{(1)}$, $\lambda_2^{(1)}$, $\nu_2^{(1)}$,$\xi_0^{(1)}$, $c_0^{(1)}$, $a^{(1)}$, $k^{(1)}$, $\nu_3^{(1)}$, $b_0^{(1)}$, $\chi^{(1)}$ are constitutive constants, and $I_0^{(1)}$ is the coefficient of inertia, $j_0^{(1)}$ is the microstretch inertia, ε_{ijk} is the Levi–Civita symbol (see [8]).

Due to the positiveness of internal energy, the coefficients of the system (2.5)–(2.9) must satisfy the following conditions:

$$\varkappa^{(1)}>0,\ \ \varkappa^{(1)}+2\mu^{(1)}>0,\ \ \varkappa^{(1)}+2\mu^{(1)}+3\lambda^{(1)}>0,$$

$$\xi_0^{(1)}(\varkappa^{(1)}+2\mu^{(1)}+3\lambda^{(1)})>3(\lambda_0^{(1)})^2,$$
$$\gamma^{(1)}>|\beta^{(1)}|,\ \ a_0^{(2)}k^{(1)}-(\nu_1^{(1)})^2>0,\ \ \beta^{(1)}+\gamma^{(1)}+3\alpha^{(1)}>0,$$
$$\chi^{(1)}>0,\ \ a^{(1)}>0,\ \ k^{(1)}>0,\ \ a_0^{(1)}>0,\ \ a_0^{(1)}(\gamma^{(1)}-\beta^{(1)})>2(b_0^{(1)})^2,$$
$$(\gamma^{(1)}-\beta^{(1)})[a_0^{(1)}k^{(1)}-(\nu_1^{(1)})^2]+4b_0^{(1)}\nu_1^{(1)}\nu_2^{(1)}-2a_0^{(1)}(\nu_2^{(1)})^2-2k^{(1)}(b_0^{(1)})^2>0,$$
$$\rho_1>0,\ \ I_0^{(1)}>0,\ \ j_0^{(1)}>0,\ \ \beta_0^{(1)}>0,$$

where ρ_1 is the mass density.

2.4. **Formulation of the interface crack boundary-transmission problem.** By W_p^r, H_p^s and $B_{p,q}^s$ with $r\geqslant 0$, $s\in\mathbb{R}$, $1<p<\infty$, $1\leqslant q\leqslant\infty$, we denote the Sobolev–Slobodetskii, Bessel potential, and Besov function spaces, respectively, (see, e.g., [29]). Recall that $H_2^r=W_2^r=B_{2,2}^r$, $H_2^s=B_{2,2}^s$, $W_p^t=B_{p,p}^t$, and $H_p^k=W_p^k$, for any $r\geqslant 0$, for any $s\in\mathbb{R}$, for any positive and non-integer t, and for any non-negative integer k.

Let $\mathcal{M}_0$ be a smooth surface without boundary. For a proper sub-manifold $\mathcal{M}\subset\mathcal{M}_0$, we denote by $\widetilde{H}_p^s(\mathcal{M})$ and $\widetilde{B}_{p,q}^s(\mathcal{M})$ the following subspaces of $H_p^s(\mathcal{M}_0)$ and $B_{p,q}^s(\mathcal{M}_0)$, respectively,

$$\widetilde{H}_p^s(\mathcal{M})=\big\{g:\ g\in H_p^s(\mathcal{M}_0),\ \operatorname{supp} g\subset\overline{\mathcal{M}}\big\},$$
$$\widetilde{B}_{p,q}^s(\mathcal{M})=\big\{g:\ g\in B_{p,q}^s(\mathcal{M}_0),\ \operatorname{supp} g\subset\overline{\mathcal{M}}\big\},$$

while $H_p^s(\mathcal{M})$ and $B_{p,q}^s(\mathcal{M})$ stand for the spaces of restrictions on $\mathcal{M}$ of functions from $H_p^s(\mathcal{M}_0)$ and $B_{p,q}^s(\mathcal{M}_0)$, respectively,

$$H_p^s(\mathcal{M})=\big\{r_{\mathcal{M}}f:\ f\in H_p^s(\mathcal{M}_0)\big\},\quad B_{p,q}^s(\mathcal{M})=\big\{r_{\mathcal{M}}f:\ f\in B_{p,q}^s(\mathcal{M}_0)\big\},$$

where $r_{\mathcal{M}}$ is the restriction operator onto $\mathcal{M}$.

Now, we formulate the interface crack boundary-transmission problem: *Find vector functions*

$$U^{(1)}=(u^{(1)},\varphi^{(1)},\phi^{(1)},\psi^{(1)},\vartheta^{(1)})^\top=(u_1^{(1)},\dots,u_9^{(1)})^\top\ :\ \Omega^{(1)}\to\mathbb{C}^9,$$
$$U^{(2)}=(u^{(2)},\phi^{(2)},\vartheta^{(2)})^\top=(u_1^{(2)},\dots,u_7^{(2)})^\top\ :\ \Omega^{(2)}\to\mathbb{C}^7,$$

belonging, respectively, to the spaces $[W_p^1(\Omega^{(1)})]^9$ *and* $[W_p^1(\Omega^{(2)})]^7$ *with* $1<p<\infty$ *and satisfying*
(i) *the systems of partial differential equations*:

$$A^{(1)}(\partial_x,\tau)\,U^{(1)}=0\ \text{ in }\ \Omega^{(1)},\tag{2.10}$$
$$A^{(2)}(\partial_x,\tau)\,U^{(2)}=0\ \text{ in }\ \Omega^{(2)},\tag{2.11}$$

(ii) *the boundary conditions*:

$$\big\{\mathcal{T}^{(1)}(\partial_x,n,\tau)\,U^{(1)}\big\}^+=Q^{(1)}\ \text{ on }\ S_N^{(1)},\tag{2.12}$$
$$\big\{\mathcal{T}^{(2)}(\partial_x,\nu,\tau)\,U^{(2)}\big\}^+=Q^{(2)}\ \text{ on }\ S_N^{(2)},\tag{2.13}$$
$$\big\{U^{(1)}\big\}^+=f^{(1)}\ \text{ on }\ S_D^{(1)},\tag{2.14}$$
$$\{u_7^{(1)}\}^+=f_7\ \text{ on }\ \Gamma_T,\tag{2.15}$$
$$\{u_8^{(1)}\}^+=f_8\ \text{ on }\ \Gamma_T,\tag{2.16}$$

(iii) *the transmission conditions on* Γ_T :

$$\Big\{u_j^{(1)}\Big\}^+-\big\{u_j^{(2)}\big\}^+=f_j\quad\text{on }\ \Gamma_T,\qquad j=\overline{1,6},\tag{2.17}$$
$$\Big\{u_9^{(1)}\Big\}^+-\big\{u_7^{(2)}\big\}^+=f_9\quad\ \text{on }\ \Gamma_T,\tag{2.18}$$
$$\big\{\big[\mathcal{T}^{(1)}(\partial_x,n,\tau)U^{(1)}\big]_j\big\}^++\big\{[\mathcal{T}^{(2)}(\partial_x,\nu,\tau)U^{(2)}]_j\big\}^+=F_j,\quad\text{on }\ \Gamma_T,\quad j=\overline{1,6},\tag{2.19}$$
$$\big\{\big[\mathcal{T}^{(1)}(\partial_x,n,\tau)U^{(1)}\big]_9\big\}^++\big\{[\mathcal{T}^{(2)}(\partial_x,\nu,\tau)U^{(2)}]_7\big\}^+=F_7,\quad\text{on }\ \Gamma_T,\tag{2.20}$$

(iv) *the interfacial crack conditions on* Γ_C:

$$\big\{\mathcal{T}^{(1)}(\partial_x,n,\tau)\,U^{(1)}\big\}^+=\widetilde{Q}^{(1)}\quad\text{on }\ \Gamma_C,\tag{2.21}$$

$$\big\{\mathcal{T}^{(2)}(\partial_x,\nu,\tau)\,U^{(2)}\big\}^+=\widetilde{Q}^{(2)}\quad\text{on }\ \Gamma_C,\tag{2.22}$$

where $n=-\nu$ *on* Γ,

$$\begin{aligned}
Q^{(1)}&=(Q^{(1)}_1,\dots,Q^{(1)}_9)^\top\in\big[\,B^{-\frac1p}_{p,p}(S^{(1)}_N)\,\big]^9,\\
\widetilde{Q}^{(1)}&=(\widetilde{Q}^{(1)}_1,\dots,\widetilde{Q}^{(1)}_9)^\top\in\big[\,B^{-\frac1p}_{p,p}(\Gamma_C)\,\big]^9,\\
Q^{(2)}&=\big(Q^{(2)}_1,\dots,Q^{(2)}_7\big)^\top\in\big[\,B^{-\frac1p}_{p,p}(S^{(2)}_N)\,\big]^7,\\
\widetilde{Q}^{(2)}&=\big(\widetilde{Q}^{(2)}_1,\dots,\widetilde{Q}^{(2)}_7\big)^\top\in\big[\,B^{-\frac1p}_{p,p}(\Gamma_C)\,\big]^7,\\
f^{(1)}&=(f^{(1)}_1,\dots,f^{(1)}_9)^\top\in\big[\,B^{1-\frac1p}_{p,p}(S^{(1)}_D)\,\big]^9,\\
f&=\big(f_1,\dots,f_9\big)^\top\in\big[\,B^{1-\frac1p}_{p,p}(\Gamma_T)\,\big]^9,\\
F&=\big(F_1,\dots,F_7\big)^\top\in\big[\,B^{-\frac1p}_{p,p}(\Gamma_T)\,\big]^7.
\end{aligned}\tag{2.23}$$

Note that, in addition, the functions F_j, $Q^{(1)}_j$, $\widetilde{Q}^{(1)}_j$, $\widetilde{Q}^{(2)}_j$ and $Q^{(2)}_j$ have to satisfy some evident compatibility conditions (see Subsection 3.1, inclusions (3.22), (3.23)).

We have the following uniqueness theorem for $p=2$.

Theorem 2.1. *Let* $\Omega^{(1)}$ *and* $\Omega^{(2)}$ *be the Lipschitz domains and either* $\tau=\sigma+i\omega$ *with* $\sigma>0$, *or* $\tau=0$. *Then the interface crack boundary transmission problem* (2.10)–(2.23) *has at most one solution pair* $(U^{(1)},U^{(2)})$ *in the space* $[W^1_2(\Omega^{(1)})]^9\times[W^1_2(\Omega^{(2)})]^7$, *provided* $\operatorname{mes}S^{(1)}_D>0$.

Proof. Proof of the theorem is quite similar to that of Theorem 1.1 in [5]. □

Later we will prove the uniqueness theorem for $p\neq2$. To prove the existence of solutions to the above formulated interface crack boundary-transmission problem, we use the potential method and the theory of pseudodifferential equations. To this end, we introduce the following single layer potentials:

$$V^{(1)}_\tau(h^{(1)})(x)=\int\limits_{\partial\Omega^{(1)}}\Gamma^{(1)}(x-y,\tau)\,h^{(1)}(y)\,d_yS,$$

$$V^{(2)}_\tau(h^{(2)})(x)=\int\limits_{\partial\Omega^{(2)}}\Gamma^{(2)}(x-y,\tau)\,h^{(2)}(y)\,d_yS,$$

where $\Gamma^{(1)}(x,\tau)$ and $\Gamma^{(2)}(x,\tau)$ are the fundamental matrices of the differential operators $A^{(1)}(\partial_x,\tau)$ and $A^{(2)}(\partial_x,\tau)$, respectively, $h^{(1)}=(h^{(1)}_1,\dots,h^{(1)}_9)^\top$ and $h^{(2)}=(h^{(2)}_1,\dots,h^{(2)}_7)^\top$ are the density vector functions. The explicit expressions of the fundamental matrices $\Gamma^{(1)}(x,\tau)$ and $\Gamma^{(2)}(x,\tau)$ and their properties can be found in [6] and [7].

We introduce also the following boundary integral operators generated by the single layer potentials

$$\mathcal{H}^{(1)}_\tau(h^{(1)})(z)=\int\limits_{\partial\Omega^{(1)}}\Gamma^{(1)}(z-y,\tau)\,h^{(1)}(y)\,d_yS,\quad z\in\partial\Omega^{(1)},\tag{2.24}$$

$$\mathcal{K}^{(1)}_\tau(h^{(1)})(z)=\int\limits_{\partial\Omega^{(1)}}\mathcal{T}^{(1)}(\partial_z,n(z),\tau)\Gamma^{(1)}(z-y,\tau)\,h^{(1)}(y)\,d_yS,\quad z\in\partial\Omega^{(1)},\tag{2.25}$$

$$\mathcal{H}^{(2)}_\tau(h^{(2)})(z)=\int\limits_{\partial\Omega^{(2)}}\Gamma^{(2)}(z-y,\tau)\,h^{(2)}(y)\,d_yS,\quad z\in\partial\Omega^{(2)},\tag{2.26}$$

$$\mathcal{K}_\tau^{(2)}(h^{(2)})(z) = \int\limits_{\partial\Omega^{(2)}} \mathcal{T}^{(2)}(\partial_z, n(z), \tau)\Gamma^{(2)}(z-y,\tau)\, h^{(2)}(y)\, d_y S, \quad z \in \partial\Omega^{(2)}. \tag{2.27}$$

Note that $\mathcal{H}_\tau^{(1)}$ and $\mathcal{H}_\tau^{(2)}$ are pseudodifferential operators of order -1, while $\mathcal{K}_\tau^{(1)}$ and $\mathcal{K}_\tau^{(2)}$ are pseudodifferential operators of order 0, i.e., singular integral operators (for details see Appendix).

Now, we formulate several auxiliary lemmas proved in [7].

Lemma 2.2. *Let* $\operatorname{Re}\tau = \sigma > 0$ *and* $1 < p < \infty$. *An arbitrary solution vector* $U^{(2)} \in [\,W_p^1(\Omega^{(2)})\,]^7$ *to the homogeneous equation* $A^{(2)}(\partial,\tau)\,U^{(2)} = 0$ *in* $\Omega^{(2)}$ *can be uniquely represented by the single layer potential*

$$U^{(2)} \;=\; V_\tau^{(2)}\Big(\big[\,P_\tau^{(2)}\big]^{-1}\chi^{(2)}\Big) \;\; in \;\; \Omega^{(2)},$$

where

$$P_\tau^{(2)} := -2^{-1}\, I_7 + \mathcal{K}_\tau^{(2)}, \quad \chi^{(2)} = \big\{\mathcal{T}^{(2)}U^{(2)}\,\big\}^+ \in \big[\,B_{p,p}^{-\frac{1}{p}}(\partial\Omega^{(2)})\,\big]^7, \tag{2.28}$$

and $\mathcal{K}_\tau^{(2)}$ *is defined by* (2.27).

For the mapping properties and invertibility of the operator $P_\tau^{(2)}$ in appropriate function spaces see Theorem 4.4.

Lemma 2.3. *Let* $\operatorname{Re}\tau = \sigma > 0$ *and*

$$P_\tau^{(1)} := -2^{-1}\, I_9 + \mathcal{K}_\tau^{(1)} + \beta\,\mathcal{H}_\tau^{(1)}, \tag{2.29}$$

where $\mathcal{K}_\tau^{(1)}$ *and* $\mathcal{H}_\tau^{(1)}$ *are defined by* (2.25) *and* (2.24)*, respectively, and* β *is a smooth real-valued scalar function on* $S^{(1)}$*, not vanishing identically and satisfying the conditions*

$$\beta \geqslant 0, \qquad \operatorname{supp}\beta \subset S_D^{(1)}. \tag{2.30}$$

Then the operators

$$P_\tau^{(1)} \;:\; \big[\,H_p^s(\partial\Omega^{(1)})\,\big]^9 \to \big[\,H_p^s(\partial\Omega^{(1)})\,\big]^9,$$
$$P_\tau^{(1)} \;:\; \big[\,B_{p,q}^s(\partial\Omega^{(1)})\,\big]^9 \to \big[\,B_{p,q}^s(\partial\Omega^{(1)})\,\big]^9$$

are invertible for all $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, *and for all* $s \in \mathbb{R}$.

As a consequence, we have the following

Lemma 2.4. *Let* $\operatorname{Re}\tau = \sigma > 0$ *and* $1 < p < \infty$. *An arbitrary solution* $U^{(1)} \in [\,W_p^1(\Omega^{(1)})\,]^9$ *to the homogeneous equation* $A^{(1)}(\partial_x,\tau)U^{(1)} = 0$ *in* $\Omega^{(1)}$ *can be uniquely represented by the single layer potential*

$$U^{(1)} = V_\tau^{(1)}\big(\,\big[P_\tau^{(1)}\big]^{-1}\chi\big) \;\; in \;\; \Omega^{(1)},$$

where

$$\chi = \{\mathcal{T}^{(1)}U^{(1)}\}^+ + \beta\,\{U^{(1)}\}^+ \in [\,B_{p,p}^{-\frac{1}{p}}(\partial\Omega^{(1)})\,]^9.$$

3. The Existence and Regularity Results

3.1. **Reduction to boundary equations.** Let us return to problem (2.10)–(2.23) and derive the equivalent boundary integral formulation. Keeping in mind (2.23), let

$$G^{(1)} := \begin{cases} Q^{(1)} & \text{on } S_N^{(1)}, \\ \widetilde{Q}^{(1)} & \text{on } \Gamma_C, \end{cases} \qquad G^{(2)} := \begin{cases} Q^{(2)} & \text{on } S_N^{(2)}, \\ \widetilde{Q}^{(2)} & \text{on } \Gamma_C, \end{cases}$$
$$G^{(1)} \in \big[\,H^{-1/2}(S_N^{(1)} \cup \Gamma_C)\,\big]^9, \qquad G^{(2)} \in \big[\,H^{-1/2}(S_N^{(2)} \cup \Gamma_C)\,\big]^7, \tag{3.1}$$

and

$$G_0^{(1)} = (G_{01}^{(1)}, \dots, G_{09}^{(1)})^\top \in \big[\,B_{p,p}^{-\frac{1}{p}}(\partial\Omega^{(1)})\,\big]^9,$$
$$G_0^{(2)} = (G_{01}^{(2)}, \dots, G_{07}^{(2)})^\top \in \big[\,B_{p,p}^{-\frac{1}{p}}(\partial\Omega^{(2)})\big]^7 \tag{3.2}$$

be some fixed extensions of the vector functions $G^{(1)}$ and $G^{(2)}$, respectively, onto $\partial\Omega^{(1)}$ and $\partial\Omega^{(2)}$ preserving the space. It is evident that arbitrary extensions of the same vector functions can then be represented as

$$G^{(1)\,*} = G_0^{(1)} + \psi + h^{(1)}, \qquad G^{(2)\,*} = G_0^{(2)} + h^{(2)},$$

where

$$\begin{aligned} \psi &:= (\psi_1, \dots, \psi_6)^\top \in \big[\, \widetilde{B}_{p,p}^{-\frac{1}{p}}(S_D^{(1)})\,\big]^9, \\ h^{(1)} &:= (h_1^{(1)}, \dots, h_6^{(1)})^\top \in \big[\, \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma_T)\,\big]^9, \\ h^{(2)} &:= (h_1^{(2)}, \dots, h_4^{(2)})^\top \in \big[\, \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma_T)\,\big]^7 \end{aligned} \tag{3.3}$$

are arbitrary vector functions.

We look for a solution pair $(U^{(1)}, U^{(2)})$ of the mixed boundary-transmission problem (2.10)–(2.23) in the form of single layer potentials

$$U^{(1)} = (u_1^{(1)}, \dots, u_9^{(1)})^\top = V_\tau^{(1)}\big([\,P_\tau^{(1)}]^{-1}\,\big[\,G_0^{(1)} + \psi + h^{(1)}\big]\,\big) \ \text{ in } \ \Omega^{(1)}, \tag{3.4}$$

$$U^{(2)} = (u_1^{(2)}, \dots, u_7^{(2)})^\top = V_\tau^{(2)}(\,[\,P_\tau^{(2)}\,]^{-1}\,\big[\,G_0^{(2)} + h^{(2)}\big]\,\big) \ \text{ in } \ \Omega^{(2)}, \tag{3.5}$$

where $P_\tau^{(1)}$ and $P_\tau^{(2)}$ are given by (2.29) and (2.28), and $h^{(1)}$, $h^{(2)}$ and ψ are the unknown vector functions satisfying inclusions (3.3).

Keeping in mind (2.30), we see that the homogeneous differential equations (2.10), (2.11), the boundary conditions (2.12), (2.13) and the crack conditions (2.21), (2.22) are satisfied automatically.

The remaining boundary and transmission conditions (2.17)–(2.20) lead to the system of pseudodifferential equations for the unknown vector functions ψ, $h^{(1)}$ and $h^{(2)}$

$$r_{_{S_D^{(1)}}}\big[\,\mathcal{H}_\tau^{(1)}\,[P_\tau^{(1)}]^{-1}\,\big(G_0^{(1)} + \psi + h^{(1)}\big)\,\big] = f^{(1)} \ \text{ on } \ S_D^{(1)}, \tag{3.6}$$

$$r_{\Gamma_T}\big[\,\mathcal{H}_\tau^{(1)}\,[P_\tau^{(1)}]^{-1}\,(\,G_0^{(1)} + \psi + h^{(1)})\,\big]_j = f_j \ \text{ on } \ \Gamma_T, \ \ j = 7, 8, \tag{3.7}$$

$$\begin{aligned} r_{\Gamma_T}\big[\mathcal{H}_\tau^{(1)}\,[P_\tau^{(1)}]^{-1}\,(G_0^{(1)} + \psi + h^{(1)})\big]_j - r_{\Gamma_T}\big[\mathcal{H}_\tau^{(2)}\,[P_\tau^{(2)}]^{-1}(G_0^{(2)} + \, h^{(2)})\big]_j &= f_j \ \text{ on } \ \Gamma_T, \\ j &= \overline{1,6}, \end{aligned} \tag{3.8}$$

$$r_{\Gamma_T}\big[\mathcal{H}_\tau^{(1)}\,[P_\tau^{(1)}]^{-1}\,(G_0^{(1)} + \psi + h^{(1)})\big]_9 - r_{\Gamma_T}\big[\mathcal{H}_\tau^{(2)}\,[P_\tau^{(2)}]^{-1}(G_0^{(2)} + h^{(2)})\big]_7 = f_9 \ \text{ on } \ \Gamma_T, \tag{3.9}$$

$$r_{\Gamma_T}[G_0^{(1)} + \psi + h^{(1)}]_j + r_{\Gamma_T}[G_0^{(2)} + h^{(2)}]_j = F_j \ \text{ on } \ \Gamma_T, \ \ j = \overline{1,6}, \tag{3.10}$$

$$r_{\Gamma_T}[G_0^{(1)} + \psi + h^{(1)}]_9 + r_{\Gamma_T}[G_0^{(2)} + h^{(2)}]_7 = F_7 \ \text{ on } \ \Gamma_T. \tag{3.11}$$

After some rearrangement we get the system of pseudodifferential equations

$$r_{_{S_D^{(1)}}}\big[\,\mathcal{H}_\tau^{(1)}\,[P_\tau^{(1)}]^{-1}\,\big(\psi + h^{(1)}\big)\,\big] = \widetilde{f}^{(1)} \ \text{ on } \ S_D^{(1)}, \tag{3.12}$$

$$r_{\Gamma_T}\big[\,\mathcal{H}_\tau^{(1)}\,[P_\tau^{(1)}]^{-1}\,(\,\psi + h^{(1)})\,\big]_j = \widetilde{f}_j \ \text{ on } \ \Gamma_T, \ \ j = 7, 8, \tag{3.13}$$

$$r_{\Gamma_T}\big[\mathcal{H}_\tau^{(1)}\,[P_\tau^{(1)}]^{-1}\,(\psi + h^{(1)})\big]_j - r_{\Gamma_T}\big[\mathcal{H}_\tau^{(2)}\,[P_\tau^{(2)}]^{-1}(\,h^{(2)})\big]_j = \widetilde{f}_j \ \text{ on } \ \Gamma_T, \ \ j = \overline{1,6}, \tag{3.14}$$

$$r_{\Gamma_T}\big[\mathcal{H}_\tau^{(1)}\,[P_\tau^{(1)}]^{-1}\,(\psi + h^{(1)})\big]_9 - r_{\Gamma_T}\big[\mathcal{H}_\tau^{(2)}\,[P_\tau^{(2)}]^{-1}(\,h^{(2)})\big]_7 = \widetilde{f}_9 \ \text{ on } \ \Gamma_T, \tag{3.15}$$

$$r_{\Gamma_T} h_j^{(1)} + r_{\Gamma_T} h_j^{(2)} = \widetilde{F}_j \ \text{ on } \ \Gamma_T, \ \ j = \overline{1,6}, \tag{3.16}$$

$$r_{\Gamma_T} h_9^{(1)} + r_{\Gamma_T} h_7^{(2)} = \widetilde{F}_7 \ \text{ on } \ \Gamma_T, \tag{3.17}$$

where

$$\widetilde{f}_k^{(1)} := f_k^{(1)} - r_{_{S_D^{(1)}}}\big[\,\mathcal{H}_\tau^{(1)}\,[P_\tau^{(1)}]^{-1}\,G_0^{(1)}\,\big]_k \in B_{p,p}^{1-\frac{1}{p}}(S_D^{(1)}), \ \ k = \overline{1,6}, \tag{3.18}$$

$$\widetilde{f}_j := f_j - r_{\Gamma_T}\big[\,\mathcal{H}_\tau^{(1)}\,[P_\tau^{(1)}]^{-1}\,G_0^{(1)}\big]_j \in B_{p,p}^{1-\frac{1}{p}}(\Gamma_T), \ \ j = 7, 8, \tag{3.19}$$

$$\widetilde{f}_j := f_j + r_{\Gamma_T}\big[\,\mathcal{H}_\tau^{(2)}\,[P_\tau^{(2)}]^{-1}\,G_0^{(2)}\big]_j - r_{\Gamma_T}\big[\,\mathcal{H}_\tau^{(1)}\,[P_\tau^{(1)}]^{-1}\,G_0^{(1)}\big]_j \in B_{p,p}^{1-\frac{1}{p}}(\Gamma_T), \ \ j = \overline{1,6}, \tag{3.20}$$

$$\widetilde{f}_9 := f_9 + r_{\Gamma_T}\,\big[\mathcal{H}^{(2)}_\tau\,[P^{(2)}_\tau]^{-1}\,G^{(2)}_0\big]_7 - r_{\Gamma_T}\,\big[\mathcal{H}^{(1)}_\tau\,[P^{(1)}_\tau]^{-1}\,G^{(1)}_0\big]_9 \in B^{1-\frac{1}{p}}_{p,p}(\Gamma_T), \tag{3.21}$$

$$\widetilde{F}_j := F_j - r_{\Gamma_T} G^{(1)}_{0j} - r_{\Gamma_T} G^{(2)}_{0j} \in r_{\Gamma_T}\widetilde{B}^{-\frac{1}{p}}_{p,p}(\Gamma_T), \quad j=\overline{1,6}, \tag{3.22}$$

$$\widetilde{F}_7 := F_7 - r_{\Gamma_T} G^{(1)}_{09} - r_{\Gamma_T} G^{(2)}_{07} \in r_{\Gamma_T}\widetilde{B}^{-\frac{1}{p}}_{p,p}(\Gamma_T). \tag{3.23}$$

Inclusions (3.22), (3.23) are the *compatibility conditions* for the mixed boundary-transmission problem under consideration. Therefore, in what follows, we assume that $\widetilde{F}_j$ are extended from Γ_T onto the manifold $\partial\Omega^{(2)}\cup\partial\Omega^{(1)}\setminus\Gamma_T$ by zero, i.e., $\widetilde{F}_j \in \widetilde{B}^{-\frac{1}{p}}_{p,p}(\Gamma_T)$, $j=\overline{1,7}$.

Let us introduce the Steklov–Poincaré type matrix pseudodifferential operators

$$\mathcal{A}^{(1)}_\tau := \mathcal{H}^{(1)}_\tau\,[P^{(1)}_\tau]^{-1}, \qquad \mathcal{A}^{(2)}_\tau := \mathcal{H}^{(2)}_\tau\,\big(P^{(2)}_\tau\big)^{-1}$$

and

$$\mathcal{B}^{(2)}_\tau := \begin{bmatrix} (A^{(2)}_\tau)_{11} & \dots & (A^{(2)}_\tau)_{16} & 0 & 0 & (A^{(2)}_\tau)_{17} \\ \dots & \dots & \dots & 0 & 0 & \dots \\ (A^{(2)}_\tau)_{61} & \dots & (A^{(2)}_\tau)_{66} & 0 & 0 & (A^{(2)}_\tau)_{67} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ (A^{(2)}_\tau)_{71} & \dots & (A^{(2)}_\tau)_{73} & 0 & 0 & (A^{(2)}_\tau)_{77} \end{bmatrix}_{9\times 9}.$$

Taking into account equations (3.16) and (3.17), we can rewrite equations (3.13), (3.14), (3.15) in a matrix form and, consequently, the whole system (3.12)–(3.17) can be rewritten as follows:

$$r_{S^{(1)}_D}\,\mathcal{A}^{(1)}_\tau\,(\psi + h^{(1)}) = \widetilde{f}^{(1)} \ \text{ on } \ S^{(1)}_D, \tag{3.24}$$

$$r_{\Gamma_T}\,\mathcal{A}^{(1)}_\tau\,(\psi + h^{(1)}) + r_{\Gamma_T}\,\mathcal{B}^{(2)}_\tau h^{(1)} = \widetilde{g} \ \text{ on } \ \Gamma_T, \tag{3.25}$$

$$r_{\Gamma_T}\,h^{(1)}_j + r_{\Gamma_T}\,h^{(2)}_j = \widetilde{F}_j \ \text{ on } \ \Gamma_T, \ \ j=\overline{1,6}, \tag{3.26}$$

$$r_{\Gamma_T} h^{(1)}_9 + r_{\Gamma_T} h^{(2)}_7 = \widetilde{F}_7 \ \text{ on } \ \Gamma_T, \tag{3.27}$$

where

$$\widetilde{f}^{(1)} := (\widetilde{f}^{(1)}_1,\dots,\widetilde{f}^{(1)}_9)^\top \in \big[\,B^{1-\frac{1}{p}}_{p,p}(S^{(1)}_D)\,\big]^9, \tag{3.28}$$

$$\widetilde{g} := (\widetilde{g}_1,\dots,\widetilde{g}_9)^\top \in \big[\,B^{1-\frac{1}{p}}_{p,p}(\Gamma_T)\,\big]^9, \tag{3.29}$$

$$\widetilde{g}_j := \widetilde{f}_j + r_{\Gamma_T}\big[\,\mathcal{H}^{(2)}_\tau[\,P^{(2)}_\tau\,]^{-1}\,\widetilde{F}\big]_j, \ \ j=\overline{1,6}, \tag{3.30}$$

$$\widetilde{g}_7 = \widetilde{f}_7, \ \ \widetilde{g}_8 = \widetilde{f}_8, \ \ \widetilde{g}_9 = \widetilde{f}_9 + r_{\Gamma_T}\big[\,\mathcal{H}^{(2)}_\tau[\,P^{(2)}_\tau\,]^{-1}\,\widetilde{F}\big]_7,$$

$$\widetilde{F} := (\widetilde{F}_1,\dots,\widetilde{F}_7)^\top \in \big[\,\widetilde{B}^{-\frac{1}{p}}_{p,p}(\Gamma_T)\,\big]^7. \tag{3.31}$$

It is easy to see that the simultaneous equations (3.12)–(3.17) and (3.24)–(3.27), where the right-hand sides are defined by equalities (3.18)–(3.23) and (3.28)–(3.31), are equivalent in the following sense: if the triplet $(\psi, h^{(1)}, h^{(2)}) \in [\widetilde{H}^{-\frac{1}{2}}(S^{(1)}_D)]^9 \times [\widetilde{B}^{-\frac{1}{p}}_{p,p}(\Gamma_T)]^9 \times [\widetilde{B}^{-\frac{1}{p}}_{p,p}(\Gamma_T)]^7$ solves system (3.24)–(3.27), then $(\psi, h^{(1)}, h^{(2)})$ solves system (3.12)–(3.17), and vice versa.

3.2. **The existence theorems and regularity of solutions.** Here, we show that the system of pseudodifferential equations (3.24)–(3.27) is uniquely solvable in appropriate function spaces. To this end, let us introduce the notation

$$\mathcal{N}_\tau := \begin{bmatrix} r_{S^{(1)}_D}\,\mathcal{A}^{(1)}_\tau & r_{S^{(1)}_D}\,\mathcal{A}^{(1)}_\tau & r_{S^{(1)}_D}\,[\,0\,]_{9\times 7} \\ r_{\Gamma_T}\,\mathcal{A}^{(1)}_\tau & r_{\Gamma_T}[\mathcal{A}^{(1)}_\tau + \mathcal{B}^{(2)}_\tau] & r_{\Gamma_T}\,[\,0\,]_{9\times 7} \\ r_{\Gamma_T}\,[\,0\,]_{7\times 9} & r_{\Gamma_T}\,I_{7\times 9} & r_{\Gamma_T}\,I_7 \end{bmatrix}_{25\times 25},$$

$$I_{7\times 9} := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{7\times 9} .$$

Further, let

$$\begin{aligned} \Phi : &= (\psi,\ h^{(1)},\ h^{(2)})^\top, \qquad Y := (\widetilde{f},\ \widetilde{g},\ \widetilde{F})^\top, \\ \mathbf{X}_p^s &:= \big[\,\widetilde{B}^s_{p,p}(S_D^{(1)})\,\big]^9 \times \big[\,\widetilde{B}^s_{p,p}(\Gamma_T)\,\big]^9 \times \big[\,\widetilde{B}^s_{p,p}(\Gamma_T)\,\big]^7, \\ \mathbf{Y}_p^s &:= \big[\,B^{s+1}_{p,p}(S_D^{(1)})\,\big]^9 \times \big[\,B^{s+1}_{p,p}(\Gamma_T)\,\big]^9 \times \big[\,\widetilde{B}^s_{p,p}(\Gamma_T)\,\big]^7, \\ \mathbf{X}_{p,q}^s &:= \big[\,\widetilde{B}^s_{p,q}(S_D^{(1)})\,\big]^9 \times \big[\,\widetilde{B}^s_{p,q}(\Gamma_T)\,\big]^9 \times \big[\,\widetilde{B}^s_{p,q}(\Gamma_T)\,\big]^7, \\ \mathbf{Y}_{p,q}^s &:= \big[\,B^{s+1}_{p,q}(S_D^{(1)})\,\big]^9 \times \big[\,B^{s+1}_{p,q}(\Gamma_T)\,\big]^9 \times \big[\,\widetilde{B}^s_{p,q}(\Gamma_T)\,\big]^7. \end{aligned}$$

Note that

$$\mathbf{X}_2^s = \mathbf{X}_{2,2}^s, \qquad \mathbf{Y}_2^s = \mathbf{Y}_{2,2}^s, \quad \forall s \in \mathbb{R}.$$

System (3.24)–(3.27) can be rewritten as follows:

$$\mathcal{N}_\tau\, \Phi = Y, \tag{3.32}$$

where $\Phi \in \mathbf{X}_p^s$ is the sought for vector function and $Y \in \mathbf{Y}_p^s$ is the given vector function.

Due to Theorems 4.3 and 4.4, the operator $\mathcal{N}_\tau$ has the following mapping properties:

$$\begin{aligned} \mathcal{N}_\tau \ &: \ \mathbf{X}_p^s \to \mathbf{Y}_p^s, \\ \mathcal{N}_\tau \ &: \ \mathbf{X}_{p,q}^s \to \mathbf{Y}_{p,q}^s, \end{aligned} \tag{3.33}$$

for all $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$. As it will become clear later, the operator (3.33) is not invertible for all $s \in \mathbb{R}$. The interval $a < s < b$ of invertibility depends on p and on some parameters γ' and γ'' (see (3.40)–(3.42)), which are determined by the eigenvalues of special matrices constructed by means of the principal homogeneous symbol matrices of the operators $\mathcal{A}_\tau^{(1)}$ and $\mathcal{A}_\tau^{(1)} + \mathcal{B}_\tau^{(2)}$. Note that the numbers γ' and γ'' determine also Hölder's smoothness exponents for the solutions to the original mixed boundary-transmission problem in the neighbourhood of the exceptional curves $\partial S_D^{(1)}$, $\partial\Gamma_C$ and $\partial\Gamma$.

We start with the following

Theorem 3.1. *Let the conditions*

$$1 < p < \infty, \quad 1 \leqslant q \leqslant \infty, \quad \frac{1}{p} - 1 + \gamma'' < s + \frac{1}{2} < \frac{1}{p} + \gamma' \tag{3.34}$$

be satisfied with γ' and γ'' given by (3.42)*. Then the operators* (3.33) *are invertible.*

Proof. We prove the theorem in several steps. First, we show that (3.33) are Fredholm operates with a zero index and afterwards we establish that the corresponding null-spaces are trivial.

Step 1. Let us note that the operators

$$\begin{aligned} r_{S_D^{(1)}}\, \mathcal{A}_\tau^{(1)} \ &: \ \big[\,\widetilde{B}^s_{p,p}(\Gamma_T)\,\big]^9 \to \big[\,B^{s+1}_{p,p}(S_D^{(1)})\,\big]^9, \\ r_{\Gamma_T}\, \mathcal{A}_\tau^{(1)} \ &: \ \big[\,\widetilde{B}^s_{p,q}(S_D^{(1)})\,\big]^9 \to \big[\,B^{s+1}_{p,q}(\Gamma_T)\,\big]^9 \end{aligned} \tag{3.35}$$

are compact, since $S_D^{(1)}$ and Γ_T are disjoint, $\overline{S_D^{(1)}} \cap \overline{\Gamma_T} = \varnothing$. Further, we establish that the operators

$$\begin{aligned} r_{S_D^{(1)}}\, \mathcal{A}_\tau^{(1)} \ &: \ \big[\,\widetilde{H}_2^{-\frac{1}{2}}(S_D^{(1)})\,\big]^9 \to \big[\,[H_2^{\frac{1}{2}}(S_D^{(1)})\,\big]^9, \\ r_{\Gamma_T}\big[\,\mathcal{A}_\tau^{(1)} + \mathcal{B}_\tau^{(2)}\,\big] \ &: \ \big[\,\widetilde{H}_2^{-\frac{1}{2}}(\Gamma_T)\,\big]^9 \to \big[\,H_2^{\frac{1}{2}}(\Gamma_T)\,\big]^9 \end{aligned} \tag{3.36}$$

are strongly elliptic Fredholm pseudodifferential operators of order -1 with a zero index. We note that the principal homogeneous symbol matrices of these operators are strongly elliptic.

Using Green's formula and Korn's inequality, for an arbitrary solution vector $U^{(1)}\in[H_2^1(\Omega^{(1)})]^9$ to the homogeneous equation

$$A^{(1)}(\partial_x,\,\tau)U^{(1)}=0 \quad \text{in} \quad \Omega^{(1)},$$

by the standard arguments, we derive (see, e.g., [6, 7])

$$\operatorname{Re}\big\langle[U^{(1)}]^+,\,[\mathcal{T}^{(1)}U^{(1)}]^+\big\rangle_{\partial\Omega^{(1)}}\geqslant c_1\,\|\,U^{(1)}\,\|^2_{[H_2^1(\Omega^{(1)})]^9}-c_2\,\|\,U^{(1)}\,\|^2_{[H_2^0(\Omega^{(1)})]^9}, \tag{3.37}$$

where $\langle\cdot,\cdot\rangle_{\partial\Omega^{(1)}}$ denotes the duality pairing between the spaces $\big[H^{\frac12}(\partial\Omega^{(1)})\big]^9$ and $\big[H^{-\frac12}(\partial\Omega^{(1)})\big]^9$.

Substitute here $U^{(1)}=V_\tau^{(1)}([P_\tau^{(1)}]^{-1}\zeta)$ with $\zeta\in[H_2^{-\frac12}(\partial\Omega^{(1)})]^9$. Due to the equality

$$\zeta=P_\tau^{(1)}[\mathcal{H}_\tau^{(1)}]^{-1}\{U^{(1)}\}^+$$

and the boundedness of the operators involved, we have

$$\|\zeta\|^2_{[H_2^{-\frac12}(\partial\Omega^{(1)})]^9}\leqslant c^*\|\{U^{(1)}\}^+\|^2_{[H_2^{\frac12}(\partial\Omega^{(1)})]^9}$$

with some positive constant c^*. By the properties of single layer potentials, we have

$$\big\{U^{(1)}\big\}^+=\mathcal{H}_\tau^{(1)}\big[P_\tau^{(1)}\big]^{-1}\zeta,\quad \big\{\mathcal{T}^{(1)}U^{(1)}\big\}^+=\Big(-\frac12 I_9+\mathcal{K}_\tau^{(1)}\Big)\big[P_\tau^{(1)}\big]^{-1}\zeta.$$

By the trace theorem, from (3.37), we deduce

$$\operatorname{Re}\langle\mathcal{H}_\tau^{(1)}[P_\tau^{(1)}]^{-1}\zeta,\,\big(-2^{-1}\,I_9+\mathcal{K}_\tau^{(1)}+\beta\,\mathcal{H}_\tau^{(1)}\big)\big[P_\tau^{(1)}\big]^{-1}\zeta\rangle_{\partial\Omega^{(1)}}\geqslant c_1'\,\|\,\zeta\,\|^2_{[H_2^{-\frac12}(\partial\Omega^{(1)})]^9}$$
$$+\big\|\,\beta\,\mathcal{H}^{(1)}[P_\tau^{(1)}]^{-1}\zeta\big\|^2_{[H_2^{\frac12}(\partial\Omega^{(1)})]^9}-c_2\,\big\|\,V_\tau^{(1)}([P_\tau^{(1)}]^{-1}\zeta)\big\|^2_{[H_2^0(\Omega^{(1)})]^9}.$$

Thus we have

$$\operatorname{Re}\langle\mathcal{H}_\tau^{(1)}[P_\tau^{(1)}]^{-1}\zeta,\,\zeta\rangle_{\partial\Omega^{(1)}}\geqslant c_1'\,\|\,\zeta\,\|^2_{[H_2^{-\frac12}(\partial\Omega^{(1)})]^9}$$
$$+\big\|\,\beta\,\mathcal{H}^{(1)}[P_\tau^{(1)}]^{-1}\zeta\big\|^2_{[H_2^{\frac12}(\partial\Omega^{(1)})]^9}-c_2\,\big\|\,V_\tau^{(1)}([P_\tau^{(1)}]^{-1}\zeta)\big\|^2_{[H_2^0(\Omega^{(1)})]^9}.$$

In particular, in view of Theorem 4.1, for arbitrary $\zeta\in[\widetilde{H}_2^{-\frac12}(S_D^{(1)})]^9$, we have

$$\|\,U^{(1)}\,\|^2_{[H_2^0(\Omega^{(1)})]^9}\leqslant c^{**}\,\|\,\zeta\,\|^2_{[\widetilde{H}_2^{-\frac32}(S_D^{(1)})]^9},$$

and, consequently,

$$\operatorname{Re}\big\langle r_{_{S_D^{(1)}}}\mathcal{H}_\tau^{(1)}[P_\tau^{(1)}]^{-1}\zeta,\,\zeta\big\rangle_{\partial\Omega^{(1)}}\geqslant c_1'\,\|\,\zeta\,\|^2_{[\widetilde{H}_2^{-\frac12}(S_D^{(1)})]^9}-c_2''\,\|\,\zeta\,\|^2_{[\widetilde{H}_2^{-\frac32}(S_D^{(1)})]^9}. \tag{3.38}$$

From (3.38), it follows that

$$r_{_{S_D^{(1)}}}\mathcal{A}_\tau^{(1)}=r_{_{S_D^{(1)}}}\mathcal{H}_\tau^{(1)}[P_\tau^{(1)})]^{-1}\ :\ \big[\,\widetilde{H}_2^{-\frac12}(S_D^{(1)})\,\big]^9\to\big[\,H_2^{\frac12}(S_D^{(1)})\,\big]^9$$

is a strongly elliptic pseudodifferential Fredholm operator with index zero (see [18, 21]).

Then the same is true for the operator (3.36), since the principal homogeneous symbol matrix of the operator $\mathcal{B}_\tau^{(2)}$ is nonnegative (see [23]). Therefore, the operator (3.33) is Fredholm with index zero for $s=-1/2$, $p=2$ and $q=2$ due to the compactness of operators (3.35).

Step 2. With the help of the uniqueness Theorem 2.1, due to the representation formulas (3.4) and (3.5) with $G_0^{(1)}=0$ and $G_0^{(2)}=0$, we can easily show that the operator (3.33) is injective for $s=-1/2$, $p=2$ and $q=2$. Since its index is zero, we conclude that it is surjective. Thus the operator (3.33) is invertible for $s=-1/2$, $p=2$ and $q=2$.

Step 3. To complete the proof, for the general case, we proceed as follows. The following blockwise lower triangular matrix pseudodifferential operator

$$\mathcal{N}_\tau^{(0)} := \begin{bmatrix} r_{_{S_D^{(1)}}} \mathcal{A}_\tau^{(1)} & r_{_{S_D^{(1)}}} [\,0\,]_{9\times 9} & r_{_{S_D^{(1)}}} [\,0\,]_{9\times 7} \\ r_{\Gamma_T} [\,0\,]_{9\times 9} & r_{\Gamma_T} [\mathcal{A}_\tau^{(1)} + \mathcal{B}_\tau^{(2)}] & r_{\Gamma_T} [\,0\,]_{9\times 7} \\ r_{\Gamma_T} [\,0\,]_{7\times 9} & r_{\Gamma_T}\, I_{7\times 9} & r_{\Gamma_T}\, I_7 \end{bmatrix}_{25\times 25}$$

is a compact perturbation of the operator $\mathcal{N}_\tau$. Let us analyze the properties of the diagonal entries

$$r_{_{S_D^{(1)}}} \mathcal{A}_\tau^{(1)} \;:\; [\,\widetilde{B}^s_{p,q}(S_D^{(1)})]^9 \to \big[\,B^{s+1}_{p,q}(S_D^{(1)})\,\big]^9,$$

$$r_{\Gamma_T} \big[\,\mathcal{A}_\tau^{(1)} + \mathcal{B}_\tau^{(2)}\,\big] \;:\; \big[\,\widetilde{B}^s_{p,q}(\Gamma_T)]^9 \to \big[\,B^{s+1}_{p,q}(\Gamma_T)\,\big]^9.$$

Let

$$\mathfrak{S}_1(x,\xi_1,\xi_2) := \mathfrak{S}(\mathcal{A}_\tau^{(1)}; x, \xi_1, \xi_2)$$

be the principal homogeneous symbol matrix of the operator $\mathcal{A}_\tau^{(1)}$ and let $\lambda_j^{(1)}(x)$ $(j = \overline{1,9})$ be the eigenvalues of the matrix

$$\mathcal{D}_1(x) := \big[\mathfrak{S}_1(x,0,+1)\big]^{-1}\, \mathfrak{S}_1(x,0,-1), \quad x \in \partial S_D^{(1)}.$$

Similarly, let

$$\mathfrak{S}_2(x,\xi_1,\xi_2) = \mathfrak{S}(\mathcal{A}_\tau^{(1)} + \mathcal{B}_\tau^{(2)}; x, \xi_1, \xi_2)$$

be the principal homogeneous symbol matrix of the operator $\mathcal{A}_\tau^{(1)} + \mathcal{B}_\tau^{(2)}$ and let $\lambda_j^{(2)}(x)$ $(j = \overline{1,9})$ be the eigenvalues of the corresponding matrix

$$\mathcal{D}_2(x) := \big[\mathfrak{S}_2(x,0,+1)\big]^{-1}\, \mathfrak{S}_2(x,0,-1), \quad x \in \partial\Gamma_T. \tag{3.39}$$

Note that the curve $\partial\Gamma_T$ is the union of the curves, where the interface intersects the exterior boundary $\partial\Gamma$, and the crack edge $\partial\Gamma_C$, $\partial\Gamma_T = \partial\Gamma \cup \partial\Gamma_C$.

Further, we set

$$\gamma_1' := \inf_{x\in\partial S_D^{(1)},\, 1\leqslant j\leqslant 9} \frac{1}{2\pi}\, \arg \lambda_j^{(1)}(x), \quad \gamma_1'' := \sup_{x\in\partial S_D^{(1)},\, 1\leqslant j\leqslant 9} \frac{1}{2\pi}\, \arg \lambda_j^{(1)}(x), \tag{3.40}$$

$$\gamma_2' := \inf_{x\in\partial \Gamma_T,\, 1\leqslant j\leqslant 9} \frac{1}{2\pi}\, \arg \lambda_j^{(2)}(x), \quad \gamma_2'' := \sup_{x\in\partial \Gamma_T,\, 1\leqslant j\leqslant 9} \frac{1}{2\pi}\, \arg \lambda_j^{(2)}(x). \tag{3.41}$$

Note that γ_i' and γ_i'' $(i = 1, 2)$ depend on the material parameters in general and does not depend on the geometry of the curves $\partial S_D^{(1)}$, $\partial\Gamma_T$. Since some of the eigenvalues equal to 1, we have $\gamma_i' \in (-\frac{1}{2}, 0]$, $\gamma_i'' \in [0, \frac{1}{2})$ $i = 1.2$ (cf. [8, 9]).

Let's introduce the notation

$$\gamma' := \min\{\gamma_1',\, \gamma_2'\}, \quad \gamma'' := \max\{\gamma_1'',\, \gamma_2''\}, \tag{3.42}$$

then

$$-\frac{1}{2} < \gamma' \leq 0 \leq \gamma'' < \frac{1}{2}. \tag{3.43}$$

From Theorem 4.5, we conclude that if the parameters $r_1, r_2 \in \mathbb{R}$, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, satisfy the conditions

$$\frac{1}{p} - 1 + \gamma_1'' < r_1 + \frac{1}{2} < \frac{1}{p} + \gamma_1', \qquad \frac{1}{p} - 1 + \gamma_2'' < r_2 + \frac{1}{2} < \frac{1}{2}\gamma_2',$$

then the operators

$$r_{_{S_D^{(1)}}} \mathcal{A}_\tau^{(1)} \;:\; \big[\,\widetilde{H}^{r_1}_p(S_D^{(1)})\,\big]^9 \to \big[\,H^{r_1+1}_p(S_D^{(1)})\,\big]^9,$$

$$r_{_{S_D^{(1)}}} \mathcal{A}_\tau^{(1)} \;:\; \big[\,\widetilde{B}^{r_1}_{p,q}(S_D^{(1)})\,\big]^9 \to \big[\,B^{r_1+1}_{p,q}(S_D^{(1)})\,\big]^9,$$

$$r_{\Gamma_T} \big[\,\mathcal{A}_\tau^{(1)} + \mathcal{B}_\tau^{(2)}\,\big] \;:\; \big[\,\widetilde{H}^{r_2}_p(\Gamma_T)\,\big]^9 \to \big[\,H^{r_2+1}_p(\Gamma_T)\,\big]^9,$$

$$r_{\Gamma_T}\big[\,\mathcal{A}_\tau^{(1)}+\mathcal{B}_\tau^{(2)}\,\big]\;:\;\big[\,\widetilde{B}_{p,q}^{r_2}(\Gamma_T)\,\big]^9\to\big[\,B_{p,q}^{r_2+1}(\Gamma_T)\,\big]^9$$

are the Fredholm operators with zero index.

Therefore, if conditions (3.34) are satisfied, then the above operators are Fredholm ones with a zero index. Consequently, operators (3.33) are Fredholm with zero index and invertible due to the results obtained in Step 2 (see [2]) . □

Now we formulate the basic existence and uniqueness results for the interface crack boundary-transmission problem under consideration.

Theorem 3.2. *Let inclusions* (2.23) *and compatibility conditions* (3.22), (3.23) *hold and let*

$$\frac{4}{3-2\gamma''}<p<\frac{4}{1-2\gamma'} \tag{3.44}$$

with γ' *and* γ'' *be defined in* (3.42)*. Then the interface crack boundary-transmission problem* (2.10)–(2.22) *has a unique solution*

$$(U^{(1)},U^{(2)})\in\big[\,W_p^1(\Omega^{(1)})\,\big]^9\times[\,W_p^1(\Omega^{(2)})\,]^7,$$

which can be represented by the formulas

$$U^{(1)}=V_\tau^{(1)}\big([P_\tau^{(1)}]^{-1}\big[\,G_0^{(1)}+\psi+h^{(1)}\big]\big)\ \ in\ \ \Omega^{(1)}, \tag{3.45}$$

$$U^{(2)}=V_\tau^{(2)}\big(\big[\,P_\tau^{(2)}\,\big]^{-1}\big[\,G_0^{(2)}+h^{(2)}\big]\big)\ \ in\ \ \Omega^{(2)}, \tag{3.46}$$

where the densities ψ, $h^{(1)}$ *and* $h^{(2)}$ *are to be determined from system* (3.6)–(3.11) (*or from system* (3.24)–(3.27))*, while* $G_0^{(1)}$ *and* $G_0^{(2)}$ *are some fixed extensions of the vector functions* $G^{(1)}$ *and* $G^{(2)}$*, respectively, onto* $\partial\Omega^{(1)}$ *and* $\partial\Omega^{(2)}$*, preserving the space* (*see* (3.1) *and* (3.2)).

Moreover, the vector functions $G_0^{(1)}+\psi+h^{(1)}$ *and* $G_0^{(2)}+h^{(2)}$ *are defined uniquely by the above systems and are independent of the extension operators.*

Proof. From Theorems 4.1, 4.2 and 3.1 with p satisfying (3.44) and $s=-1/p$ it follows immediately that the pair $(U^{(1)},U^{(2)})\in[W_p^1(\Omega^{(1)})]^9\times[W_p^1(\Omega^{(2)})]^7$ given by (3.45), (3.46) represents a solution to the interface crack boundary-transmission problem (2.10)–(2.22). Next, we show the uniqueness of solutions.

Due to inequalities (3.43), we have

$$p=2\in\Big(\frac{4}{3-2\gamma''},\frac{4}{1-2\gamma'}\Big).$$

Therefore the unique solvability for $p=2$ is a consequence of Theorem 2.1.

To show the uniqueness result for all other values of p from the interval (3.44), we proceed as follows. Let a pair

$$(U^{(1)},U^{(2)})\in\big[\,W_p^1(\Omega^{(1)})\,\big]^9\times[\,W_p^1(\Omega^{(2)})\,]^7$$

with p satisfying (3.44), be a solution to the homogeneous interface crack boundary-transmission problem. Then it is evident that

$$\big\{U^{(1)}\big\}^+\in\big[\,B_{p,p}^{1-\frac1p}(\partial\Omega^{(1)})\,\big]^9,\qquad\big\{U^{(2)}\big\}^+\in\big[\,B_{p,p}^{1-\frac1p}(\partial\Omega^{(2)})\,\big]^7,$$

$$\big\{\mathcal{T}^{(1)}U^{(1)}\big\}^+\in\big[\,B_{p,p}^{-\frac1p}(\partial\Omega^{(1)})\,\big]^9,\qquad\big\{\mathcal{T}^{(2)}U^{(2)}\big\}^+\in\big[\,B_{p,p}^{-\frac1p}(\partial\Omega^{(2)})\,\big]^7.$$

By Lemmas 2.2 and 2.3, the vectors $U^{(2)}$ and $U^{(1)}$ in $\Omega^{(2)}$ and $\Omega^{(1)}$, respectively, are representable in the form

$$U^{(2)}=V_\tau^{(2)}\big(\big[\,P_\tau^{(2)}\big]^{-1}h^{(2)}\big)\ \text{ in }\ \Omega^{(2)},\qquad h^{(2)}=\big\{\mathcal{T}^{(2)}U^{(2)}\big\}^+,$$

$$U^{(1)}=V_\tau^{(1)}\big([P_\tau^{(1)}]^{-1}\chi\big)\ \text{ in }\ \Omega^{(1)},\qquad\chi=\big\{\mathcal{T}^{(1)}U^{(1)}\big\}^++\beta\,\big\{U^{(1)}\big\}^+.$$

Moreover, due to the homogeneous boundary and transmission conditions, we have

$$h^{(2)}\in\big[\,\widetilde{B}_{p,p}^{-\frac1p}(\Gamma_T)\,\big]^7,\ \ \chi=h^{(1)}+\psi\in[B_{p,p}^{-\frac1p}(S^{(1)})\,\big]^9,\ \ h^{(1)}\in\big[\,\widetilde{B}_{p,p}^{-\frac1p}(\Gamma_T)\,\big]^9,\ \ \psi\in\big[\,\widetilde{B}_{p,p}^{-\frac1p}(S_D^{(1)})\,\big]^9.$$

By the same arguments as above, we arrive at the homogeneous system

$$\mathcal{N}_\tau\, \Phi = 0 \ \text{ with } \ \Phi := (\psi,\ h^{(1)},\ h^{(2)})^\top \in \mathbf{X}_p^{-\frac{1}{p}}.$$

Due to Theorem 3.1, $\Phi = 0$ and we conclude that $U^{(2)} = 0$ in $\Omega^{(2)}$ and $U^{(1)} = 0$ in $\Omega^{(1)}$.

The last assertion of the theorem is trivial and is an easy consequence of the fact that if the single layer potentials (3.45)and (3.46) vanish identically in $\Omega^{(2)}$ and $\Omega^{(1)}$, then the corresponding densities vanish, as well. □

The following regularity result is true.

Theorem 3.3. *Let the inclusions* (2.23) *and the compatibility conditions* (3.22), (3.23) *hold and let for* $1 < r < \infty$, $1 \leqslant q \leqslant \infty$,

$$\frac{4}{3-2\gamma''} < p < \frac{4}{1-2\gamma'}\,, \qquad \frac{1}{r} - \frac{1}{2} + \gamma'' < s < \frac{1}{r} + \frac{1}{2} + \gamma', \tag{3.47}$$

with γ' *and* γ'' *defined in* (3.42).

Further, let $U^{(1)} \in [W_p^1(\Omega^{(1)})]^9$ *and* $U^{(2)} \in [W_p^1(\Omega^{(2)})]^7$ *be a unique solution pair to the interface crack boundary-transmission problem* (2.10)–(2.22). *Then the following items hold:*

(i) *if*

$$Q_k^{(1)} \in B_{r,r}^{s-1}(S_N^{(1)}), \quad Q_j^{(2)} \in B_{r,r}^{s-1}(S_N^{(2)}), \quad f_k^{(1)} \in B_{r,r}^{s}(S_D^{(1)}), \quad f_k \in B_{r,r}^{s}(\Gamma_T), \quad F_j \in B_{r,r}^{s-1}(\Gamma_T),$$
$$\widetilde{Q}_j^{(2)} \in B_{r,r}^{s-1}(\Gamma_C), \quad \widetilde{Q}_k^{(1)} \in B_{r,r}^{s-1}(\Gamma_C), \ \ k=\overline{1,9}, \ \ j=\overline{1,7},$$

and the compatibility conditions

$$\widetilde{F}_j := F_j - r_{_{\Gamma_T}} G_{0j}^{(1)} - r_{_{\Gamma_T}} G_{0j}^{(2)} \in r_{_{\Gamma_T}} \widetilde{B}_{r,r}^{s-1}(\Gamma_T), \ \ j=\overline{1,6},$$
$$\widetilde{F}_7 := F_7 - r_{_{\Gamma_T}} G_{09}^{(1)} - r_{_{\Gamma_T}} G_{07}^{(2)} \in r_{_{\Gamma_T}} \widetilde{B}_{r,r}^{s-1}(\Gamma_T),$$

are satisfied, then

$$U^{(1)} \in [\,H_r^{s+\frac{1}{r}}(\Omega^{(1)})\,]^9, \quad U^{(2)} \in [\,H_r^{s+\frac{1}{r}}(\Omega^{(2)})\,]^7;$$

(ii) *if*

$$Q_k^{(1)} \in B_{r,q}^{s-1}(S_N^{(1)}), \quad Q_j^{(2)} \in B_{r,q}^{s-1}(S_N^{(2)}), \quad f_k^{(1)} \in B_{r,q}^{s}(S_D^{(1)}), \quad f_k \in B_{r,q}^{s}(\Gamma_T), \quad F_j \in B_{r,q}^{s-1}(\Gamma_T),$$
$$\widetilde{Q}_j^{(2)} \in B_{r,q}^{s-1}(\Gamma_C), \quad \widetilde{Q}_k^{(1)} \in B_{r,q}^{s-1}(\Gamma_C), \ \ k=\overline{1,9}, \ \ j=\overline{1,7},$$

and the compatibility conditions

$$\widetilde{F}_j := F_j - r_{_{\Gamma_T}} G_{0j}^{(1)} - r_{_{\Gamma_T}} G_{0j}^{(2)} \in r_{_{\Gamma_T}} \widetilde{B}_{r,q}^{s-1}(\Gamma_T), \ \ j=\overline{1,6},$$
$$\widetilde{F}_7 := F_7 - r_{_{\Gamma_T}} G_{09}^{(1)} - r_{_{\Gamma_T}} G_{07}^{(2)} \in r_{_{\Gamma_T}} \widetilde{B}_{r,q}^{s-1}(\Gamma_T),$$

are satisfied, then

$$U^{(1)} \in \big[\,B_{r,q}^{s+\frac{1}{r}}(\Omega^{(1)})\,\big]^9, \quad U^{(2)} \in \big[\,B_{r,q}^{s+\frac{1}{r}}(\Omega^{(2)})\,\big]^7;$$

(iii) *if* $\alpha > 0$ *is not an integer and*

$$Q_k^{(1)} \in B_{\infty,\infty}^{\alpha-1}(S_N^{(1)}), \quad Q_j^{(2)} \in B_{\infty,\infty}^{\alpha-1}(S_N^{(2)}), \quad f_k^{(1)} \in C^\alpha(\overline{S_D^{(1)}}), \quad f_k \in C^\alpha(\overline{\Gamma_T}),$$
$$F_j \in B_{\infty,\infty}^{\alpha-1}(\Gamma_T), \quad \widetilde{Q}_j^{(2)} \in B_{\infty,\infty}^{\alpha-1}(\Gamma_C), \quad \widetilde{Q}_k^{(1)} \in B_{\infty,\infty}^{\alpha-1}(\Gamma_C), \ \ k=\overline{1,9}, \ \ j=\overline{1,7},$$

and the compatibility conditions

$$\widetilde{F}_j := F_j - r_{_{\Gamma_T}} G_{0j}^{(1)} - r_{_{\Gamma_T}} G_{0j}^{(2)} \in r_{_{\Gamma_T}} \widetilde{B}_{\infty,\infty}^{\alpha-1}(\Gamma_T), \ \ j=\overline{1,6},$$
$$\widetilde{F}_7 := F_7 - r_{_{\Gamma_T}} G_{09}^{(1)} - r_{_{\Gamma_T}} G_{07}^{(2)} \in r_{_{\Gamma_T}} \widetilde{B}_{\infty,\infty}^{\alpha-1}(\Gamma_T),$$

are satisfied, then

$$U^{(1)} \in \bigcap_{\alpha'<\kappa} \big[\,C^{\alpha'}(\,\overline{\Omega^{(1)}}\,)\,\big]^9, \quad U^{(2)} \in \bigcap_{\alpha'<\kappa} \big[\,C^{\alpha'}(\,\overline{\Omega^{(2)}}\,)\,\big]^7,$$

where $\kappa = \min\{\alpha,\ \gamma' + \frac{1}{2}\} > 0$.

Proof. It is word for word repeats the proof of Theorem 5.22 in [6]. □

4. Appendix

4.1. **Properties of potentials and boundary operators.** Here, we collect some theorems describing the mapping properties of potentials and the corresponding boundary integral (pseudodifferential) operators. The proof of these theorems can be found in references [6–8, 17].

Theorem 4.1. *Let $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, $s \in \mathbb{R}$. Then the single layer potentials can be extended to the following continuous operators:*

$$V_\tau^{(2)} : \big[B_{p,q}^s(\partial\Omega^{(2)})\big]^7 \to \big[B_{p,q}^{s+1+\frac{1}{p}}(\Omega^{(2)})\big]^7, \qquad V_\tau^{(1)} : \big[B_{p,q}^s(\partial\Omega^{(1)})\big]^9 \to \big[H_p^{s+1+\frac{1}{p}}(\Omega^{(1)})\big]^9,$$
$$V_\tau^{(2)} : \big[H_p^s(\partial\Omega^{(2)})\big]^7 \to \big[H_p^{s+1+\frac{1}{p}}(\Omega^{(2)})\big]^7, \qquad V_\tau^{(1)} : \big[H_p^s(\partial\Omega^{(1)})\big]^9 \to \big[H_p^{s+1+\frac{1}{p}}(\Omega^{(1)})\big]^9.$$

Theorem 4.2. *Let $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, $s \in \mathbb{R}$, $h^{(2)} \in \big[H^{-\frac{1}{2}}(\partial\Omega^{(2)})\big]^7$, $h^{(1)} \in \big[H^{-\frac{1}{2}}(\partial\Omega^{(1)})\big]^9$. Then*

$$\big\{V_\tau^{(2)}(h^{(2)})\big\}^+ = \big\{V_\tau^{(2)}(h^{(2)})\big\}^- = \mathcal{H}_\tau^{(2)}(h^{(2)}) \ \text{ on } \ \partial\Omega^{(2)},$$
$$\big\{\mathcal{T}^{(2)}(\partial,\nu,\tau)V_\tau^{(2)}(h^{(2)})\big\}^\pm = \big[\mp 2^{-1}I_7 + \mathcal{K}_\tau^{(2)}\big](h^{(2)}) \ \text{ on } \ \partial\Omega^{(2)},$$
$$\big\{V_\tau^{(1)}(h^{(1)})\big\}^+ = \big\{V_\tau^{(1)}(h^{(1)})\big\}^- = \mathcal{H}_\tau^{(1)}(h^{(1)}) \ \text{ on } \ \partial\Omega^{(1)},$$
$$\big\{\mathcal{T}^{(1)}(\partial,n,\tau)V_\tau^{(1)}(h^{(1)})\big\}^\pm = \big[\mp 2^{-1}I_9 + \mathcal{K}_\tau^{(1)}\big](h^{(1)}) \ \text{ on } \ \partial\Omega^{(1)},$$

where I_k stands for the $k \times k$ unit matrix.

The operators $\mathcal{H}_\tau^{(1)}$, $\mathcal{H}_\tau^{(2)}$, $\mathcal{K}_\tau^{(1)}$ and $\mathcal{K}_\tau^{(2)}$ possess the following mapping and the Fredholm properties.

Theorem 4.3. *Let $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, $s \in \mathbb{R}$. The operators*

$$\mathcal{H}_\tau^{(2)} : \big[H_p^s(\partial\Omega^{(2)})\big]^7 \to \big[H_p^{s+1}(\partial\Omega^{(2)})\big]^7, \qquad \mathcal{H}_\tau^{(1)} : \big[H_p^s(\partial\Omega^{(1)})\big]^9 \to \big[H_p^{s+1}(\partial\Omega^{(1)})\big]^9,$$
$$\mathcal{H}_\tau^{(2)} : \big[B_{p,q}^s(\partial\Omega^{(2)})\big]^7 \to \big[B_{p,q}^{s+1}(\partial\Omega^{(2)})\big]^7, \qquad \mathcal{H}_\tau^{(1)} : \big[B_{p,q}^s(\partial\Omega^{(1)})\big]^9 \to \big[B_{p,q}^{s+1}(\partial\Omega^{(1)})\big]^9,$$
$$\mathcal{K}_\tau^{(2)} : \big[H_p^s(\partial\Omega^{(2)})\big]^7 \to \big[H_p^s(\partial\Omega^{(2)})\big]^7, \qquad \mathcal{K}_\tau^{(1)} : \big[H_p^s(\partial\Omega^{(1)})\big]^9 \to \big[H_p^s(\partial\Omega^{(1)})\big]^9,$$
$$\mathcal{K}_\tau^{(2)} : \big[B_{p,q}^s(\partial\Omega^{(2)})\big]^7 \to \big[B_{p,q}^s(\partial\Omega^{(2)})\big]^7, \qquad \mathcal{K}_\tau^{(1)} : \big[B_{p,q}^s(\partial\Omega^{(1)})\big]^9 \to \big[B_{p,q}^s(\partial\Omega^{(1)})\big]^9.$$

are continuous.

Theorem 4.4. *Let $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, $s \in \mathbb{R}$ and $\tau = \sigma + i\,\omega$. The operators*

$$\mathcal{H}_\tau^{(2)} : \big[H^s(\partial\Omega^{(2)})\big]^7 \to \big[H^{s+1}(\partial\Omega^{(2)})\big]^7, \qquad \mathcal{H}_\tau^{(1)} : \big[H^s(\partial\Omega^{(1)})\big]^9 \to \big[H^{s+1}(\partial\Omega^{(1)})\big]^9,$$
$$\mathcal{H}_\tau^{(2)} : \big[B_{p,q}^s(\partial\Omega^{(2)})\big]^7 \to \big[B_{p,q}^{s+1}(\partial\Omega^{(2)})\big]^7, \qquad \mathcal{H}_\tau^{(1)} : \big[B_{p,q}^s(\partial\Omega^{(1)})\big]^9 \to \big[B_{p,q}^{s+1}(\partial\Omega^{(1)})\big]^9$$

are invertible if $\sigma > 0$, or $\tau = 0$.

The operators

$$\pm 2^{-1}\,I_7 + \mathcal{K}_\tau^{(2)} : \big[H_p^s(\partial\Omega^{(2)})\big]^7 \to \big[H_p^s(\partial\Omega^{(2)})\big]^7,$$
$$\pm 2^{-1}\,I_7 + \mathcal{K}_\tau^{(2)} : \big[B_{p,q}^s(\partial\Omega^{(2)})\big]^7 \to \big[B_{p,q}^s(\partial\Omega^{(2)})\big]^7,$$
$$2^{-1}\,I_9 + \mathcal{K}_\tau^{(1)} : \big[H_p^s(\partial\Omega^{(1)})\big]^9 \to \big[H_p^s(\partial\Omega^{(1)},)\big]^9,$$
$$2^{-1}\,I_9 + \mathcal{K}_\tau^{(1)} : \big[H^s(\partial\Omega^{(1)})\big]^9 \to \big[H^s(\partial\Omega^{(1)})\big]^9$$

are invertible if $\sigma > 0$.

The operators

$$-2^{-1}\,I_9 + \mathcal{K}_\tau^{(1)} : \big[H^s(\partial\Omega^{(1)})\big]^9 \to \big[H^s(\partial\Omega^{(1)})\big]^9,$$
$$-2^{-1}\,I_9 + \mathcal{K}_\tau^{(1)} : \big[B_{p,q}^s(\partial\Omega^{(1)})\big]^9 \to \big[B_{p,q}^s(\partial\Omega^{(1)})\big]^9$$

are Fredholm ones with the index equals to zero for any $\tau \in \mathbb{C}$.

4.2. **Fredholm properties of pseudodifferential operators on manifolds with boundary.** Let $\mathcal{M}$ be a compact, n-dimensional, smooth, nonselfintersecting manifold with the smooth boundary $\partial\mathcal{M} \neq \varnothing$ and let $\mathbf{A}(x, D)$ be a strongly elliptic $N \times N$ matrix pseudodifferential operator of order $\nu \in \mathbb{R}$ on $\overline{\mathcal{M}}$. Denote by $\mathfrak{S}(\mathbf{A}; x, \xi)$ the principal homogeneous symbol matrix of the operator $\mathbf{A}(x, D)$ in some local coordinate system ($x \in \overline{\mathcal{M}}$, $\xi \in \mathbb{R}^n \setminus \{0\}$).

Let $\lambda_1(x), \dots, \lambda_N(x)$ be the eigenvalues of the matrix

$$\big[\,\mathfrak{S}(\mathbf{A}; x, 0, \dots, 0, +1)\,\big]^{-1}\,\big[\,\mathfrak{S}(\mathbf{A}; x, 0, \dots, 0, -1)\,\big], \quad x \in \partial\mathcal{M},$$

and introduce the notation

$$\delta_j(x) = \operatorname{Re}\big[\,(2\pi\, i)^{-1} \ln \lambda_j(x)\,\big], \quad j = 1, \dots, N.$$

Here, $\ln\zeta$ denotes the branch of the logarithmic function, analytic in the complex plane cut along $(-\infty, 0]$. Note that the numbers $\delta_j(x)$ do not depend on the choice of the local coordinate system and the strong inequality $-1/2 < \delta_j(x) < 1/2$ holds for all $x \in \overline{\mathcal{M}}$, $j = \overline{1, N}$, due to the strong ellipticity of $\mathbf{A}$. In a particular case, where $\mathfrak{S}(\mathbf{A}; x, \xi)$ is a positive definite matrix for every $x \in \overline{\mathcal{M}}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$, we have $\delta_1(x) = \dots = \delta_N(x) = 0$, since the eigenvalues $\lambda_1(x), \dots, \lambda_N(x)$ are positive for all $x \in \overline{\mathcal{M}}$.

The Fredholm properties of strongly elliptic pseudo-differential operators on manifolds with a boundary are characterized by the following theorem (see [2, 3, 14, 28]).

Theorem 4.5. *Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$, and let $\mathbf{A}(x, D)$ be a pseudodifferential operator of order $\nu \in \mathbb{R}$ with the strongly elliptic symbol $\mathfrak{S}(\mathbf{A}; x, \xi)$, that is, there is a positive constant c_0 such that*

$$\operatorname{Re}\mathfrak{S}(\mathbf{A}; x, \xi)\,\eta \cdot \eta \geqslant c_0\,|\eta|^2$$

for $x \in \overline{\mathcal{M}}$, $\xi \in \mathbb{R}^n$ with $|\xi| = 1$, and $\eta \in \mathbb{C}^N$.

Then the operators

$$\begin{aligned} \mathbf{A} \ &: \ \big[\widetilde{H}^s_p(\mathcal{M})\big]^N \to \big[H^{s-\nu}_p(\mathcal{M})\big]^N \\ \mathbf{A} \ &: \ \big[\widetilde{B}^s_{p,q}(\mathcal{M})\big]^N \to \big[B^{s-\nu}_{p,q}(\mathcal{M})\big]^N \end{aligned} \tag{4.1}$$

are Fredholm and have the trivial index $\operatorname{Ind}\mathbf{A} = 0$ *if*

$$\frac{1}{p} - 1 + \sup_{\substack{x\in\partial\mathcal{M},\\ 1\leqslant j\leqslant N}} \delta_j(x) < s - \frac{\nu}{2} < \frac{1}{p} + \inf_{\substack{x\in\partial\mathcal{M},\\ 1\leqslant j\leqslant N}} \delta_j(x). \tag{4.2}$$

Moreover, the null-spaces and indices of the operators (4.1) *coincide for all values of the parameter $q \in [1, +\infty]$, provided p and s satisfy inequality* (4.2)*.*

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